Velocity Distributions for Elastically Colliding Electrons

PHILIP M. MORSE, W. P. ALLIS AND E. S. LAMAR, *Massachusetts Institute of Technology*
(Received June 24, 1935)

The form of the function giving the distribution in velocity of electrons in a gas is determined by a pair of equations which correspond to the detailed balancing of energy and momentum, and take into account the variation of collision cross section with velocity. The kinetic energy ϵ of the electrons is supposed to be larger than that of the gas atoms, yet small enough so that the majority of the energy lost is by elastic collisions with the atoms. The equations are solved in detail for two cases. One case is that of electrons in a uniform electric field, where the distribution is independent of position. The distribution function is found to be proportional to $\exp(-\epsilon^2/a^2)$, instead of to $\exp(-\epsilon/a)$, as is the case for the Maxwell distribution, when the electrons are in temperature equilibrium with the gas atoms. The average energy of the electrons, the drift current, etc., are computed as a function of the field strength. The other case considered is that of a homogeneous beam of electrons of energy ϵ_0, shot into a field free space, where they lose energy to the gas atoms by collisions. The distribution function depends on z, the distance along the beam in mean free paths, and on t, the average number of collisions the electron has had before its energy decreases from ϵ_0 to ϵ. This distribution is also not Maxwellian, but depends on a solution of an equation in z and t having the form of the heat flow equation. The solution has been tested experimentally, and a quantitative check is obtained.

INTRODUCTION

IN all problems involving the motions of free electrons among the atoms of a gas, it has been customary to make one of two simplifying assumptions: either that the electrons lose no energy on collision with the atoms, so that at any point all the electrons have the same energy; or that the electronic velocities have a Maxwellian distribution whose density and temperature can vary from point to point in the gas. These two assumptions represent, in a sense, two opposite limiting conditions, and neither assumption is valid for intermediate cases.

The first assumption is only valid when the mean free path of the electrons is longer than the dimensions of the apparatus, for the electrons lose energy, even in an elastic collision, due to the recoil of the atom. This loss is small per collision, but unless the mean free path is very long, there will be enough impacts to give a noticeable "spread" to the energy distribution of the electrons.

The Maxwellian distribution, on the other hand, is only valid when the electrons are in temperature equilibrium with the gas, or when there is available some mechanism for transferring energy directly from electron to electron. These requirements are not fulfilled when there is an applied electric field, or when high energy electrons are shot into the gas from outside; unless, possibly, when the electron density is quite large. When the gas pressure is relatively large and the electronic density relatively low, one must expect a velocity distribution which is neither as "sharp" as the first assumption would demand, nor as "spread" as the second assumption would require.

It is possible to derive this intermediate distribution by balancing the electron's gain of energy, due to the applied field and to diffusion, with its loss, due to collisions with the gas atoms. In this paper only "elastically controlled" distributions will be studied, i.e. those cases where the average electronic energy is small enough so that, on the average, more energy is lost by elastic collisions than by inelastic ones. The method outlined below can be extended to cases where the inelastic collisions are more important, but the elastically controlled case is the simplest, and must be studied first.

ELASTIC COLLISIONS

The probability that an electron of velocity v will collide elastically with an atom and be scattered at an angle Θ to its primary direction is determined by the *angle scattering function* $\sigma(v, \Theta)$. The resultant probability of elastic collision for any direction of scattering can be expressed in terms of the *elastic collision cross section*

$$q(v)=2\pi\int_0^\pi\sigma(v, \Theta) \sin \Theta d\Theta. \qquad (1)$$

If it is desired to find the loss of momentum of the electron in the collision, the *momentum transfer cross section* is needed,[1]

$$Q(v)=2\pi\int_0^\pi\sigma(v,\Theta)(1-\cos\Theta)\sin\Theta d\Theta. \quad (2)$$

The two cross sections differ appreciably only if the angle scattering curve has a pronounced excess in the forward or backward direction. Both cross sections vary markedly with the electronic velocity, in most cases.

When an electron of mass m and energy $\epsilon=\frac{1}{2}mv^2$ collides elastically with an atom of mass M and is scattered at an angle Θ, it loses a certain fraction of its energy to the recoiling atom. We shall be interested in electrons whose energy although lower than the atomic excitation potentials is considerably higher than the thermal energy of the atoms, so that the atoms can be considered as being at rest. With this approximation, and neglecting the squares of the small quantity (m/M), the fraction of energy lost per collision is

$$(\Delta\epsilon/\epsilon)=(2\Delta v/v)=2(m/M)(1-\cos\Theta). \quad (3)$$

It can be seen that the expression for the average loss of energy per collision will involve the cross section Q.

