On the Dirac Electron in a Gravitational Field

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In this paper the behavior of Dirac's electron in gravitational fields is discussed for several special cases. In paragraph II a general derivation of the red shift in a gravitational field is given without reference to the special atomic system concerned. Paragraph III contains a treatment of the gyromagnetic effect.

I. INTRODUCTION

SEVERAL authors[1] have lately given a formulation of the theory of Dirac's spinning electron in a gravitational field. To obtain full covariance it was necessary to remove the restriction

$$\pi_k\gamma_i-\gamma_i\pi_k=0$$

$$(\pi_k=\text{generalized momentum vector}) \quad (1)$$

previously imposed on Dirac's matrices which made them independent of the coordinates x^l. The commutation relations

$$\mathring{\gamma}_i\mathring{\gamma}_k+\mathring{\gamma}_k\mathring{\gamma}_i=2\delta_{ik} \quad (2)$$

now take on the form

$$\gamma_i\gamma_k+\gamma_k\gamma_i=2g_{ik} \quad (g_{ik}=\text{metrical tensor}) \quad (3)$$

and by differentiation of (3) one easily obtains the fundamental relation

$$\partial\gamma_i/\partial x^k=\Gamma_{ik}{}^{\mu}\gamma_\mu+\Gamma_k\gamma_i-\gamma_i\Gamma_k. \quad (4)$$

$\Gamma_{ik}{}^{\mu}$ are the Christoffel symbols, Γ_k represent an infinitesimal contact transformation. Naturally, the conditions of integrability, $\partial^2\gamma_i/\partial x^k\partial x^l=\partial^2\gamma_i/\partial x^l\partial x^k$, or more explicitly

$$\Phi_{kl}\gamma_i-\gamma_i\Phi_{kl}=\tfrac{1}{2}(R_{kl,\ i\mu}-R_{kl,\ \mu i})\gamma^\mu, \quad (5a)$$

$$\Phi_{kl}\equiv\partial\Gamma_l/\partial x^k-\partial\Gamma_k/\partial x^l+\Gamma_l\Gamma_k-\Gamma_k\Gamma_l \quad (5b)$$

$$(R_{kl,\ \mu\nu}=\text{ Riemann's curvature tensor})$$

must be satisfied. (4) can now be brought into the form

$$\gamma_i\Gamma_k-\Gamma_k\gamma_i=(b_{k,\ i\mu}-b_{k,\ \mu i})\gamma^\mu, \quad (4a)$$

which is solved by

$$\Gamma_k=\tfrac{1}{2}b_{k,\ \mu\nu}\gamma^\mu\gamma^\nu+\varphi_k\cdot 1, \quad (4b)$$

where φ_k is arbitrary.

Dirac's equation written in its general form is

$$\gamma^k(\partial/\partial x^k-\Gamma_k)\psi=mc\psi/\hbar r, \quad (6)$$

where γ^k and Γ_k satisfy (3) and (4). It transforms covariantly both under point transformations, $x'^k=x'^k(x^i)$ (where it behaves like a scalar), and also under contact transformations (S transformations, spin transformations). For if we multiply (6) by S^{-1} matrices (det $S\neq 0$) from the left we are led to an equation of the same form (6) on using the substitutions

$$\psi\rightarrow S^{-1}\psi,\quad \gamma^k\rightarrow S^{-1}\gamma^k S,\quad \partial/\partial x^k\rightarrow\partial/\partial x^k,$$
$$\Gamma_k\rightarrow S^{-1}\Gamma_k S-S^{-1}\partial S/\partial x^k, \quad (7)$$
$$\varphi_k\equiv\tfrac{1}{4}tr\Gamma_k\rightarrow\varphi_k-\tfrac{1}{4}\partial\log\det S/\partial x^k.$$

From (6) we infer that the matrix vector Γ_k/i reduces to the covariant vector potential of Dirac's theory for the case of a Galilean metric, $g_{ik}=\delta_{ik}$, if the spinframe (as represented by the γ^k matrices) is selected independent of the coordinates.* Similarly, it can also be shown that Eq. (6) and its Hermitian conjugate lead to the general relativistic extensions of familiar conservation theorems.

In this paper we shall consider the application of the generalized theory to two important physical phenomena. Paragraph II gives a derivation of Einstein's red shift as a direct consequence of the principles of Dirac's equation. In paragraph III the equation is applied to the

[1] The most complete report on the subject is contained in a paper by W. Pauli, Ann. d. Physik **18**, 337 (1933). We follow in our notation the paper by E. Schrödinger, Berl. Ber. 105, 1932. Compare also V. Bargmann, Berl. Ber. 345, 1932.

