Radiative Capture of Protons by Carbon

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In previous notes[1] we have reported results of calculations which show that the probability of γ-ray radiation is sufficiently high to make the reaction $C^{12}+H^1\rightarrow N^{13}+\gamma$ a probable one. It turned out that the theoretical yield is larger than that observed by a factor of roughly 1000. As in all intensity calculations the overlapping of wave functions in the initial and final states is of primary importance and the theoretical result is sensitive to this overlapping. The apparent discrepancy between calculation and experiment may be taken to indicate that the interaction between C^{12} and H^1 is not represented sufficiently well by a potential function which is constant inside C^{12} and is Coulombian outside of it. In order to decrease the theoretical yield one has to change the potential in such a way as to decrease the region in which the initial and final wave functions overlap or else one must shift the region towards smaller radii. The γ-ray yield expected in the bombardment of Be^9 by protons is calculated for a nuclear radius of 0.318×10^{-12} cm and a well 9 MEV deep. It increases more slowly with the voltage than the observed intensity indicating a deeper nuclear well. A simple discussion of the method of complex eigenwerte is applied to the properties of wave functions near resonance.

1. General Relations for Capture in a Central Field

A PARTICLE of mass M_1 and charge Z_1e traveling with velocity v is incident on another particle of mass M_2 and charge Z_2e supposed to be initially at rest. The particles repel each other according to the inverse square law of force when they are separated by distances $r>r_0$. Inside r_0 the force is supposed to remain central but to change into an attractive one. The system is transformed as usual into a separable system in which the coordinates of the common center of mass and the relative coordinates of the two particles can be treated independently of each other. The kinetic energy of the center of mass is constant and the factor of the wave function which involves the relative coordinates satisfies a Schrödinger equation for a single particle with mass

$$\mu=M_1M_2/(M_1+M_2). \tag{1}$$

The kinetic energy of relative motion is $(\mu/M)T$ where $T=(\frac{1}{2})M_1v^2$ is the kinetic energy of the incident particle. The plane wave which represents v particles Z_1, M_1 per second per cm^2 traveling along the Z axis is described in the reference system of the center of mass by e^{ikz} where

$$k=\mu v/\hbar \tag{2}$$

and x, y, z are the relative coordinates of particle 1 with respect to particle 2.

If the field were Coulombian also inside $r=r_0$ the plane wave would be modified by the field into[2]

$$e^{ikz}\rightarrow\sum_0^\infty(2L+1)i^LP_L(\cos\theta)e^{i\sigma_L}F_L(\rho)/\rho;\quad \rho=kr,$$
$$\sigma_L=\arg\Gamma(L+1+i\eta);\quad \eta=1/ka=Z_1Z_2c/\text{``}137\text{''}v. \tag{3}$$

Here F_L is the solution of

$$[d^2/d\rho^2+1-2\eta/\rho-L(L+1)/\rho^2]F_L=0, \tag{4}$$

which is regular at $\rho=0$ normalized so as to be asymptotic to a sine wave of unit amplitude at $\rho=\infty$. The deviation of the field from the inverse square law modifies (3) into

$$e^{ikz}\rightarrow\sum_0^\infty(2L+1)i^LP_L(\cos\theta)e^{i\sigma_L}\bar{F}_L(\rho)/\rho, \tag{5}$$

where $\bar{F}_L$ is the solution of the radial equation in the changed field for the azimuthal number L normalized to be asympototic to a sine function of unit amplitude at $\rho\rightarrow\infty$ and regular at $\rho=0$. The radial equation is (4) for $r>r_0$ and

$$[d^2/d\rho^2+1+(2/\mu v^2)U-L(L+1)/\rho^2]\bar{F}_L=0;\quad r<r_0, \tag{4'}$$

where U is the potential energy inside the "potential well."

[1] G. Breit and F. L. Yost, Phys. Rev. **46**, 1110 (1934); **47**, 508 (1935).

[2] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, 1933), Chapter III.

