The Formation of Electron-Positron Pairs by Internal Conversion of γ-Radiation

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The production of an electron and positron by the spherical γ-wave emitted from a nucleus has been considered on the basis of different approximation methods (Section II, III, IV), which yield simple closed formulae for the probability of the process. Regarding the nucleus as a Hertzian oscillator, both the dipole and quadripole radiation are treated. The approximate methods, which may be classified according to which of a certain set of parameters (the momenta of the particles, αZ, and eventually the ratios of these) are to be considered small, form a consistent scheme (§6, where also the case of the plane γ-wave is discussed). A discussion of the qualitative features of the angular and energy distribution is given (§7 and §8) and a comparison is made with the rigorous treatment of the problem by Jäger and Hulme (§9). This shows that both the differential and total internal conversion coefficient may be computed with a fair degree of accuracy by the combined use of the Born and the Schrödinger approximation results.

I. INTRODUCTION

1.

THE Dirac theory of the positron[1] describes the formation of an electron-positron pair by γ-radiation near a nucleus as a kind of photoelectric effect. The absorption of a γ-quantum of energy $\geqq 2mc^2 = 10^6$ ev raises an electron in an occupied negative energy state to a state in the positive continuum, the "hole" left in the negative energy spectrum together with the electron in the positive state constitute the positron-electron pair. For the case that the γ-radiation is a plane wave, calculations, with approximate wave functions for the probability of the process have been made by Oppenheimer and Plesset, Bethe and Heitler and others.[2] For the case of a spherical γ-wave emitted by the nucleus, with which we shall be concerned here, there exist the calculations of Oppenheimer and Nedelsky,[3] where the Born approximation is used, and those of Jäger and Hulme,[4] in which the exact solutions of the Dirac equations for a Coulomb field have been used. Because of the lengthy numerical computations necessary in the exact theory, we have gone back to approximate methods which give simple closed formulae. We have especially examined the consistency of these methods. By comparing the results with those of Jäger and Hulme one can then obtain an idea concerning the range of applicability of these approximate methods, which may be also of value in other related problems.

2.

The calculation of the probability for pair formation can in all cases be reduced to the evaluation of the matrix elements of the perturbation due to the radiation. We write the Dirac equation for the electron in a Coulomb field with a nuclear charge $\alpha^{\frac{1}{2}}Z$.[5]

$$i\partial\psi/\partial t = \{(\boldsymbol{\alpha}\cdot\mathbf{p}) + \beta + \alpha Z/r - \alpha^{\frac{1}{2}}\mathfrak{H}\}\psi. \quad (1)$$

Here $\mathfrak{H}$ is the perturbation of the radiation field:

$$\mathfrak{H} = He^{-ikt} + \text{compl. conj.},$$
$$H = V + (\boldsymbol{\alpha}\cdot\mathbf{A}), \quad (2)$$

k is the frequency which in our units is the same as the energy of the photon of this frequency; V and $\mathbf{A}$ are the space parts of the scalar and vector potentials. In accordance with the results of Hulme, Taylor and Mott[6] for the atomic

[1] P. A. M. Dirac, Proc. Roy. Soc. **133**, 60 (1931).

[2] J. R. Oppenheimer and M. Plesset, Phys. Rev. **44**, 53 (1933) (nonrelativistic approximation); H. Bethe and W. Heitler, Proc. Roy. Soc. **146**, 83 (1934) (Born approximation, influence of screening); G. Racah, Nuovo Cimento **11**, 3 (1934) (Born approximation); Y. Nishina, S. Tomonaga and S. Sakata, Suppl. Sc. Pap. I. P. C. R. **24**, 1 (1934) (nonrelativistic and Born approximation).

[3] J. R. Oppenheimer and L. Nedelsky, Phys. Rev. **44**, 948 (1933).

[4] J. C. Jäger and H. R. Hulme, Proc. Roy. Soc. **148**, 708 (1935).

[5] In the following we will always use rational relativistic units. In this system the unit of energy is mc^2, of momentum mc and of length the Compton wavelength $\hbar/mc$. Equations in ordinary units may be written in these units by replacing $\hbar$, m and c by unity and the electron charge e by $\alpha^{\frac{1}{2}}$, where α is the fine structure constant $e^2/\hbar c$. This α is not to be confused with the Dirac matrices α_x, α_y, α_z, collectively denoted by $\boldsymbol{\alpha}$.

[6] H. R. Hulme, Proc. Roy. Soc. **138**, 643 (1932); H. M. Taylor and N. F. Mott, Proc. Roy. Soc. **138**, 665 (1932).

internal conversion, we will consider the case of the electric quadripole wave as well as that of the electric dipole wave. The fields are given by (omitting from each potential a constant factor having the meaning of a moment, since it does not enter in the final results):

Electric dipole:

$$V=(1/r)e^{ikr}(i-1/kr)\cos\vartheta;$$
$$A_z=(i/r)e^{ikr},\quad A_x=A_y=0. \tag{3}$$

Electric quadripole:[7]

$$V=(1/r)e^{ikr}\{2P_2(\cos\vartheta)(1+3i/kr-3/k^2r^2)+1\},$$
$$A_z=(3/r)e^{ikr}(1+i/kr)\cos\vartheta,\quad A_x=A_y=0. \tag{4}$$

Applying the usual method of variation of constants the solution of (1) may be expressed in terms of the probability amplitudes $a_n(t)$ of the various states. At time $t=0$ the electron is in a negative energy state W_0, so that $a_n(0)=\delta_{n0}$. One finds then from (1) for the probability $|a_n|^2$ that the electron be in state W_n at the time t:

$$|a_n|^2=4\alpha|(n|H|0)|^2\frac{\sin^2(W_n-W_0-k)t/2}{(W_n-W_0-k)^2}, \tag{5}$$

using the Dirac notation for the matrix elements. One has now to sum $|a_n|^2$ over the spin, energy and direction of momentum (or quantum numbers corresponding thereto) of both particles. The summation over the energy of one of the particles will give then as usual the conservation of energy. The manner in which the summation has further to be carried out will depend on the type of wave functions used. We will distinguish between three cases.

(*a*) *Both particles are represented by plane waves at infinity.* This is most convenient when the Born approximation or parabolic coordinate wave functions are used. To obtain the energy distribution a summation of (5) over the direction of the momenta and over the spin has to be carried out. The result is:

$$\sigma(W_+)dW_+=dW_+(2\alpha/(2\pi)^5n)(p_+p_-W_+W_-/k)\iint d\Omega_-d\Omega_+\sum_{s_+}|(\mathbf{p}_-s_-|H|\mathbf{p}_+s_+)|^2. \tag{A}$$

Here quantities referring to the electron are denoted by the index $-$ and those referring to the positron by the index $+$;[8] $d\Omega_-$, $d\Omega_+$ are elements of solid angle of the direction of emission of the two particles. The matrix element has to be formed with wave functions normalized to one particle per unit volume. The four term summation over the spin variables s_- and s_+ reduces to twice a two term sum over one, say s_+, because although there is a preferred direction (the z axis) there is no preferred sense. Finally nk is the number of γ-quanta which in unit time are emitted by the nucleus; n depends on the type of radiation;

$n=4/3$ for electric and magnetic dipole,
$n=12/5$ for electric quadripole.