Detailed Balancing

Let the number of electrons in the volume element $d\tau=dxdydz$, whose velocities fall in the range $d\gamma=d\xi d\eta d\zeta=v^2\sin\omega dvd\omega d\varphi$ be $f(x,y,z;\xi,\eta,\zeta)d\tau d\gamma$. This defines the *distribution function* f. For reasons of simplicity we shall usually discuss distributions which are homogeneous and isotropic in the yz plane so that f is a function only of x, v and ξ (or x, v and ω). It is possible to generalize the argument to three dimensional cases, however.

The distribution function f will be determined by a method similar to that given by Lorentz,[2] but extended to include the variation of the cross section with the velocity and the loss of energy at collisions. With Lorentz we assume that f can be expanded in a series of Legendre functions of $\cos\omega=(\xi/v)$,

$$\begin{aligned}f(x,v,\omega)&=f_0(x,v)+P_1(\cos\omega)f_1(x,v)\\&\quad+P_2(\cos\omega)f_2(x,v)+\cdots\\&=f_0(x,v)+(\xi/v)f_1(x,v)+\cdots.\end{aligned} \quad (4)$$

The function f_0 determines the random distribution in velocity, and f_1 determines the electron drift. The higher terms in the series are nearly always very small and do not correspond to any simple physical property of the distribution, but serve simply to improve the form of the distribution function. Consequently, we shall neglect all except the first two terms of series (4), and expect that the result will give very nearly correct values for the random and drift velocities, even though the function f thus calculated does, in some cases, actually become negative for certain values of the parameters.

The fundamental formula determining f is obtained by fixing the attention on an element $d\tau d\gamma$ of phase space. The number of electrons leaving this element due to the applied field E and to diffusion is

$$cd\tau d\gamma=\left(\frac{eE}{m}\frac{\partial f}{\partial\xi}+\xi\frac{\partial f}{\partial x}\right)d\tau d\gamma.$$

When the first two terms of series (4) are substituted in this expression, two terms in ξ^2 appear. Since we are neglecting spherical harmonics of order higher than the first, ξ^2 must be replaced by its average value $(v^2/3)$. To this approximation, then,

$$\begin{aligned}cd\tau d\gamma=\Bigg[&\frac{eE}{m}\cos\omega\frac{\partial f_0}{\partial v}+\frac{eE}{m}\frac{1}{3v^2}\frac{\partial(v^2f_1)}{\partial v}\\&+v\cos\omega\frac{\partial f_0}{\partial x}+\frac{v}{3}\frac{\partial f_1}{\partial x}\Bigg]d\tau d\gamma.\end{aligned} \quad (5)$$

The number leaving the element of phase space due to collisions is

$$ad\tau d\gamma=Nv\int 2\pi f\sigma\sin\Theta d\Theta d\tau d\gamma=Nqvfd\tau d\gamma, \quad (6)$$

where N is the number of gas atoms per cc.

So far, all is in accord with Lorentz; the difference comes in the discussion of the number scattered into the element $d\tau d\gamma$. These electrons had, before the collision, velocities in velocity element $d\gamma'$ (Fig. 1). Their initial velocity $v'=v+(mv/M)(1-\cos\Theta)$ is larger than the final

[1] Houston, Zeits. f. Physik 48, 449 (1928).
[2] Lorentz, *Theory of Electrons* (Stechert), p. 267.

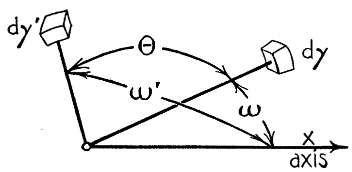

FIG. 1.

velocity as is seen from Eq. (3), and it can be shown from the same equation that the size of $d\gamma'$ is given by $d\gamma'=(v'/v)^3 d\gamma$. Hence, the total number of electrons scattered into the element $d\tau d\gamma$ per second is

$$bd\tau d\gamma = Nv\int_0^\pi 2\pi f(v',\omega')\sigma(v',\Theta) \times \sin\Theta d\Theta (v'/v)^4 d\tau d\gamma,$$

and hence the net number coming into the element $d\tau d\gamma$ by collisions is

$$(b-a)d\tau d\gamma = (2\pi N/v^3)\int_0^\pi [v'^4 f(v',\omega')\sigma(v',\Theta) - v^4 f(v,\omega)\sigma(v,\Theta)]\sin\Theta d\Theta d\tau d\gamma.$$

