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On the Cross Section of Heavy Nuclei for Slow Neutrons

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The cross section for slow neutrons is computed with the statistical model of the nucleus and neutron-proton interaction of the Majorana form. The constants in the potential are taken from Feenberg's calculations on the binding energies of light nuclei. It is rather surprising that the scattering problem is reducible to one particle form despite the fact that the Majorana operators permute the coordinates of neutrons and protons. The formulas for the cross section are applicable even when there is neutron-neutron interaction, as suggested by L. Young, provided a simple change is made in the constants. It is shown that the anomalously large cross sections in Cd and Hg cannot be interpreted on the basis of an additional node of the neutron's wave function within the nucleus for Hg as compared with Cd.

FERMI and collaborators,[1] also Dunning and Pegram,[2] have found that the cross sections of certain nuclei (notably B, Cd, W, Hg) for slow neutrons are larger than the geometrical cross section.[3] Fermi,[1] Bethe[4] and others[5] have shown that such abnormally large values are not really a mystery but can be naturally interpreted by our usual quantum mechanics. Namely, the quantum mechanical cross section for slow particles depends markedly on the phase at the edge of the nucleus, or in other words on the fractional number of de Broglie waves of centro-symmetric type ($l=0$) contained in the nucleus. The determining factor is thus the value of the expression

$$I=2\int_0^R[2M(W-V)]^{\frac{1}{2}}dr/h, \qquad (1)$$

where $V(r)$ is the potential function inside the nucleus and R is the latter's radius. Since we are considering very slow neutrons, we may equate their energy W to zero in (1). When I is a half-integer, the cross section is readily shown[4, 5] to be unusually large, of the order of the square of the de Broglie wavelength of the incident neutron outside the nucleus. This situation is presumably the cause of many of the abnormally large cross sections and is, so to speak, the opposite extreme from that in the Ramsauer effect in electron scattering, wherein I is a whole integer and the cross section hence abnormally small. The purpose of the present note is to evaluate (1) and so discover, if possible, what particular half-integers should be correlated with the observed maxima. We do not explicitly compute the cross section but this can be determined by the standard quantum mechanics of the scattering by a "potential hole" if the latter's radius and the value of the phase integral (1) are known.

The deflections of neutrons, either elastic or inelastic, presumably owe their existence mainly to neutron-proton forces, including the forces exerted by the protons bound in the alpha-particles contained in the nucleus. Consequently we shall omit neutron-neutron forces until near the end of the article. The form of the potential function $J(r_{np})$ for the interaction of neutrons with protons will be taken from the work of Feenberg.[6] Do not confuse $J(r_{np})$, which represents the coupling of one neutron and one proton, with $V(r)$, which is the potential for a neutron due to the totality of protons in the nucleus.

Wigner Theory. We shall first make the calculation under the assumption that J does not involve permutation operators and so is a potential of the ordinary or so-called Wigner type. Of course it will immediately be objected that the Wigner theory[7] is unsatisfactory since it is

[1] E. Amaldi, O. D'Agostino, E. Fermi, B. Pontecorvo, F. Rasetti and E. Segrè, Proc. Roy. Soc. **A149**, 522 (1935).

[2] J. R. Dunning and G. B. Pegram, Phys. Rev. **47**, 640A (1935).

[3] The cross section may be either elastic or inelastic, as in either case the dominant question is whether or not the phase integral (1) has a value which permits the neutron to penetrate the nucleus appreciably. We throughout use the term scattering in a generalized sense to include absorption of neutrons in the nucleus as well as true scattering. The abnormally large cross section usually appears to be associated mostly with absorption (cf. Dunning, Pegram, Fink and Mitchell, Phys. Rev. **47**, 796, 970 (1935)).

[4] H. Bethe, Phys. Rev. **47**, 747 (1935).

[5] Perrin and Elsasser, Comptes rendus **200**, 450 (1935); Beck and Horsley, Phys. Rev. **47**, 510 (1935).

[6] E. Feenberg, Phys. Rev. **47**, 850 (1935).