(*b*) *Both particles represented by spherical waves at infinity.* This is most convenient in the case of the nonrelativistic approximation. Because the Dirac equations for a central field are separable *only* in polar coordinates,[9] this will also be the natural representation when the exact wave functions are used. Here the integration over the direction of the momenta is replaced by a summation over the azimuthal and magnetic quantum numbers of both particles. The energy distribution is then:

[7] We have taken the z direction along the multipole axis and we will also take the axis of quantization in this direction. That this is no loss in generality can be shown by averaging the results for an arbitrarily orientated multipole field over all directions of its axes. In the same way one shows that our restriction in (4) to a zonal quadripole is not essential. In the strict development of the electromagnetic field of an oscillating charge distribution there appears together with the electric quadripole the so-called magnetic dipole terms:

$$A_x=\sin\vartheta\sin\varphi(1+i/kr)(1/r)e^{ikr},\quad V=A_z=0 \tag{4a}$$
$$A_y=-\sin\vartheta\cos\varphi(1+i/kr)(1/r)e^{ikr}$$

which we shall also consider (cf. H. M. Taylor and J. B. Fisk, Proc. Roy. Soc. **146**, 178 (1934)).

[8] The energy W_0 and momentum $\mathbf{p}_0$ of an electron in a negative energy state are related to the corresponding quantities for a positron by $W_0=-W_+$ and $\mathbf{p}_0=-\mathbf{p}_+$.

[9] This is the reason why the internal conversion problem is simpler than the problem of pair formation by a plane wave.

$$\sigma(W_+)dW_+=dW_+(4\alpha/\pi n)(W_+W_-/p_+p_-k)\sum_{l_+m_+}\sum_{l_-m_-}\sum_{s_+}|(l_-m_-s_-|H|l_+m_+s_+)|^2. \tag{B}$$

For the fields (3), (4) or (4a) there will be selection rules which eliminate the summation over one set of quantum numbers l and m. The matrix element has here to be formed with wave functions normalized to one particle in a sphere of unit radius.

(*c*) *One particle represented by a plane wave and the other by a spherical wave at infinity.* This we will use in order to determine the form of the energy distribution curve at the ends, that is when one of the particles receives almost all the energy of the γ-quantum. In the case that the positron is represented by a plane wave at infinity the energy distribution is:

$$\sigma(W_+)dW_+=dW_+(\alpha/2\pi^3 n)(p_+W_+W_-/p_-k)\int d\Omega_+\sum_{l_-m_-}\sum_{s_+}|(l_-m_-s_-|H|\mathbf{p}_+s_+)|^2. \tag{C}$$

By interchanging the indices + and − one obtains the formula for the case that the electron is represented by a plane wave. Integration over W_+ from 1 to $k-1$ yields the total internal conversion coefficient σ.

II. The Born Approximation

3.

For light nuclei or for high energies of the particles we can neglect the nuclear field and take for the wave functions plane waves. The criterion for the validity of this approximation is that the Born parameters $\alpha Z/p_+$ and $\alpha Z/p_-$ are both small compared to unity. Then

$$\psi_\pm=u_\pm[\exp\mp i(\mathbf{p}_\pm\cdot\mathbf{r})]$$

and the amplitudes u_+ and u_- are:

$$u_-^{(1)}=\left(\frac{1+W_-}{2W_-}\right)^{\frac{1}{2}}\begin{vmatrix}-\dfrac{p_{-z}}{1+W_-}\\[2ex]-\dfrac{p_{-x}+ip_{-y}}{1+W_-}\\[2ex]1\\0\end{vmatrix},\qquad u_-^{(2)}=\left(\frac{1+W_-}{2W_-}\right)^{\frac{1}{2}}\begin{vmatrix}-\dfrac{p_{-x}-ip_{-y}}{1+W_-}\\[2ex]\dfrac{p_{-z}}{1+W_-}\\[2ex]0\\1\end{vmatrix} \tag{6a}$$

$$u_+^{(1)}=\left(\frac{1+W_+}{2W_+}\right)^{\frac{1}{2}}\begin{vmatrix}1\\0\\-\dfrac{p_{+z}}{1+W_+}\\[2ex]-\dfrac{p_{+x}+ip_{+y}}{1+W_+}\end{vmatrix},\qquad u_+^{(2)}=\left(\frac{1+W_+}{2W_+}\right)^{\frac{1}{2}}\begin{vmatrix}0\\1\\-\dfrac{p_{+x}-ip_{+y}}{1+W_+}\\[2ex]\dfrac{p_{+z}}{1+W_+}\end{vmatrix} \tag{6b}$$

The superscripts (1) and (2) refer to the two spin directions. Oppenheimer and Nedelsky[3] have given the results for the energy distribution for the electric dipole and quadripole. We shall here outline the method of calculation. The matrix element in (A) will have the form:

$$(\mathbf{p}_-s_-|H|\mathbf{p}_+s_+)=(u_-^*u_+)\int[\exp-i(\mathbf{P}\cdot\mathbf{r})]Vd\tau+(u_-^*\alpha_z u_+)\int[\exp-i(\mathbf{P}\cdot\mathbf{r})]A_z d\tau,$$

in which $\mathbf{P}=\mathbf{p}_-+\mathbf{p}_+$ and V and A_z are given by (3) or (4). The angular integrations are elementary and lead to radial integrals of the form:

$$\int_0^\infty J_{l+\frac{1}{2}}(Pr)e^{ikr}r^{\nu+\frac{1}{2}}dr,$$

where $P=|\mathbf{P}|$ and l and ν are integers. These may diverge at infinity. One has then, as in the derivation of the Rutherford scattering formula using the Born approximation, to introduce a convergence factor exp $(-ar)$ and after the integration set $a=0$. One obtains in the case of the dipole:

$$|(\mathbf{p}_-s_-|H|\mathbf{p}_+s_+)|=(16\pi^2/k^2(k^2-P^2)^2)\{P_z{}^2|u_-{}^*u_+|^2+2P_zkR(u_-{}^*u_+)(u_+{}^*\alpha_zu_-)+k^2|u_-{}^*\alpha_zu_+|^2\},$$