The incident wave in accordance with Eq. (5) consists of a superposition of states with different orbital angular momenta L. The transition probability from any of these states to a lower stable state can be calculated by the formula

$$\frac{64\pi^4\nu^3}{3hc^3}\sum_n\{|\mathfrak{M}_{mn}{}^x|^2+|\mathfrak{M}_{mn}{}^y|^2+|\mathfrak{M}_{mn}{}^z|^2\},\quad (6)$$

where $\mathfrak{M}$ is the dipole moment due to the relative displacement of the two particles and m, n are, respectively, the magnetic quantum numbers of the incident and bound state. Eq. (6) is well known for two stationary states and the matrix elements entering it are supposed to be calculated by using wave functions normalized to unity. Introducing a fundamental volume V one has to divide expression (5) by $V^{\frac{1}{2}}$ and using that expression in (6) one obtains the transition probability when the incident state corresponds to v/V particles/sec./cm^2. The number of captures which occur per second due to a single nucleus exposed to the bombardment of N particles/sec./cm^2 will be defined as $N\sigma$ and σ will be called the collision cross section. Using (6) in the above manner one obtains therefore $v\sigma/V$. This amounts to using (5) directly in the calculation of matrix elements and equating the result to $v\sigma$. Due to any term with a definite L in (5) it is possible to obtain a transition to a lower stable term having azimuthal quantum number $L+1$ or $L-1$. We let the radial wave function of the stable state be R and we normalize it so as to have

$$\int_0^\infty R^2r^2dr=1. \quad (7)$$

Performing the angular integrations in (6) one finds

$$\sigma_{L,\,L\pm1}=\frac{256\pi^5e^2\nu^3}{3hc^3v}(L+\tfrac{1}{2}\pm\tfrac{1}{2})A^2\cdot k^{-2}|\int\bar{F}_LRr^2dr|^2 \quad (8)$$

$$\left.\begin{aligned}&=30.3(c/v)A^2k^{-2}\lambda^{-3}|\int\bar{F}_LRr^2dr|^2(L+\tfrac{1}{2}\pm\tfrac{1}{2}),\\ A&=(M_2Z_1-M_1Z_2)/(M_1+M_2).\end{aligned}\right\}\quad (8')$$

It is often desirable to express $\bar{F}_L$ inside the barrier as a multiple of a function $\bar{u}$ having unit amplitude at $r=r_0$ and proportional to $\bar{F}_L$. One can determine the value of $\bar{F}_L$ at $r=r_0$ in terms of the logarithmic derivative of $\bar{F}_L$ at $r=r_0-0$ and the values of F_L and G_L at $r=r_0$. Here G_L is the solution of Eq. (4) which is asymptotic to $\cos[\rho-L\pi/2-\eta\ln 2\rho+\sigma_L]$ while F_L is asymptotic to $\sin[\rho-L\pi/2-\eta\ln 2\rho+\sigma_L]$. Requiring that for $r>r_0$, $\bar{F}_L$ be expressible as a linear combination of F_L and G_L and that for large ρ the function $\bar{F}_L$ should consist of F_L and a term in $e^{i\rho}$ one satisfies the condition of having (5) equal to a sum of the modified plane wave (3) and a set of diverging waves. It is found by an easy calculation that

$$\bar{F}_L=(F_L/(1-F_LG_L\delta_L-iF_L{}^2\delta_L))_{r=r_0}\cdot\bar{u},\quad (9)$$

$$\delta_L=(F_L'/F_L-\bar{F}_L'/\bar{F}_L)_{r=r_0},\quad (9')$$

where the last differentiations are with respect to ρ. The connection with usual formulas of the theory of anomalous scattering is given by the asymptotic form of $\bar{F}_L$:

$$\bar{F}_L\sim e^{iK_L}\sin[\rho-L\pi/2-\eta\ln 2\rho+\sigma_L+K_L],\quad (10)$$

where K_L is determined by

$$e^{2iK_L}=[(1-F_LG_L\delta_L+iF_L{}^2\delta_L)/(1-F_LG_L\delta_L-iF_L{}^2\delta_L)]_{r=r_0}.\quad (10')$$

The phase K_L is the usual phase shift which is responsible for anomalous scattering.[3] Formula (9) determines the amplitude of $\bar{F}_L$. The main contributions to σ come usually from the region inside the nuclear radius r_0 and a comparable region $r_0<r<3r_0$. Inside $r<r_0$ the shape of $\bar{u}$ is determined by (4') and for $r>r_0$ the function $\bar{u}$ can be usually determined sufficiently accurately by approximate methods. The quotient $\bar{F}_L/\bar{u}$ requires the knowledge of F_L and G_L. It is possible to express F_L in the following way:[4]

[3] H. M. Taylor, Proc. Roy. Soc. **A134**, 103 (1931); **A136**, 605 (1932).