[7] E. Wigner, Phys. Rev. **43**, 252 (1933).

known to give the wrong dependence of nuclear binding energy on atomic number. However, the calculation with the Wigner[7] form of potential is very easy and serves as an illuminating and more or less necessary introduction to the computation with the preferable Majorana theory.[8] With the former, we have simply to regard the scattered neutrons as subject to the "time-exposure charge cloud" of the totality of protons in the nucleus. We may suppose that these protons are on the time average uniformly distributed in a sphere of radius R, so that the charge cloud is centro-symmetric and of uniform density $\rho=3Z/4\pi R^3$, where Z is the number of protons in the nucleus. The relation connecting V and J is clearly

$$V=\iiint J\rho dv. \tag{2}$$

Now according to Feenberg's calculations,[6] or any others[7] devised to explain the high stability of alpha-particles as compared with deuterons, the neutron-proton forces are of short range ($\sim 1.4\times 10^{-13}$ cm) compared to the radius ($\sim 7\times 10^{-13}$ cm) of a heavy nucleus. Hence we can neglect the "edge effect," i.e., we can assume that the potential suddenly vanishes at the boundary of the nucleus. Then $V(r)$ vanishes for $r>R$, while for $r<R$ it is constant and has the value

$$V=3ZC/4\pi R^3, \tag{3}$$

with $C=\iiint J(r_{np})dv=4\pi\int_0^\infty J(r_{np})r_{np}^2dr_{np}$. (4)

Because of the short range, it has been permissible to integrate to infinity in (4) rather than over the nucleus. If the nuclear volume is proportional to the atomic number, then one has $R^3=ZR_0^3$, where R_0 is independent of Z, and (1) becomes

$$I=Z^{\frac{1}{3}}[-6CM/\pi h^2R_0]^{\frac{1}{2}}. \tag{5}$$

If we use an exponential interaction function $J(r)=Ae^{-\alpha r^2}$, the expression (4) equals $A(\pi/\alpha)^{\frac{3}{2}}$. Feenberg's best values of A, α for the Wigner model are, respectively,

$$A=-165mc^2,\quad 1/\alpha^{\frac{1}{2}}=1.34\times 10^{-13}\text{ cm} \tag{6}$$

while Gamow's value of the radius of the lead nucleus gives $R_0=7.8\times 10^{-13}/(82)^{\frac{1}{3}}$ cm. With the constants thus evaluated, the expression (1) has the value 3.11 for Cd.

Impossibility of Attributing Successive Maxima to Additional Nodes. In view of the inaccurate status of nuclear models, especially those of Wigner type, the fact that 3.11 is closer to an integer than a half-integer, i.e., the cross section closer to a minimum than to the desired maximum need not cause concern. However, the point that we do want to make is that it does not appear possible to explain the maxima in W and Hg as due to an additional node of the wave function in the nucleus, i.e., increase of (1) by an integer, as compared with Cd. If the constants C and R_0 are adjusted so that (5) equals 3.5 in Cd, the values for W and Hg are, respectively, 4.05 and 4.15, considerably short of the next resonance value 4.5. If 2.5 rather than 3.5 is assigned to Cd the situation is even worse as the value for Hg becomes 2.97, almost exactly the wrong extreme. The corresponding value for boron is 1.18 but is less significant, since our theory is intended primarily for heavy atoms. To locate successive maxima at Cd and Hg, the value of I in Cd would have to be at least 5.5 and this is an impossibly high value since nothing like this much leeway is allowable in the choice of the parameters C, R_0, which fortunately enter in (5) only to the power one-half. Of course it is to be said that so far we have used the Wigner rather than Majorana theory but inclusion of the Majorana permutation operators makes interaction possible only where the wave functions overlap and so will reduce the magnitude of the effective potential $V(r)$. In fact, we will later see that with a Majorana potential with the same values (6) of the constants as we have used in the Wigner form, the value of I for Cd becomes 2.21 rather than 3.11. Thus with the Majorana version it is, if anything, still clearer, that the situation is quite unlike that in the Ramsauer effect in electron impact, where the periodic Ramsauer minima have been identified[9] with increasing numbers of nodes of the electron's wave function inside the atom as Z increases.

If the "additional node" hypothesis will not work, how, then, are the frequent anomalously large cross sections to be explained? A very sug-

[8] E. Majorana, Zeits. f. Physik **82**, 137 (1933).