where R denotes "real part of." Using (6a) and (6b), summing over the spin of one of the particles and carrying out in (A) the integrations over all the angles but the one between $\mathbf{p}_+$ and $\mathbf{p}_-$, designated by θ, we obtain the angular distribution. Writing $J(\theta, W_+)$ for the number of pairs per unit energy range and per unit solid angle, one finds:

$$J(\theta, W_+)=\frac{\alpha}{4\pi^2k^3}p_-p_+\left\{\frac{k^2}{(q^2-p_+p_-\cos\theta)^2}+\frac{W_+{}^2+W_-{}^2}{(q^2-p_+p_-\cos\theta)}+1\right\}, \tag{7}$$

where $q^2=1+W_+W_-$. The energy distribution is given by:

$$\sigma(W_+)=2\pi\int_0^\pi J(\theta, W_+)\sin\theta d\theta=(\alpha/\pi k^3)\{2p_+p_-+(W_+{}^2+W_-{}^2)\log b\}, \tag{D$_1$}$$

$$kb=1+W_+W_-+p_+p_-.$$

When k is large one finds for the total internal conversion coefficient:

$$\sigma=\int_1^{k-1}\sigma(W_+)dW_+\leqq(2\alpha/3\pi)(\log 2k-\tfrac{3}{5})+O(1/k). \tag{8}$$

The calculation for the quadripole is quite analogous and one finds:

$$J(\theta, W_+)=\frac{\alpha}{4\pi^2k^3}p_+p_-\left\{\frac{k^2}{(q^2-p_+p_-\cos\theta)^2}+\frac{p_+{}^2+p_-{}^2}{(q^2-p_+p_-\cos\theta)}\right.$$

$$\left.+\frac{1}{3k^2}(16W_+W_--3k^2)-\frac{8}{3k^2}(q^2-p_+p_-\cos\theta)\right\}, \tag{9}$$

$$\sigma(W_+)=(\alpha/3\pi k^5)\{8p_+p_-(W_+W_--1)+3k^2(p_+{}^2+p_-{}^2)\log b\}, \tag{Q$_1$}$$

$$\sigma=(2\alpha/3\pi)(\log 2k-61/30)+O(1/k). \tag{10}$$

A discussion of these results will be given in section V.[10]

III. The Schrödinger Approximation

4.

When the energy of the γ-quantum is so small that throughout the entire allowed energy range $p_+\ll1$, $p_-\ll1$ and when further $\alpha Z\ll1$ the so-called Schrödinger approximation of the Dirac wave functions may be used. Then:

[10] It has been suggested by Taylor and Fisk, Reference 7, for the process of atomic internal conversion that one should perhaps consider besides the electric quadripole the magnetic dipole terms. We have therefore carried out the calculation of the pair formation by this field (Eq. 4a) and find:

$$\sigma(W_+)=(\alpha/\pi k^3)(p_+{}^2+p_-{}^2)\log b.$$

This is slightly smaller than Q_1, so that since the magnetic dipole moment is almost certainly much smaller than the electric quadripole moment, the contribution from this source may be neglected.

$$\psi_-^{(1)}=\begin{vmatrix} \frac{i}{2}\frac{\partial u_-}{\partial z} \\ \frac{i}{2}\left(\frac{\partial u_-}{\partial x}+i\frac{\partial u_-}{\partial y}\right) \\ u_- \\ 0 \end{vmatrix}, \qquad \psi_-^{(2)}=\begin{vmatrix} \frac{i}{2}\left(\frac{\partial u_-}{\partial x}-i\frac{\partial u_-}{\partial y}\right) \\ -\frac{i}{2}\frac{\partial u_-}{\partial z} \\ 0 \\ u_- \end{vmatrix}, \tag{11a}$$

$$\psi_+^{(1)}=\begin{vmatrix} u_+ \\ 0 \\ -\frac{i}{2}\frac{\partial u_+}{\partial z} \\ -\frac{i}{2}\left(\frac{\partial u_+}{\partial x}+i\frac{\partial u_+}{\partial y}\right) \end{vmatrix}, \qquad \psi_+^{(2)}=\begin{vmatrix} 0 \\ u_+ \\ -\frac{i}{2}\left(\frac{\partial u_+}{\partial x}-i\frac{\partial u_+}{\partial y}\right) \\ \frac{i}{2}\frac{\partial u_+}{\partial z} \end{vmatrix}. \tag{11b}$$

The superscripts again denote the two directions of the spin. The elements occurring as derivatives will be referred to as the small components since they enter with a coefficient $1/c$ (not apparent in our units). The u_- and u_+ are solutions of the Schrödinger equation:

$$\Delta u_\pm+(p_\pm^2\mp 2\alpha Z/r)u_\pm=0. \tag{12}$$

The solution of (12) for the electron in a state of the continuous spectrum ($W_->1$) is:[11]

$$u_-=N_\Omega P_l^m(\vartheta,\varphi)f_l^-(p_-r),$$

where the angular normalization factor is:

$$N_\Omega=((2l+1)/4\pi)^{\frac{1}{2}}(l+m)\,!^{-\frac{1}{2}}(l-m)\,!^{-\frac{1}{2}}.$$

Of the radial wave function $f_l^-(p_-r)$ we shall need only the expansion around the origin:

$$f_l^-(p_-r)=N_l^-r^l\left\{1-\frac{\alpha Z}{l+1}r+\frac{2\alpha^2Z^2-(l+1)p_-^2}{2(l+1)(2l+3)}r^2+\cdots\right\}. \tag{13}$$

Normalizing to one particle in a sphere of unit radius, we get:[12]

$$|N_l^-|^2=e^{\pi\alpha Z/p_-}(2p_-)^{2l+2}|\Gamma(l+1+i\alpha Z/p_-)|^2/2(2l+1)\,!^2 \tag{14}$$

The corresponding expressions for the positron may be obtained from these by changing the sign of Z and replacing the index $-$ by $+$. From the expressions for the electromagnetic fields (3) and (4) one can show that the matrix elements in (B) will vanish unless the following selection rules are fulfilled:

for the dipole	$\lvert l_--l_+\rvert=0,\ 2,$	$\lvert m_--m_+\rvert=0,\ 1,\ 2;$
for the quadripole	$\lvert l_--l_+\rvert=1,\ 3,$	$\lvert m_--m_+\rvert=0,\ 1,\ 2.$

For the internal conversion, first considering the dipole, it is evident from the normalization factors (14) that transitions involving high l's will introduce the parameters of smallness, p_+ and p_- to a high power. Indeed a calculation shows that it is necessary to consider only the $s-s$ transition and the large components in the wave functions to get the main contribution. Then:

[11] We omit here the index $-$ on the l and m.
[12] See e.g. H. Bethe, *Handbuch der Physik*, Vol. 24-1, p. 292.