* One always can select "natural" frames of reference in which the γ's depend only on coordinates explicitly appearing in the g_{ik}. Examples of this are Eq. (25) and the ordinary Dirac equation.

treatment of the Dirac electron in an "artificial" gravitational field; namely, to the gyromagnetic effect.

II. The Red Shift in a Gravitational Field

According to the original line of argument given by Einstein for the red shift of light coming from heavy celestial bodies, the mechanism of this process has to be understood in the following way: The emitting atom sets up a train of waves of definite frequency which transport this frequency with them to the observer. This statement is correct only if the system of coordinates is a so-called "static" system as given by the Schwarzschild metric. The shift in frequency in this system of coordinates is due to the slowing down of the atomic vibrations in the source which shares this retardation with any type of "clock." Since the theory did not possess a model of an atomic clock, this statement had to be taken, according to Einstein, as a postulate which would have to be supplanted by proofs showing that the atoms actually satisfy the condition imposed by the general theory.[2]

Quantitatively the red shift in the approximation always used is given by the relation

$$\nu=\nu_0 g_{44}^{\frac{1}{2}}, \tag{8}$$

in which ν_0 stands for the frequency at zero gravitational potential. It is remarkable that the red shift is determined by one component g_{44} of the metrical tensor only. It was, of course, a tacit assumption in the derivation of (8) that the gravitational field should have no other effect on the atomic clock apart from slowing it down.

From the point of view of the generalized Dirac theory the question presents itself in the following form: Since the frequencies of the emitted light are determined by the eigenvalues of the Dirac equation, it is necessary, in order to obtain conformity with (8), that all eigenvalues in a static gravitational field shall be connected with those at a zero gravitational potential by the relation

$$E=E_0 g_{44}^{\frac{1}{2}}. \tag{9}$$

This relation should hold true independent of the electromagnetic fields of force in which the electron is moving, and only to the approximation mentioned above, that is, to the first order in the gravitational potentials. That this is actually the case shall be proved in what follows.

The Schwarzschild metric for a static gravitational field with the origin of the system of coordinates in the center of the heavy mass is given by the expression

$$ds^2=(1/g_{44})[(dx^1)^2+(dx^2)^2+(dx^3)^2] - g_{44}(dx^4)^2, \tag{10}$$

in which the actual value of

$$g_{44}=1-2m/((x^1)^2+(x^2)^2+(x^3)^2)^{\frac{1}{2}} \tag{10a}$$

need not concern us. *If we neglect the change of the gravitational field over atomic dimensions*, and put the center of the system of coordinates into the nucleus around which the electron is moving, (10) takes on the form

$$ds^2=(1/g'_{44})[(dx^1)^2+(dx^2)^2+(dx^3)^2]-g'_{44}(dx^4)^2, \tag{11}$$

in which g_{44}' is now a constant depending on the distance of the nucleus from the center of the heavy mass, i.e., on the average position of the electron. To the same approximation we are justified in neglecting the nondiagonal terms of the Γ_k and the $\Gamma_{ik}{}^{\mu}$. The γ^k also become constants on account of (3),

$$\gamma^4=\mathring{\gamma}^4(g'^{44})^{\frac{1}{2}}, \quad \gamma^k=\mathring{\gamma}^k/(g'^{44})^{\frac{1}{2}}. \tag{12}$$

The Dirac equation therefore takes on the form (omitting the primes):

$$\left[(g^{44})^{\frac{1}{2}}\mathring{\gamma}^4\left(\frac{\partial}{\partial x^4}-\Gamma_4\right) + \frac{1}{(g^{44})^{\frac{1}{2}}}\sum_1^3 \mathring{\gamma}^k\left(\frac{\partial}{\partial x^k}-\Gamma_k\right)\right]\psi=\frac{mc}{\hbar}\psi. \tag{13}$$

In (13) the quantities $\partial/\partial x^k$ are connected with their respective values in the zero gravitational field by the tensor transformations

$$\partial/\partial\mathring{x}^k=(\partial x^i/\partial\mathring{x}^k)(\partial/\partial x^i), \tag{14a}$$

$$\Gamma_k=(\partial\mathring{x}^i/\partial x^k)\mathring{\Gamma}_i. \tag{14b}$$

[2] Cf. W. Pauli, *Relativitätstheorie*, §53, b.