As v' differs little from v, the integrand may be written

$$v^4[f(v,\omega')-f(v,\omega)]\sigma(v) + \Delta v(\partial/\partial v)[v^4 f(v,\omega')]\sigma(v,\Theta).$$

The first term is that obtained by Lorentz, and gives on integration $-NQvf_1(v,\omega)\cos\omega$. The second term follows from the inclusion of the energy loss, and is small due to the factor (m/M) in Δv. Consequently, in it f can be replaced by f_0, and $(b-a)$ becomes finally

$$(b-a)d\tau d\gamma = \left[-NQvf_1\cos\omega + \frac{m}{M}\frac{N}{v^2}\frac{\partial}{\partial v}(v^4 Qf_0)\right]d\tau d\gamma. \quad (6a)$$

The condition for a steady state is now given by equating c to $(b-a)$. Equating separately the terms in $\cos\omega$ and those which do not contain $\cos\omega$ gives two equations:

$$eE\partial f_0/\partial v + mv\partial f_0/\partial x = -NQmvf_1, \quad (7)$$

$$\frac{eE}{2v}\frac{\partial}{\partial v}(v^2 f_1) + \tfrac{1}{2}mv^2\frac{\partial f_1}{\partial x} = \frac{m^2}{M}\frac{3N}{2v}\frac{\partial}{\partial v}(v^4 Qf_0). \quad (8)$$

The first of these equations is the one obtained by Lorentz, and represents the balance between the gain of momentum, due to diffusion and drift down the field, and the loss of momentum, due to collisions. The quantity $(1/NQ)$ is of course the electronic mean free path (for momentum transfer). The second equation represents the balance between the gain of energy, because of diffusion and drift down the field, and the loss of energy by collisions.

Since Lorentz had only Eq. (7), he had to assume the form of f_0 to be Maxwellian and then to find f_1 in terms of it. Pidduck[3] introduced an average balance of energy by determining the temperature of the Maxwell distribution in such a way that the total energy gained by drift and diffusion was equal to the total energy lost by collisions. As will be noted later, such a distribution predicts entirely too many fast electrons to agree with the experimental facts. The correct distribution must have less "spread" than the Maxwell distribution.

The addition of Eq. (8) secures a detailed balance of energy in each velocity range, and not merely a balance of the totals; and enables one to solve for both f_0 and f_1 without any further assumptions. In the present paper these two functions will be obtained for two special cases.

HOMOGENEOUS DISTRIBUTION

When f is independent of x, Eq. (8) integrates immediately to

$$eE\epsilon f_1 = 6(m/M)NQ\epsilon^2 f_0 - B \quad (9)$$

or, on multiplication by $(8\pi/3m^2)d\epsilon$,

$$\frac{8\pi eE}{3m^2}\epsilon f_1 d\epsilon - \frac{16\pi NQ}{m^2}\frac{m}{M}\epsilon^2 f_0 d\epsilon = -jd\epsilon, \quad (10)$$

where the constant B, or j, is a constant of integration. Eq. (10) gives the balance of energy, for the current carried by electrons having energies between ϵ and $\epsilon+d\epsilon$ (i.e., in the velocity element $d\gamma=(4\pi/m)vd\epsilon$) is $dJ=e\overline{\xi f}d\gamma=(ev/3)f_1 d\gamma$, and the energy taken from the field per sec. per cc is $EdJ=(8\pi eE/3m^2)\epsilon f_1 d\epsilon$. This is the first term of Eq. (10). The second term is the energy lost by collisions. If these energies are to balance, the constant B, or j, must be zero. On the other hand, if these energies do not balance and, say, the energy gained is less than that lost, then the

[3] Pidduck, Proc. Roy. Soc. **A88**, 296 (1913).

electrons are losing energy and j is a measure of that loss. In fact, j will be seen to be equal to the number of electrons per cc per sec. losing energy and passing through the energy value ϵ from higher energies to lower energies. Suppose electrons of high energy to be introduced into the system and allowed to lose energy by collisions, then j will equal the number of electrons so introduced per cc per sec. having energies higher than ϵ. If, on the other hand, high energy electrons are suffering inelastic impacts, and are thus being removed from the high energy range and replaced with a low energy, $(-j)$ will equal the number per cc per sec., originally having energies larger than ϵ, which are being stopped by an inelastic collision, and which must subsequently gain energy until they make another inelastic collision. This will be treated more fully in a later part of the paper.