$$(0, 0, \tfrac{1}{2}|H|0, 0, \tfrac{1}{2}) = \int \psi_-^{(1)} \alpha_z A_z \psi_+^{(1)} d\tau = i \int_0^\infty r e^{ikr} f_0^- f_0^+ dr,$$

only the vector potential and the wave functions with parallel spin contributing. Because $f_0^- f_0^+$ varies slowly compared to the oscillations of exp (ikr) the main contribution to the integral comes from a region near the origin. We may therefore insert for f_0^- and f_0^+ the first term of (13). In fact one sees that the terms in (13) involving higher powers of r would give contributions of higher order in p_+, p_- and αZ; therefore it would be inconsequent to include these terms. Thus:

$$\sum_{l_- m_-} \sum_{l_+ m_+} \sum_{s_+} |(l_- m_- s_-|H|l_+ m_+ s_+)|^2 \cong |(0, 0, \tfrac{1}{2}|H|0, 0, \tfrac{1}{2})|^2 \cong |N_0^-|^2 |N_0^+|^2 / k^4, \tag{15}$$

where as before we have introduced a convergence factor exp $(-ar)$ in the integrand and after integration set $a=0$. The energy distribution is then from (B):

$$\sigma(W_+) dW_+ = dW_+ \frac{48\pi\alpha^3 Z^2 W_+ W_-}{k^5 (e^{2\pi\alpha Z/p_+} - 1)(1 - e^{-2\pi\alpha Z/p_-})}. \tag{16}$$

To be consistent with the approximation here considered one should further replace W_+ and W_- by 1 and k by 2, so that:

$$\sigma(W_+) dW_+ = dW_+ \frac{3\pi\alpha^3 Z^2}{2(e^{2\pi\alpha Z/p_+} - 1)(1 - e^{-2\pi\alpha Z/p_-})}. \tag{D_2}$$

The calculation for the quadripole involves the $s-p$ and $p-s$ transitions. It is also necessary to consider the small components in those terms where they are multiplied with a large component. We will give only the result:

$$\sigma(W_+) dW_+ = dW_+ \frac{20\pi\alpha^3 Z^2}{3k^5} \left(1 - \frac{4}{k} + \frac{12}{k^2}\right) \frac{W_+ W_- (p_+^2 + p_-^2 + 2\alpha^2 Z^2)}{(e^{2\pi\alpha Z/p_+} - 1)(1 - e^{-2\pi\alpha Z/p_-})}. \tag{17}$$

Or again replacing k by 2:[13]

$$\sigma(W_+) dW_+ = dW_+ \frac{5\pi\alpha^3 Z^2}{12} \frac{p_+^2 + p_-^2 + 2\alpha^2 Z^2}{(e^{2\pi\alpha Z/p_+} - 1)(1 - e^{-2\pi\alpha Z/p_-})}. \tag{Q_2}$$

By using polar coordinate wave functions as in (B) it is impossible to obtain an angular distribution of the particles formed. For this one has to use wave functions which behave like plane waves at infinity as in (A). In the Schrödinger approximation this can be done by means of the well-known parabolic coordinate wave functions.[14] It is then found that in this approximation the distribution function $J(\theta, W_+)$ is independent of θ. For a discussion of these results see section V.

IV. Endpoint Formulae

5.

It is possible to find an expression for the energy distribution near the ends of the curve, that is, when one of the particles receives almost all the energy of the γ-quantum, by taking for the fast particle plane wave functions (6) and for the slow particle Schrödinger approximation wave functions (11). In case the positron is the fast particle the parameters of smallness are therefore $\alpha Z/p_+$, p_- and αZ. Then for the dipole we need only to consider the electron in an s state. The square of the matrix element in (C) summed over the spin s_+ becomes:

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[13] The corresponding result for the magnetic dipole is 3/5 of Q_2. Cf. footnote 10.

[14] Cf. Bethe, *Handbuch der Physik*, Vol. 24-1, p. 299.

$$\sum_{s_+}|(0,\,0,\,s_-|H|\mathbf{p}_+s_+)|^2=(2\pi^2/k)|N_0^-|^2\{1-[(k^2-4)/k^2]\cos^2\theta_+\},$$

where θ_+ is the angle between $\mathbf{p}_+$ and the z axis. Consistently with our development in powers of p_- we have set $p_+^2\cong k(k-2)$ and $W_+\cong k-1$. Inserting in (C) and integrating over θ_+ one obtains:

$$\sigma(W_+)dW_+=dW_+2\alpha^2Z(k^2+2)p_+/k^4(1-e^{-2\pi\alpha Z/p_-}). \tag{D$_3$}$$

Upon interchanging p_+ and p_- and changing the sign of Z we get the formula for the case that the positron is the slow particle:

$$\sigma(W_+)dW_+=dW_+2\alpha^2Z(k^2+2)p_-/k^4(e^{2\pi\alpha Z/p_+}-1). \tag{D$_3'$}$$

For the quadripole we have again to consider only the slow particle formed in an s state. It suffices to give the result for the positron the fast particle:

$$\sigma(W_+)dW_+=dW_+2\alpha^2Z(3k^2+8)p_+^3/3k^6(1-e^{-2\pi\alpha Z/p_-}). \tag{Q$_3$}$$

The case in which the electron is the fast particle is to be obtained as before.

V. Structure and Discussion of the Formulae

6. Consistency of the approximation methods

In the foregoing we have given the energy distribution for the electric dipole and quadripole in three different regions of approximation:

$$D_1 \text{ and } Q_1 \text{ for: } \alpha Z/p_+,\ \alpha Z/p_-\ll 1,$$
$$D_2 \text{ and } Q_2 \text{ for: } p_+,\ p_-,\ \alpha Z\ll 1,$$
$$D_3 \text{ and } Q_3 \text{ for: } \alpha Z/p_+,\ p_-,\ \alpha Z\ll 1.$$

That these approximation methods be consistent with each other it is necessary that the following limit equations be fulfilled:

$$\lim_{p_+,\,p_-\to 0} D_1=\lim_{\alpha Z/p_+,\,\alpha Z/p_-\to 0} D_2,\quad \lim_{p_-\to 0} D_1=\lim_{\alpha Z/p_-\to 0} D_3,\quad \lim_{\alpha Z/p_+\to 0} D_2=\lim_{p_+\to 0} D_3 \tag{18}$$

and the corresponding equations with Q_1, Q_2 and Q_3. Reference to the results given for $\sigma(W_+)$ in II, III and IV shows that all six limit equations are satisfied and therefore, that there are no non-uniformities in the approximation scheme.