[4] Th. Sexl, Zeits. f. Physik **81**, 163 (1933); J. A. Wheeler, F. L. Yost and G. Breit, in preparation for publication.

$$F_L = C_L \rho^{L+1} \Phi_L;$$

$$C_L = \frac{2^L}{(2L+1)!} \{2\pi\eta[L^2+\eta^2][(L-1)^2+\eta^2]\cdots[1+\eta^2]/[e^{2\pi\eta}-1]\}^{\frac{1}{2}},$$

$$G_L = D_L\{[\rho^{-L}+a_{-L+1}\rho^{-L+1}+\cdots+0\cdot\rho^{L+1}+a_{L+2}\rho^{L+2}+\cdots]+[p\ln 2\rho+q]F_L/C_L\},$$

$$(2L+1)C_L D_L = 1; \quad p = \frac{2^{2L+1}}{(2L)!(2L+1)!}[L^2+\eta^2][(L-1)^2+\eta^2]\cdots[1+\eta^2]\eta, \tag{11}$$

$$q = p\left[\frac{L}{L^2+\eta^2}+\cdots+\frac{1}{1+\eta^2}-\left(1+\frac{1}{2}+\cdots+\frac{1}{2L+1}\right)+2\gamma+\text{R.P.}\frac{\Gamma'(-i\eta)}{\Gamma(-i\eta)}\right]$$

$$+(-)^{L+1}\frac{2^L}{(2L)!}\sum_{n=-L}^{L}\frac{2^n}{(L+n)!(L-n+1)}\text{I.P.}\ (i\eta+n-1)\cdots(i\eta-L).$$

Here Φ_L is a power series in ρ the first term of which is 1 and γ is Euler's constant. The above form of F, G was worked out by Wheeler and the tables the computation of which was begun by him were used for the values of Φ_L and the second power series entering G_L. Introducing

$$\Psi = rRk^{-\frac{1}{2}}, \quad \int \Psi^2 d\rho = 1,$$

we have $$\sigma = B^2\sigma_0,$$

$$\sigma_0 \doteq 30.3(c/v)A^2k^{-5}\lambda^{-3}F_L^2$$
$$\times\left|\int_0^\infty \rho\Psi\bar{u}d\rho\right|^2(L+\tfrac{1}{2}\pm\tfrac{1}{2}), \tag{12}$$

$$B^2 = \{1/[(1-F_LG_L\delta_L)^2+F_L^4\delta_L^2]\}_{r=r_0}. \tag{12'}$$

If $F_LG_L\delta_L=1$ there is resonance of the nucleus to the L component of the incident wave. In this case $F_L^2B^2=G_L^2(kr_0)$, which is large for high barriers on account of the presence of $e^{2\pi\eta}-1$ in D_L. The functions F_L, G_L satisfy $F_L'G_L-F_LG_L'=1$. Using this relation the condition for resonance can be expressed as

$$\bar{F}_L'/\bar{F}_L = G_L'/G_L, \tag{13}$$

which shows together with $B^2=G_L^2(kr_0)$ that resonance is obtained when the function $\bar{F}_L$ has the proper phase to join itself on to the irregular solution G_L at $r=r_0$.

Although it is convenient to define the resonance velocity arbitrarily as that velocity of incident particles for which $F_LG_L\delta_L=1$ it should be remembered that neither B^2 nor σ, considered as functions of the velocity, have maxima precisely when this condition is satisfied. Thus the maximum of $B^2F_L^2$ should be reached at slightly lower velocities. For sharp resonance however the condition used is sufficiently accurate.