[9] Cf. Mott and Massey, *The Theory of Atomic Collisions* (Oxford University Press, 1933), p. 142.

gestive tentative answer to this question will be given in a future paper of Bethe and Smith. They propose that sometimes the p, d, $\cdots$ partial cross sections ($l=1$, 2, $\cdots$ etc.) are effective as well as the s in causing anomalous scattering or absorption. To be sure, the centrifugal force usually keeps all but s neutrons out of the nucleus since we are dealing with slow exterior velocities. However, near resonance the behavior is somewhat exceptional, especially since we do not have rigorously a one particle system, and so p, d, $\cdots$ wave functions may acquire some s characteristics inasmuch as l is not a good quantum number. Hence many of the subsidiary maxima may be associated with other than the s cross sections which we study.[9a]

Majorana Theory. We now give the details of the calculation with the Majorana theory, which can be used without any undue difficulty provided that one applies the Thomas-Dirac statistical theory to the protons embedded in the nucleus (but not, of course, to the scattered neutron). This approximation is essentially the analog of the use of the uniform charge cloud in the Wigner form. In the Majorana theory the potential coupling the scattered neutron with the protons in the nucleus is

$$\textstyle\sum_i J(r_{np}{}^i)T(n;p^i) \quad (r_{np}{}^i=|\mathbf{x}_n-\mathbf{x}_p{}^i|), \quad (7)$$

where $\mathbf{x}_n$, $\mathbf{x}_p{}^i$ are, respectively, the position vectors of the neutron and a typical proton i and where $T(n;p^i)$ is a permutation operator which interchanges $\mathbf{x}_n$ and $\mathbf{x}_p{}^i$ in the arguments of the complete wave function. The wave equation is reduced, if possible, to that of one particle, the scattered neutron, by integrating over the coordinates of the protons. In this fashion one obtains the equation

$$(-h^2/8\pi^2M)\nabla^2\psi(\mathbf{x}_n)+W\psi(\mathbf{x}_n)=Q, \quad (8)$$

where

$$Q=\textstyle\sum_i \iiint \varphi_i{}^*(\mathbf{x}_p)J(r_{np}{}^i)\varphi_i(\mathbf{x}_n)\psi(\mathbf{x}_p)d\mathbf{x}_p \quad (9)$$

and where ψ denotes the wave function of the neutron, and φ_i that of the proton i. We have assumed that the complete wave function is expressible as the product $\psi(\mathbf{x}_n)\Pi\varphi_i(\mathbf{x}_p{}^i)$ of the wave function of the scattered neutron and those of the individual protons in the nucleus. (To satisfy the Pauli principle, the usual antisymmetric linear combination of the protonic wave functions must be taken, but our calculation still holds with this modification.) Because of this factorization it has been possible to make the integration in (9) three rather than $3Z$ dimensional, since with any given neutron-proton potential term, the integration over $3(Z-1)$ protonic coordinates is trivial. Note especially that the integrand of (9) contains the product $\varphi_i(\mathbf{x}_n)\psi(\mathbf{x}_p)$ rather than $\varphi_i(\mathbf{x}_p)\psi(\mathbf{x}_n)$ as in the usual Wigner theory, because of the permutation operator T involved in (7).

We now use the approximate relation

$$\textstyle\sum_i \varphi_i{}^*(x_p)\varphi_i(x_n) = (2/h^3)\iiint \exp\left[2\pi i\mathbf{p}\cdot(\mathbf{x}_n-\mathbf{x}_p)/h\right]d\mathbf{p}. \quad (10)$$

The left side of this equation is called the Dirac density function, which we will denote by ρ_{np} and the right side is an approximate expression which he derives[10] for it by means of the Thomas-Fermi statistical theory. The integration in (10) is over a sphere of radius P given by

$$2(4\pi P^3/3)(4\pi R^3/3)=Zh^3 \quad (11)$$

and it is supposed that the terminal point of $\frac{1}{2}(\mathbf{x}_n+\mathbf{x}_p)$ is inside the nucleus; otherwise the the density function vanishes. Since our forces are of short range compared with the nuclear diameter, it will be allowable to consider that (10) applies when the neutron is inside the nucleus and that the left side of (10) vanishes otherwise. The integration in (10) may be performed in the usual way[10] by introducing polar coordinates whose axis coincides in direction with r. Then (10) becomes

$$\rho_{np}=(-kr_{np}\cos kr_{np}+\sin kr_{np})/\pi^2r_{np}{}^3, \quad (12)$$

where

$$k=(9\pi Z/4R^3)^{\frac{1}{3}}. \quad (13)$$

One now assumes that the neutron's wave

[9a] Since the present paper was written, Dunning, Pegram, Fink and Mitchell find that the rare earths Sm and Tb (which lie between Cd and Hg) also have anomalously large cross sections, even greater than that of Cd (Phys. Rev. **47**, 970 (1935)). The occurrence of such frequent anomalies, also the complete absorption,[3] makes it appear somewhat doubtful whether the large cross sections are really all to be interpreted as resonance phenomena of the usual type, even when one considers the effect of states with $l>0$.