It is of interest to consider in this connection whether a similar situation exists in the case of the problem of pair formation by plane γ-radiation. The result for the differential cross section using the Born approximation as given by Bethe and Heitler[15] becomes (in our units) when p_+ and p_- are made small:

$$\sigma(W_+)dW_+=dW_+(\alpha^3Z^2/6)p_+p_-(p_+^2+p_-^2). \tag{19}$$

The Schrödinger approximation has been considered by Oppenheimer and Plesset[2] using parabolic coordinates and taking only the big components in the wave functions. The result is:[16]

$$\sigma(W_+)dW_+=dW_+\frac{\pi^2\alpha^5Z^4}{3}\frac{p_+^2+p_-^2+2\alpha^2Z^2}{(e^{2\pi\alpha Z/p_+}-1)(1-e^{-2\pi\alpha Z/p_-})}. \tag{20}$$

[15] Reference 2, p. 90, Eq. (21). The + sign in front of the third line in this equation has to be changed to a − sign.

[16] A correction by a factor two has been made here (see Fermi and Uhlenbeck, Phys. Rev. **44**, 510 (1933)). The same result (20) follows, using polar coordinates and considering again only the big components for the s–p and p–s transitions, which give the largest contribution.

When $\alpha Z/p_+$ and $\alpha Z/p_-$ are small, (20) reduces to one-half of (19). However, we believe (20) is not correct, since, using parabolic coordinates one can show that the small components as well as the next approximation to the large ones give contributions of the same order of magnitude. A result different from (20) has been given by Nishina, Tomonaga and Sakata:[2]

$$\sigma(W_+)dW_+ = dW_+ \frac{\pi^2\alpha^5 Z^4}{8} \frac{(16/3)(p_+^2+p_-^2)+\alpha^2Z^2(4+\pi^2/2)}{(e^{2\pi\alpha Z/p_+}-1)(1-e^{-2\pi\alpha Z/p_-})}. \tag{21}$$

This *does* reduce to (19) if $\alpha Z/p_+$ and $\alpha Z/p_-$ are small. We have not been able to check this formula.[17] We have made a nonrelativistic calculation using the wave functions of Sommerfeld and Maue.[18] They are an expansion of the Dirac wave functions for the Coulomb field in powers of αZ and may be written:

$$\psi = \{f_0 + i(\boldsymbol{\alpha}\cdot\boldsymbol{G})\}u,$$

where, if one uses parabolic coordinates, f_0 is the solution of the scalar Schrödinger equation in these coordinates and the u's are the columns given in (6). The term with $\boldsymbol{G}$ is the first order correction and is proportional to αZ. Using f_0 only one gets the Oppenheimer-Plesset result. However the correction term, even in the nonrelativistic limit, gives a contribution of the same order of magnitude. We find then:

$$\sigma(W_+)dW_+ = dW_+ \frac{2\pi^2\alpha^5 Z^4}{3} \frac{p_+^2+p_-^2+\alpha^2Z^2}{(e^{2\pi\alpha Z/p_+}-1)(1-e^{-2\pi\alpha Z/p_-})}, \tag{22}$$

which has the correct limit.

7. Angular distribution

It is readily seen that for both the electric dipole and quadripole at each energy W_+ the angular distribution $J(\theta, W_+)$ as given by (7) and (9) is a monotonically decreasing function of θ, so that there is always a tendency for the particles to be emitted in the *same* direction. This is most marked at the middle of the energy distribution, i.e., $W_+ = W_- = k/2$ (see Fig. 1). As p_+ or p_- approach zero the emission becomes more and more spherically symmetric. All these properties may be seen in a qualitative manner from the matrix element which governs the angular distribution, that is

$$(\mathbf{p}_-s_-|H|\mathbf{p}_+s_+) \sim \int d\tau \exp(ikr - i(\mathbf{p}_+ + \mathbf{p}_-)\cdot\mathbf{r}). \tag{23}$$

In the exact theory one may expect that the same qualitative features are true with the exception that the strongest parallel emission ($\theta = 0$) will be shifted away from the center toward higher positron energies. The angular distribution will in fact qualitatively depend on a matrix element of the form (23) where now p_+ and p_- are functions of r. For each energy

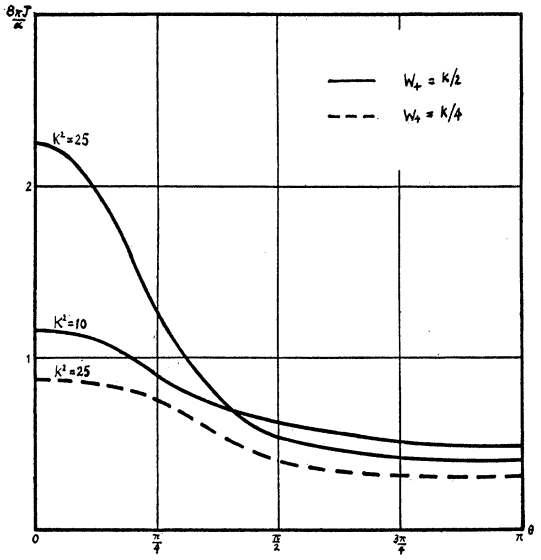

FIG. 1. Angular distribution for the dipole in the Born approximation.

[17] The angular distribution (see their Eq. (1)) does not upon integration over the angles give (21) (=their Eq. (2)).

[18] A Sommerfeld and A. W. Maue, Ann. d. Physik **22**, 629 (1935).

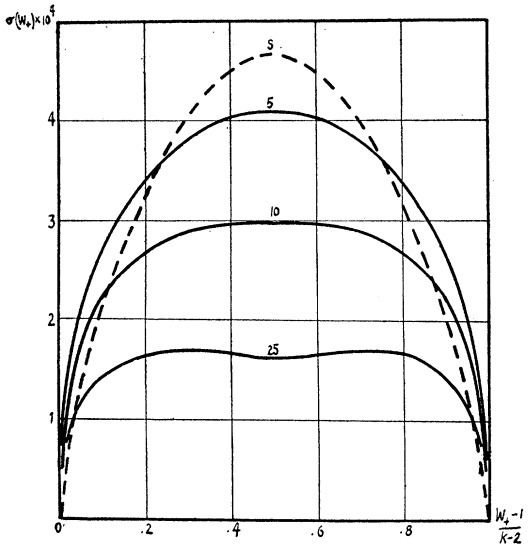

FIG. 2. Energy distribution for the dipole in the Born approximation. The numbers affixed to the curves refer to the value of k. In the statistical distribution S, computed for $k=5$ a constant multiplicative factor has been chosen so that the areas under this curve and under the corresponding Born curve are the same.