At present, we will consider B, or j, to be zero. Substituting Eq. (9) into Eq. (7) when f is independent of x, and when Q is considered to be constant, one obtains

$$f_0 = Ae^{-h^4v^4} = A\exp\left[-\frac{3m}{M}\left(\frac{NQ\epsilon}{eE}\right)^2\right], \quad (11)$$

where $h^4 = (3m/M)(NQm/2eE)^2$ and the constant $A = [nh^3/\pi\Gamma(\frac{3}{4})]$. The number n is the density of electrons per cc.

The natural units of energy and velocity for this case are the energy $\epsilon_e = (eE/NQ)$ and velocity $v_e = (2eE/mNQ)^{\frac{1}{2}}$ gained by an electron in falling, from rest, down the field for a mean free path. In terms of these units the average kinetic energy and mean drift velocity of the electrons are

$$\bar{\epsilon} = \frac{\Gamma(5/4)}{\Gamma(3/4)}\left(\frac{M}{3m}\right)^{\frac{1}{2}}\epsilon_e = 0.4270\left(\frac{M}{m}\right)^{\frac{1}{2}}\epsilon_e, \quad (12)$$

$$u = \frac{\pi^{\frac{1}{2}}}{3\Gamma(3/4)}\left(\frac{3m}{M}\right)^{\frac{1}{4}}v_e = 0.6345\left(\frac{m}{M}\right)^{\frac{1}{4}}v_e. \quad (13)$$

It is seen that the mean energy increases as (M/m) increases, because the heavier the gas atoms are, the less energy can be lost per collision. On the other hand, the average drift velocity decreases as (M/m) increases. The average speed, $(2\bar{\epsilon}/m)^{\frac{1}{2}}$, bears to u the fixed ratio $1.03(M/m)^{\frac{1}{2}}$, which is as large as 63 even for hydrogen gas.

The relative size of f_1 and f_0 at the mean energy is obtained from Eq. (9), $f_1/f_0 = 2.56(m/M)^{\frac{1}{2}}$. Since this ratio is never larger than 0.07, it is seen that for the range of energy which is important, f_1 is only a small correction to the spherically symmetric term f_0. For electronic energies larger than the mean energy, f_1 becomes relatively more important, but the exponential in Eq. (11) renders the whole distribution function negligibly small before f_1 can be the same size as f_0.

It is of interest to compare the results of this distribution with those obtained by Pidduck.[3] If one neglects Eq. (8), sets $f_0 = Be^{-\epsilon/kT}$, and requires an average energy balance, it turns out that $kT = (M/3m)^{\frac{1}{2}}(\epsilon_e/2)$. The average kinetic energy is 0.4320 $(M/m)^{\frac{1}{2}}\epsilon_e$, only 1.014 times the average given in Eq. (12). The average drift velocity is 0.9900 $(m/M)^{\frac{1}{4}}v_e$, greater than the average given in Eq. (13) by a factor of 1.560. Hence average values are not much different for the two distributions. However, the number of fast electrons is markedly less for the new distribution, and quantities which depend only on fast electrons, such as rates of excitation and ionization, will be considerably less for the new than for the Maxwell distribution. This is in better accord with experiment, for the Maxwell distribution predicts several times more ionization than is measured by Townsend.[4] The difference between the two functions is shown in Fig. 2.

The distribution given in Eq. (11) is valid only if the momentum transfer cross section is nearly independent of velocity. If it cannot be assumed that Q is constant, then the distribution function must be written

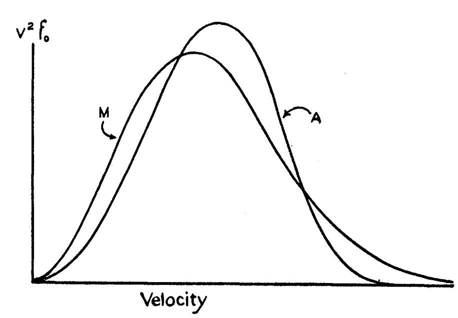

FIG. 2. Distribution in random velocity of electrons under the influence of an electric field (A), compared to a Maxwell distribution for the same average energy (M).