[10] See P. A. M. Dirac, Proc. Camb. Phil. Soc. **26**, 376 (1930).

function is of the usual spherical type,

$$\psi(\mathbf{x}_n)=\sin k_n r/r, \tag{14}$$

so that

$$\psi(\mathbf{x}_p)=(e^{ik_n r_p}-e^{-ik_n r_p})/2ir_p, \tag{15}$$

where

$$r_p^2=r^2+r_{np}^2-2r_{np}r\cos\theta, \tag{16}$$

$$\cos\theta=\cos(r_{np},r).$$

When one substitutes (15) and (16) in (9), the integration over the angular part of the volume element $d\mathbf{x}_p=r_{np}^2\sin\theta dr_{np}d\theta d\varphi$ can be performed in polar coordinates. One thus finds that

$$Q=\psi(\mathbf{x}_n)\int_0^\infty 4\pi k_n^{-1}\rho_{np}J(r_{np})(\sin k_n r_{np})r_{np}dr_{np} \tag{17}$$

with ρ_{np} as in (12). Hence the wave equation (8) has the one particle form $Q=V\psi(\mathbf{x}_n)$ provided V equals the integral in (17). An equivalent statement is that (3) and (5) are still valid provided C has the value

$$C=12\pi^3k^{-3}k_n^{-1}\int_0^\infty \rho_{np}J(r_{np})\times(\sin k_n r_{np})r_{np}dr_{np}, \tag{18}$$

instead of (4).

It must be regarded as rather surprising that *the scattering of the neutron can thus be treated by a one particle equation* despite the fact that the Majorana theory permutes the coordinates of the scattered neutron with those of the proton in the nucleus. Furthermore, Mr. C. H. Fay will prove in a later paper that this result is also true of the other partial cross sections $l>0$; the "effective potential" there, too, proves to be given by (3) and (18). One's conjecture would have been that because of the permutation effects, Q in (8) would persistently involve $\psi(\mathbf{x}_n)$ through some sort of an integral operator (cf., for instance, Fock's equation) rather than finally turning out directly proportional to $\psi(\mathbf{x}_n)$. Thus far we have not explicitly specified the value of the neutron's wavelength $2\pi/k_n$ inside the nucleus. It is, of course, that given by the relation

$$-k_n^2=8\pi^2MV/h^2=6\pi ZCM/R^3h^2 \tag{19}$$

appropriate to the effective potential V. We must emphasize that it is only because there has turned out to be an effective potential constant over the nucleus that we have been justified in assuming that the neutron's wave function is of the form (14). This assumption has been necessary in our proof of (18). Hence problems with nonconstant potentials probably would not be reducible to a one particle affair.

By eliminating C between (18) and (19) an equation is obtained for the determination of k_n. This equation must be solved by trial and error before (18) and (5) can be used to calculate (1). Eq. (18) becomes identical with (4), as we should expect, when the range of the interaction force is small compared both to $2\pi/k_n$, the neutron's wavelength while traversing the nucleus and to $2\pi/k$, the shortest wavelength of the proton embedded in the nucleus.

With the exponential form $J(r)=Ae^{-\alpha r^2}$, Eq. (18) becomes[11]

$$C=(3\pi^2/k^3)A[E(w_+)-E(w_-)+2(\alpha/\pi k_n^2)^{\frac{1}{2}}(e^{-w_+^2}-e^{-w_-^2})], \tag{20}$$

where $w_\pm=(k_n\pm k)/2\alpha^{\frac{1}{2}}$, $E(w)=(2/\pi^{\frac{1}{2}})\int_0^w e^{-x^2}dx$.