But here, on account of the constancy of the g_{ik} in the neighborhood of the nucleus, (14) becomes equivalent to

$$\partial/\partial\mathring{x}^k = (1/(g^{44})^{\frac{1}{2}})(\partial/\partial x^k) \quad (k=1,2,3), \tag{15a}$$

$$\Gamma_4 = (g^{44})^{\frac{1}{2}}\Gamma_4, \quad \mathring{\Gamma}_k = (1/(g^{44})^{\frac{1}{2}})\Gamma_k. \tag{15b}$$

Correspondingly we have the relation

$$\partial/\partial x^4 = E. \tag{15c}$$

E in (15c) stands for the new energy of the electron in the gravitational field. With these simplifications (13) reduces to

$$\mathring{\gamma}^4 E(g^{44})^{\frac{1}{2}}\psi = \left[\frac{mc}{\hbar} + \mathring{\gamma}^4\mathring{\Gamma}_4 - \sum_1^3 \mathring{\gamma}^k\left(\frac{\partial}{\partial\mathring{x}^k} - \mathring{\Gamma}_k\right)\right]\psi. \tag{16}$$

In (16) the right side obviously is the same as it was in the absence of the gravitational field, namely

$$\mathring{E}\mathring{\gamma}^4\psi. \tag{17}$$

The factor of $\mathring{\gamma}^4\psi$ on the left side of (16) determines the eigenvalue for the system which must be equal to the value $\mathring{E}$ in the absence of a gravitational field. We therefore have the relation

$$E(g^{44})^{\frac{1}{2}} = \mathring{E} \tag{18}$$

or correspondingly

$$E = \mathring{E}(g_{44})^{\frac{1}{2}} \quad (g_{44} = 1/g^{44}). \tag{9}$$

(9) constitutes the expression for the red shift as derived from the Dirac equation for the electron in a gravitational field. (8) and (9) are, of course, in agreement. This concludes the proof that the Dirac electron constitutes an atomic clock in the sense used by Einstein.

III. The Gyromagnetic Effect[3]

While in the derivation of the red shift a non-Euclidean metric was essential for the result obtained, we shall here treat a problem of rotating axes for which, obviously, the metric remains Euclidean though the g_{ik} differ from their Galilean values.

The gyromagnetic effect, as is well known, consists in the magnetization of substances containing spinning electrons and is due to a rotation of the matter in bulk. The Coriolis force set up by the rotation acts on the electrons like a magnetic field. The frequency of the rotation is connected with the intensity of the equivalent magnetic field by the relation

$$\omega/H = e/mc. \tag{19}$$

In (19) we have written down the relation for the case of interaction between spin and rotation. For orbital motion and rotation the ratio is given by

$$\omega/H = e/2mc. \tag{20}$$

The difference between (19) and (20) is known as the gyromagnetic anomaly of the spinning electron. The experiment leads to a factor which, though slightly smaller, lies close to (19).

The theoretical treatment of this problem is straightforward on the basis of the general theory of the Dirac electron. We introduce a frame of reference given by the relations:

$$\begin{aligned} x^1 &= \mathring{x}^1 \cos\omega t + \mathring{x}^2 \sin\omega t, \\ x^2 &= -\mathring{x}^1 \sin\omega t + \mathring{x}^2 \cos\omega t, \\ x^3 &= \mathring{x}^3, \\ x^4 &\equiv ict = \mathring{x}^4, \end{aligned} \tag{21}$$

from which we obtain for the g_{ik} the following scheme:

$$g_{ik} = \begin{Bmatrix} 1 & 0 & 0 & i\omega' x^2 \\ 0 & 1 & 0 & -i\omega' x^1 \\ 0 & 0 & 1 & 0 \\ i\omega' x^2 & -i\omega' x^1 & 01 & -\omega'^2[(x^1)^2+(x^2)^2] \end{Bmatrix} \cdot (\omega' = \omega/c). \tag{22}$$

Limiting ourselves to a first approximation, i.e., taking the quantity ω as small compared with all atomic frequencies, we can, with the help of (3) and (22) easily determine the quantities

$$\begin{aligned} \gamma_1 &= \mathring{\gamma}_1 + i\omega' x^2 \mathring{\gamma}_4, \quad \gamma_3 = \mathring{\gamma}^{\prime 8} \\ \gamma_2 &= \mathring{\gamma}_2 - i\omega' x^1 \mathring{\gamma}_4, \quad \gamma_4 = \mathring{\gamma}_4. \end{aligned} \tag{23}$$

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[3] Cf. the comprehensive report by L. T. Barnett, Rev. Mod. Phys. 7, 129 (1935).