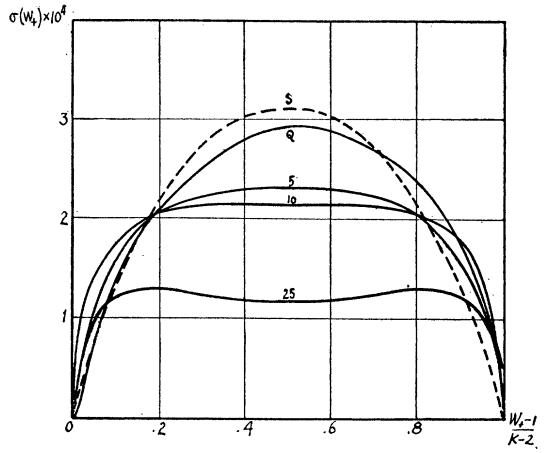

FIG. 3. Energy distribution for the quadripole. The curves marked 5, 10, 25 are the Born approximation distribution for those values of k. The curve Q is the Schrödinger approximation (17) for $k=5$, $Z=6$ while the statistical distribution S, $k=5$ has the same area as Q.

the parallel emission will preponderate over the antiparallel emission ($\theta=\pi$) and this contrast will be greatest when $p_+(r)=p_-(r)$ for then the integrand oscillates the least. Since the main contribution comes from $r\leqq 1$, i.e., the Compton wavelength, then when $p_-(1)=p_+(1)$ the difference of the energies of electron and positron at infinity will be about $2\alpha Z$. That is, the maximum preponderance of the parallel over the antiparallel emission may be expected to occur at a point of the energy distribution which is displaced from the center toward higher positron energies by an amount $2\alpha Z$, i.e., 0.6 MEV for $Z=80$.

For any fixed value of W_+ Eqs. (7) and (9) show that the angular distribution as a function of the energy of the γ-quantum k will become more and more spherically symmetric as k approaches 2. This will also be true in the exact theory, and is in accord with the result mentioned in the Schrödinger approximation.

For increasing k the parallel emission increases while the antiparallel emission decreases so that the maximum at $\theta=0$ becomes more and more sharp. In fact for large k the average value of θ is:

$$\bar{\theta}\sim(\log 2k)^{-\frac{1}{2}}.$$

This holds for both dipole and quadripole fields. Indeed from (7) and (9) it follows that for large k the angular distributions and therefore the energy distributions and the total internal conversion coefficients for both fields become asymptotically equal as may be verified from (D_1), (Q_1) and (8), (10). This of course is to be expected, since the pairs are formed in a region around $r\gtrsim 1$ which for increasing k will lie more and more in the radiation zone of the multipole field. In this radiation zone the field strengths E and H, normalized so as to give unit energy flux, will become the same for all multipoles and therefore the probability for pair formation for k large become the same. The asymptotic forms of (7), (D_1) and (8) will therefore be valid independently of the Born approximation and for all multipole fields.

8. Energy distribution

In the interpretation of all the expressions found for the energy distribution it is well to distinguish between two factors. The first may be called the statistical factor and is simply proportional to the volume in phase space of the particle formed.[19] In our case where the

[19] Cf. G. E. Uhlenbeck and S. Goudsmit, *Zeeman Jubilee Papers*, Nijhoff, The Hague, 1935, p. 201.

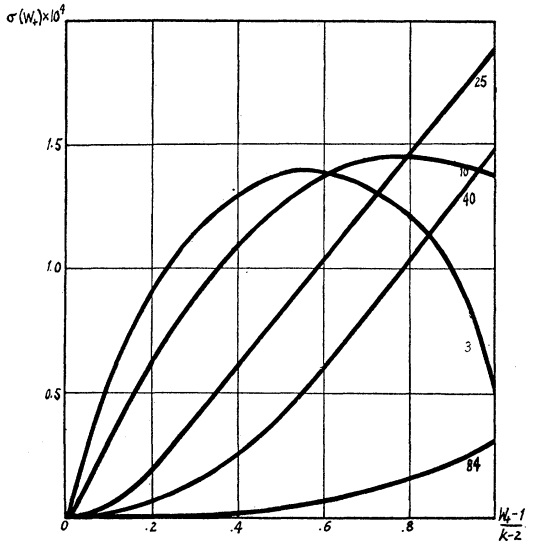

FIG. 4. Energy distribution for the dipole in the Schrödinger approximation for $k=2.2$. The numbers affixed to the curves refer to the value of Z.

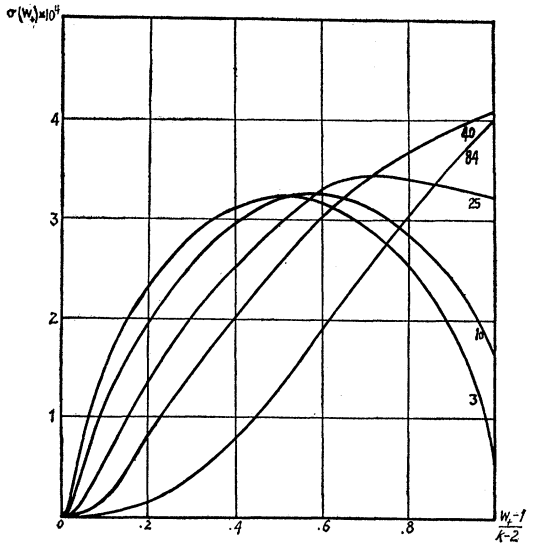

FIG. 5. Energy distribution for the dipole in the Schrödinger approximation for $k=3$ and the values of Z indicated.

energy is distributed between two particles with no conservation of momentum this factor becomes $\sim W_+W_-p_+p_-$. The second factor, which is the more interesting one, is characteristic of the mechanism of the process. We will refer to it as the "mechanism factor." It will depend essentially on the matrix element of the pair-producing field. The division into these two factors is quite evident in (A) for example.

In Figs. 2 and 3 we have given the energy distribution for different values of k for both the dipole and quadripole as follows from the Born approximation formulae D_1 and Q_1.[20] As is to be expected the curves are symmetrical between the two particles; since the nuclear field has been neglected there is physically no difference between the particles. For energies below $k\leqq 5$ the form of the distribution is given almost exactly by the statistical factor, so that the mechanism factor is practically constant. For higher k the distribution becomes flatter than the statistical factor and eventually develops a minimum at the center, just as the distribution in the case of the plane γ-wave.[21] This is probably due to the spin of the particles since in the Pauli-Weiszkopf theory[22] such a minimum does *not* appear.

In Figs. 4 and 5 we have given the energy distribution for two values of k and several values of Z as follows from the Schrödinger approximation formula (16) for the dipole. Because of the influence of the nuclear field, which is contained in the mechanism factor and is expressed by $|N_0^-|^2|N_0^+|^2$ (see Eq. (15)), the distribution is no longer symmetrical. This factor $|N_0^-|^2|N_0^+|^2$ is the product of the densities of the two particles at the nucleus and for large Z or small k essentially determines the form of the distribution. Expressing the repulsion of the positron by the nucleus $|N_0^+|^2$ will be small and will make the distribution function vanish exponentially when p_+ goes to zero. Due to the attraction of the electron by the nucleus, expressed by $|N_0^-|^2$ the distribution will be finite at the other end. As a consequence for large Z or small k the distribution function will be monotonically increasing with W_+. For smaller Z or larger k the curve will develop a maximum which shifts more and more to the center, and approaches the statistical distribution. This is evident from

[20] Among the γ-rays found for light elements there are several of energies around $k=5$, 10 and 25. For example, $_5B^{11}$ (2.4 MEV, $k=4.7$), $_6C^{13}$ (2.7 MEV, $k=5.3$), $_6C^{12}$ (5.5 MEV, $k=10.8$), $_8O^{16}$ (5.4 MEV, $k=10.6$), $_3Li^7$ bombarded with protons ($\sim$12 MEV, $k=24$).