A series expansion

$$C=A\left(\frac{\pi}{\alpha}\right)^{\frac{3}{2}}e^{-k_n^2/4\alpha}\left[1-\frac{1}{10}\frac{k^2}{\alpha}\left(\frac{3}{2}-\frac{k_n^2}{4\alpha}\right)+\frac{3k^4}{224\alpha^2}\left(1-\frac{k_n^2}{3\alpha}+\frac{k_n^4}{60\alpha^2}\right)+\cdots\right] \tag{21}$$

is convenient for small values of k^2/α. With Feenberg's[6] best choice of constants for the Majorana form, *viz.*

$$A=-174mc^2,\quad \alpha^{-\frac{1}{2}}=1.29\times10^{-13}\text{ cm}, \tag{22}$$

one finds that $k_n=1.06\cdot10^{13}$ and that $I=2.21$. It makes no appreciable difference whether (6) or (22) is used.

Inclusion of Neutron-Neutron Interaction. As shown by L. Young,[12] considerations of the relative stability of nuclei with different atomic and mass numbers indicate that there must be neutron-neutron forces (and by symmetry similar proton-proton forces) besides the usual neutron-proton and Coulombic proton-proton forces. The neutron-neutron coupling is presumably small compared to the neutron-proton, but still not negligible. We shall show in the following paragraphs that inclusion of neutron-neutron inter-

[11] In deriving (20), a useful relation is obtained by integrating formula 508 of Peirce's tables with respect to the parameter b.

[12] L. Young, Phys. Rev. **47**, 972 (1935).

action is approximately equivalent merely to altering the constant A in (20) or (21).

The most general potential which depends on the relative alignment of the spins $\mathbf{s}_i$, $\mathbf{s}_j$ (measured in multiples of $h/2\pi$) of two neutrons is expressible as a linear function

$$M(r_{nn})+N(r_{nn})\mathbf{s}_i\cdot\mathbf{s}_j \qquad (23)$$

of their scalar product $\mathbf{s}_i\cdot\mathbf{s}_j$ inasmuch as the latter has only two characteristic values. Eq. (23) gives the "direct" or essentially Wigner type of coupling between two neutrons. With two neutrons, however, there is exchange degeneracy, a complication not found in neutron-proton interaction. The exchange effect converts a Wigner force into one of Majorana type, or *vice versa.* (Hence the present discussion is general enough to include both the Wigner and Majorana models.) If we use the Dirac vector model,[13, 14] which shows that the exchange is formally equivalent to insertion of a spin-spin coupling factor $-\frac{1}{2}(1+4\mathbf{s}_i\cdot\mathbf{s}_j)$, we see that the exchange potential associated with (23) is

$$-\tfrac{1}{2}T_{nn}[M+N\mathbf{s}_i\cdot\mathbf{s}_j](1+4\mathbf{s}_i\cdot\mathbf{s}_j) = -\tfrac{1}{2}T_{nn}\{M(1+4\mathbf{s}_i\cdot\mathbf{s}_j)+N(\tfrac{3}{4}-\mathbf{s}_i\cdot\mathbf{s}_j)\}, \qquad (24)$$

where T_{nn} is a Majorana permutation operator. The second form of (24) follows from the first in virtue of the matrix identity[15] $16(\mathbf{s}_i\cdot\mathbf{s}_j)^2+8\mathbf{s}_i\cdot\mathbf{s}_j-3=0$. If we consider the interaction of a neutron with two other neutrons i, i' in a closed shell (i.e., in equivalent orbits with anti-parallel spin), then the part of (23) or (24) proportional to $\mathbf{s}_i\cdot\mathbf{s}_j$ disappears since $(\mathbf{s}_i+\mathbf{s}_{i'})\cdot\mathbf{s}_j=0$. Thus on the average (23) equals M and (24) equals $-\frac{1}{2}T_{nn}(M+\frac{3}{4}N)$, provided we disregard forces between neutrons, neither of which are in a closed shell, or else are in the same orbit. It is known that if the direct potential persists on the average in heavy atoms, then an excessive binding energy is obtained. (This is essentially the difficulty that Wigner terms give an energy which varies too rapidly with Z.) Hence M is zero, or at least very small, and for our purposes we can consider the neutron-neutron forces to be entirely of the Majorana type, neglecting a small correction for neutrons not in closed shells to be considered later. The effective potential is thus the sum $V+V_{nn}$ of two expressions V, V_{nn}, where V is given by (3) and (18) and where V_{nn} is given by expressions similar to (3) and (18) except that Z is replaced by Z_n, the number of neutrons in the nucleus, and that J, k must be replaced, respectively, by $-\frac{3}{8}N$ and $k'=(9\pi Z_n/4R^3)^{\frac{1}{3}}$ (cf. Eq. (13)). In the neutron-neutron version of (18) or (20) the change due to the substitution of k' for k manifests itself mainly in the factor k^{-3} (rather than in ρ_{np}) if w is moderately large. Since the neutron-neutron forces are subsidiary it will be a sufficient approximation if we ignore the distinction between k and k' except in this factor. The resulting change in (18) or (20) is then just counterbalanced by the substitution of Z_n for Z in (3). Hence to a sufficient approximation V_{nn} differs from V merely in the substitution of $-\frac{3}{8}N$ for J. Thus one may allow for neutron-neutron coupling by taking the apparent neutron-proton interaction to be $J-\frac{3}{8}N$.