If resonance is sharp then the observed yield in thick targets is primarily due to the captures of particles retarded by the stopping power of the target to approximately the resonance velocity. The yield can be then calculated approximately by integrating the contributions which result from (12) in the neighborhood of resonance. The quantity $1-F_LG_L\delta_L$ can be then supposed to vary linearly with the energy in the energy range giving important contributions around resonance. Changes in all other quantities entering σ will be neglected in this approximation, and their values at resonance will be used. According to the 3/2 power law of variation of range with the kinetic energy T the distance dl laid off by the particle in the target while the energy changes by dT is $dl=(3/2)ldT/T$. By means of this relation and (12) it is found that the number of captures per N_1 incident particles bombarding a target having N_2 nuclei per cm^3 is

$$Y = (3\pi/2)(\Delta T/T)N_1N_2l_0\sigma_0/F_L^4\delta_L^2, \tag{14}$$

where l_0 is the range of particles having the resonance velocity and ΔT is one-half of the half value breadth of resonance so that the resonance curve is represented by $1/[(\Delta T)^2+(T-T_0)^2]$. For high barriers the dependence of Y is most critical with respect to $\Delta T/T$ and the factors $1/F_L^4\delta_L^2$ as well as the F_L^2 involved in σ_0. The two last factors together give a net factor G_L^2 so that Y is proportional to $G_L^2(\Delta T/T)$. Here G_L^2 increases with the barrier height but $\Delta T/T$ decreases and the net effect will be seen to leave Y nearly independent of the barrier height.

2. Relations Between Wave Functions Near Resonance

It was shown by Casimir[5] that Gamow's use of solutions exponentially decaying with time and exponentially increasing with distance can be interpreted as an approximation to a straightforward calculation by means of wave packets and that the actual solution satisfies the necessary conditions of finiteness at infinite distances. The same fact together with relations between wave functions near resonance was previously reported by one of us.[6] In view of the fact that the method used allows one to see the connection between capture and spontaneous disintegration most clearly it is described below.

We consider solutions of the radial equation

$$d^2\Psi/dr^2+(2m/\hbar^2)(E-V)\Psi=0 \qquad (15)$$

and we suppose that at infinity the function V decreases with r faster than $1/r$. This restriction is of a purely mathematical character because at sufficiently large distances the Coulombian field can be changed into a more rapidly decreasing one without changing the physical results. We shall also require that the function V is such as to allow regular solutions for Ψ. Otherwise the function V can be arbitrary. The energy E is supposed to be positive. Asymptotically for large r we have

$$\Psi(E)=Ae^{-ikr}+Be^{+ikr}, \quad k=(2mE)^{\frac{1}{2}}/\hbar. \qquad (16)$$

If Ψ is defined in the neighborhood of $r=0$ by the value of its first nonvanishing derivative C the differential Eq. (15) establishes its values for all r and hence the coefficients A, B are determined as functions $A(C, E)$, $B(C, E)$. Physical applications should be made with real E. It is nevertheless useful to examine the functions A, B also for complex E. Gamow's complex eigenwerte method simply amounts to making use of the properties of A, B as functions of E in the neighborhood of the root of

$$A(C, E)=0. \qquad (16')$$

A root of this equation we call $E_0-i\lambda\hbar/2$ where E_0, λ are real. The constant λ has no relation to the wavelength which it stood for in the first section and will be interpreted as the disintegration constant. For real C, E the function Ψ must be real and hence

$$B=A^* \quad (E \text{ real}). \qquad (16'')$$

The roots of (16′) are therefore complex, and $\lambda\neq 0$. As in Gamow's paper one shows that for small $|\lambda|$ only $\lambda>0$ need be considered. For real E close to E_0 and for small λ one may expand by Taylor's series around $E_0-i\lambda\hbar/2$

$$\begin{aligned} A(E)&=(E-E_0+i\lambda\hbar/2)(dA/dE)_{A=0}; \\ B(E)&=(E-E_0-i\lambda\hbar/2)(dA/dE)^*_{A=0}. \end{aligned} \qquad (17)$$

By using these values in (16) it follows that for large r

$$\overline{|\Psi|^2}=2|(dA/dE)_{E=0}|^2[(E-E_0)^2+(\lambda\hbar/2)^2]. \qquad (17')$$

For small r the function Ψ varies only slowly with E and this is particularly so if there is a "well" at small r in the potential V. Thus according to (17′) the relative probability of the particle being in the region of small r, i.e., inside the nucleus, is proportional to

$$1/[(E-E_0)^2+(\lambda\hbar/2)^2]. \qquad (18)$$

This shows that half value breadth is $\lambda\hbar$ while the quantity ΔT of the first section is $\lambda\hbar/2$.