With the help of (22), (23) and (4) we can then calculate the explicit expressions for $\Gamma_{ik}{}^{\mu}$ and Γ_k. The matrix terms in Γ_k are given by

$$\begin{aligned}\Gamma_1&=(i\omega'/4)(\gamma^2\gamma^4-\gamma^4\gamma^2)=-\omega'\alpha_y/2,\\ \Gamma_2&=(i\omega'/4)(\gamma^4\gamma^1-\gamma^1\gamma^4)=\omega'\alpha_x/2,\\ \Gamma_3&=0,\\ \Gamma_4&=-(i\omega'/4)(\gamma^1\gamma^2-\gamma^2\gamma^1)=\omega'\sigma_z/2,\end{aligned}\tag{24}$$

and we arrive at the new form of the Dirac equation, which is accurate to the first power in ω':

$$H\psi\equiv[H_0\psi+\omega'(\alpha_xy-\alpha_yx)\pi_t+\tfrac{1}{2}\hbar\omega'\sigma_z]\psi=0.\tag{25}$$

H_0 stands for the Hamiltonian in a nonrotating system of coordinates, $H_0=-\pi_t+\alpha\cdot\pi+mc\beta$. The other symbols have their conventional meaning. H differs from H_0 by the terms

$$(\omega/c)(\alpha_xy-\alpha_yx)\pi_t+(\hbar\omega/2c)\sigma_z.\tag{26}$$

If the electron moved in a homogeneous magnetic field given by the vector potential

$$A_x=\tfrac{1}{2}Hy,\quad A_y=-\tfrac{1}{2}Hx,\tag{27}$$

and in a system of coordinates with a Galilean metric, the terms due to the vector potential would be equal to

$$(\bar{\omega}/c)(\alpha_xy-\alpha_yx)mc,\quad (\bar{\omega}=eH/2mc).\tag{28}$$

Comparing (26) and (28) we notice that the magnetic field which would produce the same additional terms as the rotating system of coordinates depends on the state of the electron. In other words, there does not exist a universal gyromagnetic relation between frequencies of rotation and intensity of the magnetic field.

It is justified in our approximation to write in (26)

$$\pi_t=mc,$$

because the difference between π_t and mc is of relativistic order of magnitude and can therefore be omitted *in the small correction term.* Noting furthermore, that

$$H\pi_t-\pi_tH=0,\tag{29}$$

$$\text{i.e., } \pi_t=\text{const.}=E,$$

and that

$$H_0\sigma_z-\sigma_zH_0\backsimeq 0\tag{30}$$

except for terms of relativistic order of magnitude, we can form a second order equation out of (25)

$$H_0{}^2\psi-2m\omega(L_z+\tfrac{1}{2}\hbar\sigma_z)\psi=0.\tag{31}$$

Here L_z stands for the z component of the orbital angular momentum. $L_z+\frac{1}{2}\hbar\sigma_z$ is an integral of the equations of motion. In the case of a magnetic field given by the vector-potential (27) the equation analogous to (31) would take the form

$$H_0{}^2\psi-2m\bar{\omega}(L_z+\hbar\sigma_z)\psi=0.\tag{32}$$

(31) and (32) permit us to verify (19) and (20). For $L_z=0$ we would obtain equivalence of (31) and (32) for

$$\omega=2\bar{\omega},$$

while for $L_z\gg\hbar\sigma_z$ one obtains

$$\omega=\bar{\omega}.$$

Intermediate stages are given by the formula analogous to Landé's:

$$\omega/\bar{\omega}=(L_z+\hbar\sigma_z)/(L_z+\tfrac{1}{2}\hbar\sigma_z).\tag{33}$$

An alternative way of treating the problem[4] goes as follows: Starting from the original form of Dirac's equation and transforming the momentum operators to rotating coordinates with the help of the relations

$$\partial/\partial\mathring{t}=\partial/\partial t+\omega y\partial/\partial x-\omega x\partial/\partial y,\quad\text{etc.,}\tag{34}$$

one arrives at the following (exact) Dirac equation

$$\bar{H}\Psi=[-\pi_t-\omega'L_z+\bar{\alpha}\cdot\pi+mc\beta]\Psi=0,\tag{35}$$

where

$$\begin{aligned}\bar{\alpha}_x&=\alpha_x\cos\omega t+\alpha_y\sin\omega t,\\ \bar{\alpha}_y&=-\alpha_x\sin\omega t+\alpha_y\cos\omega t,\\ \bar{\alpha}_z&=\alpha_z.\end{aligned}\tag{36}$$