[4] Townsend, *Electricity in Gases*, p. 295.

$$f_0=A\exp\left[-\left(\frac{6m}{M}\right)\int^{\epsilon}\left(\frac{NQ\epsilon}{eE}\right)d\epsilon\right],$$
$$f_1=\frac{6m}{M}\frac{NQ}{eE}\epsilon f_0. \quad (14)$$

The general effect of a variable cross section is easily seen. The cross section Q can be considered as a scale factor for the variable ϵ, so that where the cross section is small the curve is flatter, where it is large the curve is steeper, than it otherwise would be. The change is small except when the cross section exhibits a Ramsauer effect, in which case the derived mobility, rate of ionization, etc., may be considerably affected.

Homogeneous Distribution without Field

The distribution discussed in the last section is maintained by the electric field E and will not hold if E falls low enough for the mean kinetic energy of the electrons to be comparable to that of the atoms. It is possible, however, to maintain a distribution without a supporting field by continuously introducing electrons of high energy ϵ_0, as, for instance, by ionizing the gas by x-rays. These electrons gradually lose energy by collisions and thus drift down the energy scale to end up in a large accumulation of electrons of low energy.

The stationary distribution must be such that the number of electrons falling in unit time below any energy level is equal to the rate j at which they are introduced. But the electrons crossing the energy level ϵ are just those in the range $\Delta v=vm/M$ above ϵ which have collisions. Thus, $j=NQvf\cdot 4\pi v^2(m/M)v$, or

$$f=f_0=(M/m)(jm^2/16\pi NQ\epsilon^2). \quad (15)$$

This is, of course, the solution of Eq. (10) with E set equal to zero. The distribution is isotropic unless j varies with x. The distribution f goes to infinity as ϵ approaches zero, but Eq. (15) is no longer valid for electronic energies less than the temperature energy of the gas molecules.

Inhomogeneous Distribution

Imagine electrons of energy ϵ_0 to be liberated in a limited region of a gas, and to diffuse away from this region, at the same time losing energy due to collisions. The function giving the distribution both in space and in energy will be a solution of Eqs. (7) and (8) with $E=0$. When Q can be considered constant, it is convenient to introduce new variables, defined as follows:

$$z=NQx, \quad t=(M/2m)\ln(\epsilon_0/\epsilon),$$
$$R=4\pi v^4 f=R_0+\cos\omega R_1. \quad (16)$$

The variable z is the distance measured in mean free paths, and the relation $d\epsilon=-(2m/M)\epsilon dt=\overline{\Delta\epsilon}dt$ shows that in order to lose the energy $d\epsilon$ the electron must make, on the average, dt collisions. Hence, t measures the average number of collisions the electron has suffered since it started.

Making all these changes, Eqs. (7) and (8) reduce to

$$R_1=-\partial R_0/\partial z, \quad 3\partial R_0/\partial t=\partial^2 R_0/\partial z^2. \quad (17)$$

By generalizing Eqs. (7) and (8), it can be shown that the three dimensional variation of R is governed by the equation

$$\nabla^2 R_0=3(\partial/\partial t)R_0, \quad (18)$$

where the unit of length is again a mean free path. In this case, the drift current due to electrons in the velocity range dv is the vector

$$\mathbf{dJ}=-(1/3v)\,\text{grad}\,(R_0)dv.$$

Eq. (18) or the second Eq. (17) is formally the same as the heat flow equation, R_0 corresponding to the temperature and t to the time; it shows that the electrons spread out in space as their energy decreases exactly as heat spreads out with time.

A solution of the one dimensional equation is the Green's function

$$R_0=(M/2NQm)(3/\pi t)^{\frac{1}{2}}e^{-3z^2/4t},$$

which is here normalized to correspond to the introduction of one electron, with the energy ϵ_0, per second at the point $z=0$. The singularity in the solution at $t=0$ corresponds to the highest energy in the distribution, ϵ_0.