[21] Bethe and Heitler, Reference 2, Fig. 5.

[22] W. Pauli and V. Weiszkopf, Helv. Phys. Acta **7**, 709 (1934).

(16) and (17) when $\alpha Z/p_+$ and $\alpha Z/p_- \ll 1$. It becomes therefore similar to the Born approximation distribution for small k (cf. Fig. 3). For a particular value of k and W_+ the probability as a function of Z will have a maximum which occurs at larger Z the greater W_+. This is due to the interplay of the densities $|N_0^-|^2$ and $|N_0^+|^2$ which are increasing and decreasing functions of Z, respectively.

The behavior of the distribution function near the ends as given by the endpoint formulae in section IV is qualitatively the same as that found from the Schrödinger approximation, since they also contain the density factors. One might hope that especially at the important end of the distribution, i.e., $W_- \cong 1$, (D_3) and (Q_3) would give quantitatively better results. Comparison with the exact theory (see §9) shows that this is *not* the case, (D_3) and (Q_3) giving too large values. This must mean that replacing the positron wave function by a plane wave overestimates its "interference effect" on the probability. At the other end of the distribution, i.e., $W_+ \cong 1$, (D_3') and (Q_3') give the same behavior as the Schrödinger approximation. The strong exponential decrease of the distribution function at this end must be expected to be generally valid.

9. Comparison with the exact theory of Jäger and Hulme

Using the exact Dirac wave functions for the Coulomb field Jäger and Hulme have given the theory for the internal conversion process for both the dipole and the quadripole. They have obtained numerical results for $\sigma(W_+)$ at five points on the energy range in the case $k=3$, $Z=84$. Fig. 6 shows the comparison of these results with the corresponding Schrödinger approximation curves.[23]

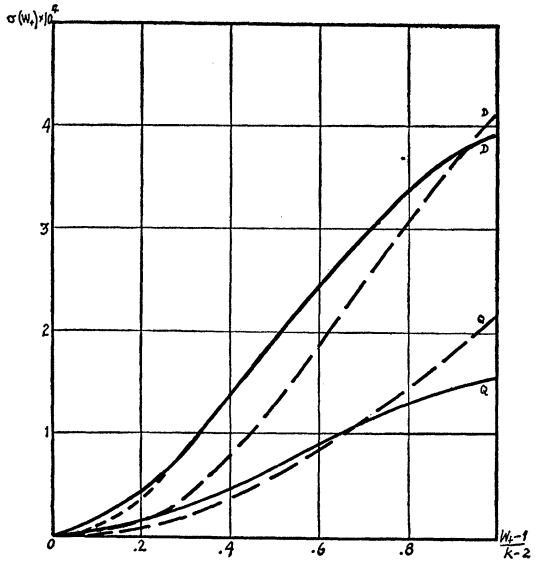

FIG. 6. Energy distribution for $k=3$, $Z=84$, in the exact theory (full lines) and in the Schrödinger approximation (dashed lines). The dipole and quadripole curves are designated by D and Q, respectively.

The agreement is no doubt partly accidental. This becomes clear if one tries to improve the Schrödinger approximation formulae (16) and (17) by removing the restriction $\alpha Z \ll 1$. This can be done by using the exact Dirac wave functions, developing in p_+ and p_- only and considering just those transitions which give the main contribution. These correspond to the $s-s$ (dipole) and $s-p$, $p-s$ transitions (quadripole) in the Schrödinger approximation. The result in case of the dipole for the sum of the matrix elements in (B) is quite analogous to (15). One gets from this sum:

$$|M_-|^2|M_+|^2(\Gamma^2(2\gamma_0)/k^{4\gamma_0}), \qquad (24)$$

where $\gamma_0=(1-\alpha^2Z^2)^{\frac{1}{2}}$ and $|M_-|^2$, $|M_+|^2$ are radial normalization factors:

$$|M_-|^2=\frac{(1+W_-)(2p_-)^{2\gamma_0}e^{\pi\alpha ZW_-/p_-}|\Gamma(\gamma_0+i\alpha ZW_-/p_-)|^2(\gamma_0+1)^2}{4W_-\Gamma^2(2\gamma_0+1)}$$

and $|M_+|^2$ is to be obtained from this by changing the sign of Z and by replacing $-$ by $+$. The difference in the dependence on k between (15) and (24) is due to the fact that in

[23] The lowest value of W_+ which Jäger and Hulme have computed is $W_+=1.25$. In accordance with what has been pointed out concerning the behavior of the distribution function near $W_+=1$, we believe it more plausible to extrapolate their curve towards $W_+=1$ as we have indicated by the dotted curve.

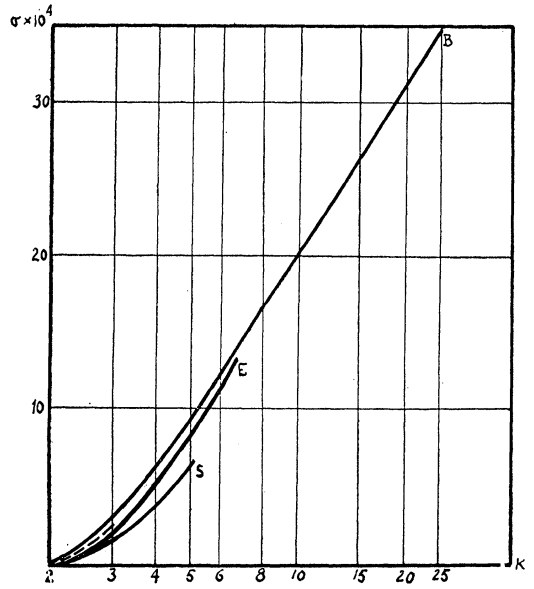

FIG. 7. Total internal conversion coefficient for the dipole. B is the Born approximation curve, E the curve of Jäger and Hulme and S the Schrödinger approximation result all for $Z=84$. The dotted curve is the Schrödinger approximation curve for $Z=10$.