Feenberg uses a neutron-neutron interaction of exponential form and of the same range as the neutron-proton, so that $J(r)=Ae^{-\alpha r^2}$, $N(r)=Be^{-\alpha r^2}$. He finds[15a] that the following choice of constants gives the proper binding energies for light atoms:

$$A=-64\ mc^2,\quad \tfrac{3}{4}B=26\ mc^2, \quad 1/\alpha^{\frac{1}{2}}=2.4\times10^{-13}\ \text{cm}. \qquad (25)$$

Using (3) and (20), except that A is replaced by $A-\frac{3}{8}B$ in accordance with the preceding paragraph, one finds that $k=1.05\times10^{13}$, $I=2.19$. These values do not differ appreciably from those obtained with the constants (22) appropriate to a model with no neutron-neutron coupling. The similarity of the results with (22) and (25) may perhaps be a coincidence, since the constants (25), notably the range $1/\alpha^{\frac{1}{2}}$, differ considerably from (22). Still it suggests that our estimate of (1) will probably not be altered greatly by neutron-neutron complications as long as the constants are adjusted to fit the binding energies of light atoms. This fact is comforting, as the ratio of neutron-neutron to neutron-proton forces is not known at present with any precision.

Comparison with Bethe's Semi-empirical Method. Bethe[4] has estimated the value of the integral (1) by a method somewhat different from ours. He subtracts the mean kinetic energy of the neutrons bound in the nucleus, computed by the Thomas statistical method from the observed binding energy per neutron. The balance is what we may term a semi-empirical potential to be used in (1). He thus obtains the value 2.7

[13] Our rather brief treatment here assumes some familiarity with the Dirac vector model (Proc. Roy. Soc. **A123**, 714, 1929), whose use is more fully explained by the writer in Phys. Rev. **45**, 405 (1934) or in reference 15.

[14] Another and more complete discussion of neutron-neutron interaction which does not employ the language of the Dirac vector model (given here largely as an interesting alternative) will be published by Feenberg.

[15] J. H. Van Vleck, *The Theory of Electric and Magnetic Susceptibilities*, pp. 341 and 318.

[15a] These calculations of Feenberg will be published later. Since the present paper was written, he has extended them, and finds that the neutron-neutron interaction is probably smaller and in (25). This development does not matter for our purposes since even inclusion of as much neutron-neutron interaction as in (25) does not change the value of (1) materially. It is clearly to be understood that the reason the results are not affected by inclusion of neutron-neutron interaction is that the latter alters the values of A and α in such a way as largely to compensate the effect of making $B\neq0$.

for (1) in Cd.[16] That his value 2.7 is smaller than our Wigner value 3.1 is immediately intelligible, as the Wigner model gives too large an attractive potential energy for heavy atoms. It is also clear why his estimate is larger than our Majorana value 2.2. Namely, Bethe's procedure tacitly assumes forces of the Wigner type and so is not accurately applicable to the Majorana model since, in the latter, forces are effective only where the wave functions overlap and so are less potent for scattered neutrons than for the average neutron bound in the nucleus, because of the longer wavelength of the latter. Hence the effective potential for scattering is of smaller absolute magnitude than that for binding.[17]