The problem to which we wish to apply this theory, and which has been studied experimentally, is the following: a beam of electrons is projected with kinetic energy ϵ_0 into a region free from electric fields, it is desired to find the

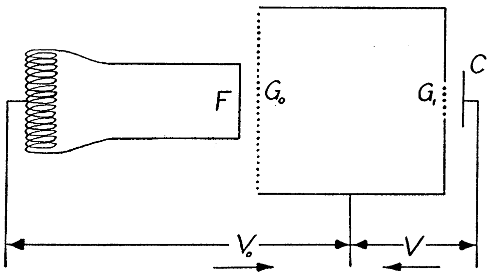

FIG. 3. Schematic diagram of apparatus.

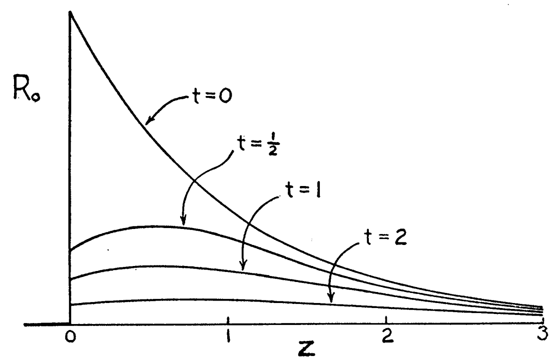

FIG. 4. Curves of $R_0=4\pi v^4 f_0$, for a beam of electrons entering a field-free space, as function of z the distance from entry in mean free paths, for different values of t the average number of collisions since entry.

distribution in velocity of the electrons at various points inside the region.

A schematic diagram of the experimental apparatus used is given in Fig. 3. Electrons from a filament F are accelerated by a potential V_0 to the entrance grid G_0 through which they enter into a field-free region surrounded by a metal cylinder. Samples of the distribution are taken through the exit grid G_1, L cm from G_0, by varying the retarding potential V to the collector C and by measuring the resulting collector current.

The electrons in any cc of the field-free region can be classified in two groups: those in the primary beam having the energy ϵ_0, and those, having suffered one or more collisions since their entrance into the region, which therefore have energies less than ϵ_0. Because of collisions with gas atoms, the primary current will decrease exponentially, its value at a distance x from the entrance grid being $I=I_0e^{-z}$, $z=NQx$. The steady state distribution of electrons of lower energy is maintained by the electrons scattered out of the primary beam, of which there are NQI per sec. per cc. According to the discussion above, this distribution is given in terms of a function R_0, a solution of the heat flow Eq. (18) which satisfies the boundary conditions at the surface of the region considered, and which satisfies the "initial condition" that at $t=0$ (i.e., at $\epsilon=\epsilon_0$) R_0 is equal to $(M/m)I$. The "initial value" of R_0 is obtained from Eq. (15) with j set equal to NQI.

The exit grid samples only the central part of the beam and if the diameter of the metal cylinder is greater than two or three mean free paths, so that we can neglect the boundaries at the sides, the problem may be treated as a one dimensional one. In this case the function R_0 is given in terms of the Green's function of Eq. (19).

$$R_0=(M/2m)I_0(3/\pi t)^{\frac{1}{2}}\int_0^{z_1}e^{-\lambda}e^{-3(z-\lambda)^2/4t}d\lambda$$
$$=I_0\frac{M}{2m}e^{-z+(t/3)}\left\{\Phi\left[\left(\frac{3}{t}\right)^{\frac{1}{2}}\left(\frac{t}{3}+\frac{z_1-z}{2}\right)\right]-\Phi\left[\left(\frac{3}{t}\right)^{\frac{1}{2}}\left(\frac{t}{3}-\frac{z}{2}\right)\right]\right\}, \quad (19)$$

where $z_1=NQL$, and $\Phi(x)$ is the error integral $(4/\pi)^{\frac{1}{2}}\int_0^x e^{-u^2}du$. This function has been plotted for several values of t in Fig. 4 for the case $z_1=\infty$. It is seen that the curves smooth out after very few collisions, so that low velocity electrons predominate everywhere except within a mean free path of the entrance grid.