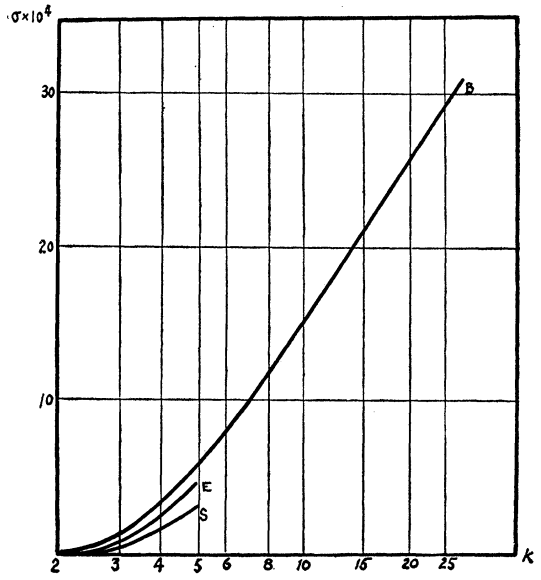

FIG. 8. Total internal conversion coefficient for the quadripole, $Z=84$.

the exact theory the wave functions considered behave like r^{γ_0-1} at the origin. Combining (24) with (B) one obtains an expression for the energy distribution which for $\alpha Z \ll 1$ becomes the same as (16). This might be expected to give better numerical results. Actually this is *not* the case; the αZ arbitrary distribution lies entirely below that obtained from (16). The infinity of the wave functions at the origin, expressed in the k-dependence, does tend to increase the results but this is entirely overcome by the fact that the positron normalization factor $|M_+|^2$ is so much smaller than the non-relativistic $|N_0^+|^2$. Of course it is not surprising that the αZ arbitrary distribution lies completely below the true one of Jäger and Hulme, since one has also to consider other transitions. It appears then that the overestimate of the positron normalization factor $|N_0^+|^2$ in the Schrödinger approximation rather accidentally compensates for the neglect of the other transitions.

One may expect that in a certain range of energy this compensation will always be more or less the case. Since in addition the Schrödinger approximation formulae (16) and (17) have the right qualitative behavior, we believe that for k say up to 5 and for all Z, (16) and (17) will give also quantitative results correct within a fair degree of accuracy. For k larger than 5 and Z small (say ≤ 20) the Born approximation will no doubt give nearly correct results. For large Z however, the Born approximation even for values of k as high as 6 or 7 will probably not give the correct form of the energy distribution, although the total area σ may be given quite accurately.

10. The total internal conversion coefficient

In Figs. 7 and 8 we have given the total internal conversion coefficient as a function of k both for dipole and quadripole. The Born approximation result has been computed by graphical integration from (D_1) and (Q_1).[24] The Schrödinger approximation results have been obtained in the same way from (16) and (17). For k near 2 (16) and (17) can be integrated analytically, giving:

$$\sigma \leqq (3/4)\alpha^2 Z g^3 e^{-2\pi\alpha Z/g},$$

$$\sigma \leqq (5/12)\alpha^4 Z^3 g^3 e^{-2\pi\alpha Z/g},$$

respectively, if $g^2=2(k-2)$. This shows the be-

[24] We have not made use of the asymptotic formulae (8) and (10) since the approach to them is very slow.

havior for k near 2; for large k (8) and (10) show that σ increases like log k.

These results have to be compared with the results of Jäger and Hulme which are also indicated in the figures. They have given curves for the dipole up to $k=7$ and for the quadripole up to $k=5$ in the case $Z=84$. One sees again that by the combined use of the Born and Schrödinger approximation results the total internal conversion coefficient may be obtained for all k and all Z to within a fair degree of accuracy, say of the order 15 percent.

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The Spectrum of Molybdenum V

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The spectrum of Mo V has been excited in a vacuum spark and photographed with a vacuum spectrograph. With the aid of the irregular doublet law applied to the Sr I-like isoelectronic sequence, lines involving combinations of $4d^2\,{}^3(FP)$, $4d5s\,{}^{1,3}D$, $4d5d\,{}^1(PD)$, and $4d5d\,{}^3(DF)$ with $4d5p\,{}^{3,1}(F°D°P°)$ and $4d4f\,{}^3(G°F°)$, and lying in the spectral region from 2100A to 400A have been identified. Estimates of the absolute term values have been made from a Moseley diagram.

THE spectra of the first four elements in the Sr I-like isoelectronic sequence have been classified by various investigators; Sr I by Russell and Saunders and by others,[1] Y II by Meggers and Russell,[2] Zr III by Kiess and Lang,[3] and Cb IV by Gibbs and White[4] and, more fully, by Lang.[5] The present paper extends the sequence to include Mo V.

The spectrum of Mo V was excited in a vacuum spark between solid metal electrodes and photographed with a vacuum spectrograph containing a concave grating of 150-cm radius of curvature and ruled with 15,000 lines per inch. The vacuum spark between aluminum electrodes furnished the standard lines, either those of aluminum itself or those of nitrogen and oxygen brought out in the discharge.

Higher order lines were carefully noted. The lines of Mo IV and Mo VI which lie in the region investigated (2100A to 400A) were sorted out on the basis of the classifications by Eliason[6] and by the author.[7]

The identification of the multiplet transitions $4d5p\,{}^3(F°D°P°)$ into $4d^2\,{}^3(FP)$ and $4d5s\,{}^3D$ furnished the key to the classification of the triplet lines. First, the triplet transition $4d5s\,{}^3D - 4d5p\,{}^3(F°D°P°)$ was looked for with the help of a linear extrapolation of the wave numbers of the corresponding lines for the preceding members of the sequence. Fig. 1 is a diagram showing the relative positions of the lines in these multiplets. In this diagram, only the stronger lines, in general, are connected from one element to the next.

Since there was a large number of lines in the region of the spectrum occupied by the lines of the $4d5s\,{}^3D - 4d5p\,{}^3(F°D°P°)$ multiplets, it was only by finding lines lying in the region of $4d^2\,{}^3F - 4d5p\,{}^3(F°D°P°)$ (as indicated by the linear extrapolation of wave numbers) and having the same wave number separations as the lines of the $4d5s - 4d5p$ transition that a solution of the problem was obtained. Difficulty in identifying these latter lines was experienced because

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[1] Saunders, Astrophys. J. **56**, 73 (1922); Russell and Saunders, Astrophys. J. **61**, 39 (1925).

[2] Meggers and Russell, Bur. Standards J. Research **2**, 733 (1929).

[3] Kiess and Lang, Bur. Standards J. Research **5**, 305 (1930)

[4] Gibbs and White, Phys. Rev. **31**, 520 (1928).

[5] Lang, Phys. Rev. **44**, 325 (1933). Also, private communication to Professor R. C. Gibbs.

[6] Eliason, Phys. Rev. **43**, 745 (1933).

[7] Trawick, Phys. Rev. **46**, 63 (1934).