This equation permits of an interpretation similar

[4] O. Halpern, Phys. Rev. **37**, 1719 (1931).

to that which we have given to (25) except for the ambiguity in (35). π_t *is no longer an integral of the equation of motion* since it does not commute with $\bar{H}$. This constitutes a special case illustrating a general feature of the covariant form of Dirac's equation. It is due to the fact that the α's and γ's are now functions of the coordinates. Two equations which are equivalent to each other are not interpretable with equal ease because, as a result of the special choice of γ^i, one does not admit the same integrals as the other one. Since (35) and (25) are equivalent, there must exist, of course, a spin transformation which connects the two equations with each other. The transformation is given by the matrices (accurate to first order terms in ω)

$$S=\cos\frac{\omega t}{2}\cdot 1-i\sin\frac{\omega t}{2}\cdot\sigma_z-\frac{\omega'}{2}\left(y\cos\frac{\omega t}{2}+x\sin\frac{\omega t}{2}\right)\alpha_x+\frac{\omega'}{2}\left(x\cos\frac{\omega t}{2}-y\sin\frac{\omega t}{2}\right)\alpha_y,$$
$$S^{-1}=\cos\frac{\omega t}{2}\cdot 1+i\sin\frac{\omega t}{2}\cdot\sigma_z+\frac{\omega'}{2}\left(y\cos\frac{\omega t}{2}+x\sin\frac{\omega t}{2}\right)\alpha_x-\frac{\omega'}{2}\left(x\cos\frac{\omega t}{2}-y\sin\frac{\omega t}{2}\right)\alpha_y. \tag{37}$$

The Hamiltonians are connected by the relation $H=S^{-1}\bar{H}S$ (7̄a)

and the eigenfunctions by $\psi=S^{-1}\Psi.$ (7̄b)

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The Efficiency of the Tube Counter

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The efficiency of counting ionizing particles by the tube counter is investigated, taking account of the electrical behavior of the counter and the random nature of the arrival of the particles. Expressions are obtained for the efficiency of counting, average recovery time, average voltage impulse delivered to the amplifier, and the average number of particles counted per unit time, N, in terms of the number of particles arriving per unit time, N_0, the time constant of the counter, and one other parameter, d, which is constant for a given counter. Methods for measuring d (which for most counters will lie between 1.3 and five) are indicated. The efficiency of counting decreases with an increase of N_0, rapidly at first, then more slowly, so as to have the asymptotic value $1/d$. The determination of N_0 from N, and the general question of counter efficiency are discussed.

THE behavior of Geiger-Müller and similar types of counters has been studied experimentally by various investigators.[1] As the negative potential on the tube, V_T, is increased from zero, the average number of counts per unit time, the source being constant, behaves as in Fig. 1. Counting begins at the "threshold voltage," V_S, rises rapidly to a value N, and remains nearly constant for a range of voltage which depends on the individual counter. In a well-built counter, the flat region of the curve extends over a considerable range of voltage. The counter is operated in this region, since any slight variation of supply voltage will then have negligible effect on the rate of counting. At higher voltages, the

[1] (The first two references give a good bibliography of the previous work.) Burger Scheidlin, Ann. d. Physik **12**, 283 (1932); Schulze, Zeits. f. Physik **78**, 92 (1932); Medicus, Zeits. f. Physik **74**, 350 (1932); Curtis, Bur. Standards J. Research **10**, 229 (1933); Brunetti and Ollano, Nuovo Cimento **10**, 92 (1933); Greiner, Zeits. f. Physik **81**, 543 (1933); Hummel, Zeits. f. Physik **76**, 483 (1932); Physik. Zeits. **34**, 331 (1933); Kuhn, Zeits. f. Instrumentenk. **54**, 415 (1934); Danforth, Phys. Rev. **46**, 1026 (1934); J. Frank. Inst. **219**, 108 (1935); Wernow, Trav. de l'Inst. d'Etat de Radium **2**, 30 (1933) (abstracted in Physik. Berichte **15**, 1448 (1934)); H. Teichmann, Physik. Zeits. **35**, 637 (1934); Bosch, Ann. d. Physik **19**, 65 (1934); Janossy, Zeits. f. Physik **88**, 372 (1934); Henning and Schade, Zeits. f. Physik **90**, 597 (1934); A relevant article on photon counters is Werner, Zeits. f. Physik **90**, 384 (1934); **92**, 705 (1934).