To find, from this expression, the current which will reach the collector through the exit grid, against a retarding potential V, requires some further assumptions. The random current is given by

$$2\pi\int_0^{v_0}v^2dv\int_0^{\pi/2}\xi f\sin\omega d\omega$$
$$=(m/2M)\int_0^\infty dt\int_0^{\pi/2}R\cos\omega\sin\omega d\omega.$$

However, the exit grid does not allow electrons to pass whose velocity makes a large angle ω with the axis of the tube, so that the integral over ω extends only up to a limiting angle ω_1. Since R_1 is small compared to R_0 everywhere except at the entrance grid, we can write for the random current passing the exit grid

$$(m/4M)\sin^2\omega_1\int_0^\infty R_0(z_1)dt.$$

The retarding potential does not allow all electrons passing the grid to reach the collector, but only those whose normal velocity is great enough to overcome the retarding potential eV. The correct integration here would be difficult, but if the angle ω_1 is not large, no great error is made if we simply integrate over all energies ϵ greater than eV. The current to the collector is then

$$J_c = I + (m/4M)\sin^2\omega_1 \int_0^T R_0(z_1)dt$$

$$= I_0 e^{-z_1} + CI_0 \left\{ T^{\frac{1}{2}}(e^{-3z_1^2/4T} - e^{-z_1}) + \tfrac{1}{2}(3\pi)^{\frac{1}{2}}(1-z_1)\left[1 - \Phi\left(\frac{z_1}{2}\left(\frac{3}{T}\right)^{\frac{1}{2}}\right)\right]\right.$$

$$\left. + \tfrac{1}{2}(3\pi)^{\frac{1}{2}} e^{-z_1+(T/3)}\left[\Phi\left(\left(\frac{T}{3}\right)^{\frac{1}{2}}\right) - \Phi\left(\left(\frac{T}{3}\right)^{\frac{1}{2}} - \frac{z_1}{2}\left(\frac{3}{T}\right)^{\frac{1}{2}}\right)\right]\right\},$$

where $C = (3/16\pi)^{\frac{1}{2}} \sin^2 \omega_1$, and $T = (M/2m) \times \ln(V_0/V)$. Due to the factor $(M/2m)$, T becomes quite large as soon as the collector voltage V is made somewhat smaller than the accelerating voltage V_0, so that over most of the range of V the asymptotic form for J_c can be used:

$$J_c \rightarrow I_0 e^{-z_1} + I_0 A[\ln(V_0/V)]^{\frac{1}{2}}(1-e^{-z_1}). \quad (20)$$

The constant A is left in this expression to be determined by experiment. There are several factors whose effects cannot be exactly calculated, as the electron beam is not infinitely wide and the exact influence of the walls of the tube and of the grids is unknown. If these effects do not depend too markedly on the velocity, they should be taken care of by adjusting A.

An Experimental Check

The experimental tube was built to correspond to Fig. 3. It contained a nickel cylinder, a filament, and a plate collector. The cylinder was five cm long and five cm in diameter. The end of the cylinder nearest the filament was covered by the gauze grid G_0, and the other end with sheet nickel, except for a central circular window one cm in diameter, covered with nickel gauze, the grid G_1. The filament was ten-mil tungsten two cm long, situated five mm from the grid G_0. The plate C was a disk of sheet nickel and was five mm from the grid G_1. This was done to eliminate, as far as possible, collisions between electrons and gas atoms everywhere except in the field-free space between the grids.

The filament was heated by alternating current obtained from a low voltage transformer. Since the ends of the filament were cooled by conduction along the leads, only its central portion contributed to the emission current. An examination of the energy distribution of the electrons emitted by the filament showed a total spread which did not exceed one volt.

For a fixed accelerating voltage V_0 between filament and entrance grid, the current to the plate C was measured as a function of the retarding potential V between collector and exit grid. The first tests in high vacuum showed a wide-spread distribution in normal components of energy of the electrons reaching C. It was decided that this spread resulted from reflection of the electrons by the grids and from the walls of the cylinder. The tube was taken apart, therefore, and the cylinder, grids and collector were covered with amorphous carbon to reduce electron reflection. The energy distribution subsequently obtained in high vacuum was extremely flat over most of the range, and showed only a small residue of the previously observed effect.

After having made the run in high vacuum, runs were made with helium gas in the tube over a range of pressures such as to allow from one to ten mean free paths between the grids. Helium was chosen because its effective cross section varies least with velocity, and an accelerating potential of only 18.5 volts was used to insure that the number of inelastic collisions would be negligible.

The experimental results are shown as circles in Fig. 5. The solid curves are plots of Eq. (20). I_0 was taken as the value of the current when the tube was evacuated, and was 0.478 microampere, or 143 mm galvanometer deflection.