Concluding Remarks. It would be satisfying if (1) could be estimated sufficiently accurately so that one could decide whether the resonance in Cd corresponds to the value 2.5 or 1.5 of I or whether I is really not a half integer in Cd, so that the anomaly is to be blamed on effects involving $l>0$. Unfortunately we are unable to achieve this goal. Some corrections which might be applied to our calculations are the following: The error due to considering the range of the forces as negligible compared with the radius of the nucleus and to neglecting the gradual tailing off of the nucleus may be crudely estimated by assuming that the potential decreases linearly with r from $R-L$ to $R+L$ instead of terminating suddenly at $r=R$. If $L=2\times 10^{-13}$ cm, the value of (1) is raised about 10 percent.[18] A small increase of about 2 percent may be expected for each neutron of uncompensated spin in the nucleus, as such neutrons exert forces of the ordinary rather than Majorana type. On the other hand, a reduction of 10 or 15 percent in our estimates of (1) is quite conceivable because there is empirical evidence that the neutron-proton forces in scattering are smaller than those effective between bound particles. This distinction may be due either to a dependence of force on velocity, as suggested by Feenberg,[19] or to a dependence of force on spin alignment, as proposed by Massey and Mohr,[20] Wigner (unpublished) and others. The scattered neutron's spin is parallel to only half the protonic spins in heavy nuclei, whereas the situation is different in calculations made on the binding energy of the deuton, where the proton's spin and the neutron's spin are parallel.

In closing, we may mention that the effective potential given by (3) and (18), which reduces the Majorana theory to a one electron problem, will doubtless be useful in connection with other nuclear phenomena besides scattering.

The writer is much indebted to Dr. E. Feenberg for valuable suggestions and the privilege of seeing his manuscripts in advance of publication. He also wishes to thank Professor H. Bethe for interesting discussion.

[16] In comparing with Bethe's work, one must note that his integral (8) differs from our (1) by an additive constant as well as a factor π (see footnote 26b of his paper). Bethe's value of I is raised from 2.7 to 2.9 when a correction is made for the effect of the Coulombic forces on binding energy.

[17] Bethe's method suggests that a possible semi-empirical modification of our procedure is to adjust the constant A so as to obtain the right binding energy for heavy nuclei when the latter is computed by the statistical method with Gamow's empirical radius (i.e., is deduced from a calculation similar to that in Majorana's paper[8] except that the range of the forces is not assumed large compared to the wavelength). The value of I is then raised from 2.2 to 2.8 if α is given by either (22) or (25) (with neutron-neutron forces included in the latter case). One might suspect that this modified procedure had elements of truth, as the apparent A deduced from an incorrect statistical calculation of binding energies might be better for our purposes than the true A. Thus the difference between 2.2 and 2.8 may be an estimate of the error inherent in our statistical method. However, it is possible that this is an excessively pessimistic view and that with (22) or (25) the actual value of I is closer to 2.2 than 2.8, since more reliance can probably be placed in our use of the statistical method for scattering than in the calculation of binding energies, especially when the latter is performed in the worst possible way with the nuclear radius as given rather as a variable parameter. That such a calculation of binding energies is certainly bad is shown by the fact that if A is given by either (22) or (25), the absolute magnitude of the potential energy turns out to be less than that of the kinetic energy, so that nuclei cannot exist, and the proper stability is achieved only if A is multiplied by a factor $(2.8/2.2)^2$. An accurate calculation with (22) or (25) probably yields about the right binding energy for heavy as well as light atoms.

[18] The value of L can be less than $1/\alpha^{\frac{1}{2}}$, since in the Majorana theory the forces are effective only where the wave functions overlap.

[19] E. Feenberg, Phys. Rev. **47**, 857 (1935).

[20] Cf. Massey and Mohr, Proc. Roy. Soc. **A148**, 213 (1935). In the original Heisenberg theory (Zeits. f. Physik **77**, 1 (1932)), the neutron-proton forces are of opposite sign for parallel as compared with anti-parallel spins. In the Majorana form there is no dependence on spin alignment. The dependence on this alignment cannot be as drastic as in the Heisenberg theory since the latter is unable to explain the singular stability of the alpha particle and is therefore incorrect. There might, however, be a moderate dependence, so that the forces are a linear combination of the Heisenberg and Majorana types, with the latter weighted more heavily. Our calculations apply to such a modified theory if the constant A is suitably changed.