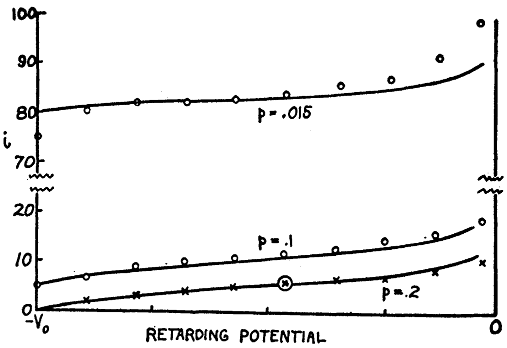

FIG. 5. Comparison of experimental results (circles and crosses) with calculated curves for the apparatus shown in Fig. 3, for different pressures of helium, in mm of mercury. Accelerating voltage V_0 is 18.5 v, and collector current i for vacuum is 143 mm galvanometer deflection. The single arbitrary constant is adjusted to make the curves fit the point indicated by the heavy circle.

The value of NQ, the number of collisions per cm path at 18.5 volts, is obtained from Normand's measurements of the cross section of helium.[5] It is 8 collisions per cm per mm gas pressure. Since the distance between grids, L, is 5 cm, the quantity z_1 is $40p$, where p is the gas pressure in mm of mercury. The only unknown parameter is A, and it was determined by fitting the single point indicated by the heavy circle in the figure. The agreement for all three curves is quite satisfactory.

FURTHER APPLICATIONS

The Green's function method can also be applied to the lateral diffusion from a narrow beam of electrons in a field-free space. For instance, the distribution function for a beam defined by a slit whose width is a mean free paths is given in terms of the function

$$R_0=\left(\frac{M}{4m}\right)I_0e^{-z+(t/3)}\left[1-\Phi\left(\frac{3}{t}\right)^{\frac{1}{2}}\left(\frac{t}{3}-\frac{z}{2}\right)\right]\times\left[\Phi\left(\frac{a+2x}{4}\left(\frac{3}{t}\right)^{\frac{1}{2}}\right)+\Phi\left(\frac{a-2x}{4}\left(\frac{3}{t}\right)^{\frac{1}{2}}\right)\right], \quad (21)$$

where z is the distance, in mean free paths, along the beam, and x is the distance at right angles to the beam. Fig. 6 shows R_0 as a function of x for different values of z and of t. As the electrons have more than 0.997 of their initial energy when $t=3$, it is seen how rapidly the beam loses definition as the electrons lose energy.

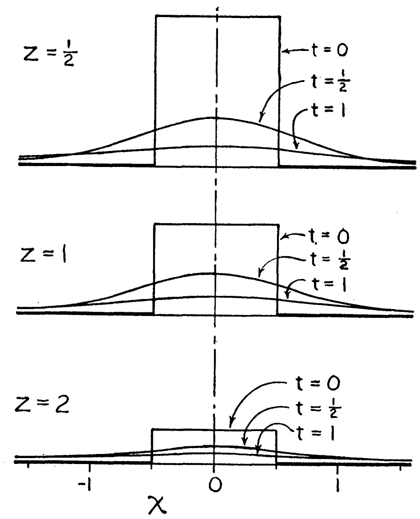

FIG. 6. Dispersion of a narrow beam of electrons by passage through a gas. Curves give values of $R_0=4\pi v^4 f_0$, as function of x the distance across the beam in mean free paths, for different values of z the distance from entry, and of t the average number of collisions since entry.

Although the discussion above was based on the assumption that the cross section Q was practically independent of the electronic velocity, it is possible to obtain a solution when Q cannot be considered constant. We set $S=4\pi v^4 NQf$, and Eq. (7), for $E=0$, becomes $S_1=-(1/NQ)(\partial S_0/\partial x)$. Choosing a new variable

$$\mu=(M/m)\int_v^{v_0}[dv/(NQ)^2v],$$

Eq. (8) becomes $(\partial^2 S_0/\partial x^2)=3(\partial S_0/\partial\mu)$, which is similar to Eq. (17). S can therefore be obtained by the methods discussed above. However, since the relation between μ and v is a complicated one if Q varies markedly with v, average values of current, etc., must be obtained by numerical integration.

The foregoing investigation originated from a suggestion, made by Dr. K. T. Compton in connection with a problem of sputtering, that the effects due to particles which had suffered several collisions would be as important as those due the primary beam. We are indebted to him and to Dr. Irving Langmuir for several helpful discussions.

[5] Normand, Phys. Rev. 35, 1217 (1930).