Superposing solutions (16) one obtains a wave packet

$$\Psi=\int_0^\infty\Psi(E)e^{-iEt/\hbar}f(E)dE \qquad (19)$$

and one can arrange $f(E)$ so as to have Ψ initially vanish at $r=\infty$. Approximately this is accomplished by

$$f(E)=(\lambda\hbar/2\pi)/[(E-E_0)^2+(\lambda\hbar/2)^2]. \qquad (19')$$

The calculation of the function (19) by using (16), (19′) is easily made approximately by using $K-K_0\cong(E-E_0)/\hbar v_0$ where v_0 is the velocity at

[5] H. Casimir, Physica **1**, 193 (1934); cf. also H. Bethe, Ann. d. Physik **4**, 443 (1930).
[6] G. Breit, Phys. Rev. **40**, 127 (1932).

resonance and by changing the range of integration from $0 \to \infty$ into $-\infty \to +\infty$. The integration reduces then to

$$\int_{-\infty}^{+\infty} e^{-ixt'/\hbar}[x-i\lambda\hbar/2]^{-1}dx = 0(t'>0); \quad = 2\pi i e^{\lambda t'/2}(t'<0)$$

or

$$\int_{-\infty}^{+\infty} e^{-ixt'/\hbar}[x+i\lambda\hbar/2]^{-1}dx = -2\pi i e^{-\lambda t'/2}(t'>0); \quad = 0(t'<0).$$

It is found that in the region where (16) holds

$$\Psi = B(E_0 - i\lambda\hbar/2)\exp[-i(E_0t - mv_0r)/\hbar - \lambda(t - r/v_0)/2] \quad (t>r/v_0), \quad (20)$$

$$\Psi = 0 \quad (t<r/v_0),$$

where

$$B(E_0 - i\lambda\hbar/2) = -i\lambda\hbar(dA/dE)^*_{A=0} \quad (20')$$

in accordance with the fact that (17) applies for complex as well as real values of E. Similarly for small r one finds, using (19′) and the fact that $\Psi(E)$ is practically independent of E in the region where $f(E)$ is appreciable:

$$\Psi = \Psi(E_0)\exp[-iE_0t/\hbar - \lambda t/2]. \quad (20'')$$

Eqs. (20), (20″) show that to within the approximations made Gamow's solution represents the wave packet in the region $r<v_0t$ but that for $r>v_0t$ the wave packet gives zero probability of finding the particle. In higher approximations the head of the wave packet at $r=v_0t$ becomes rounded off and for $r>v_0t$ there is a finite though small probability of finding the particle.

Eq. (18) shows that one may define E_0 as that real energy for which the probability of the particle being inside the nucleus is a maximum taken relatively to the probability of its being very far away. Eqs. (17), (19), (19′) show that the wave packet which is represented by Gamow's solution contains stationary states with different energies and that the relative probabilities of these energies vary in proportion with the same resonance factor (18) which determines the relative probabilities of capture for particles of these energies; in comparing relative probabilities of particles of different energies we use the same amplitude at $r=\infty$ for all energies.

According to Gamow's conservation theorem argument one can determine the disintegration constant by

$$\lambda = v|B(E_0 - i\lambda\hbar/2)|^2 / \left(\int|\Psi(E_0)|^2dr\right)_{E=E_0-i\lambda\hbar/2}, \quad (21)$$

where the integral is to be taken only through the nucleus. Here the numerator represents the number of particles leaving the nucleus and the denominator similarly represents the number of particles in the nucleus. This formula of Gamow's is often inconvenient in applications because it presupposes the knowledge of the solution of the wave equation for complex energies. Using Eq. (17) we have

$$B(E_0 - i\lambda\hbar/2) = 2B(E_0). \quad (21')$$

Substituting this into Eq. (21) we see that

$$\lambda = v/\int|\bar{G}|^2dr, \quad (21'')$$

where $\bar{G}$ is the solution of the wave equation for a *real* energy normalized so as to be asymptotic to a sine function of unit amplitude at ∞. Remembering that $\Delta T = \lambda\hbar/2$ we also have

$$\Delta T/T = 1/\int|\bar{G}|^2d\rho; \quad (\rho = mvr/\hbar). \quad (22)$$

The above general properties of wave functions near resonance are seen to be verified by the special functions used in section 1. Thus according to Eq. (17) the phase of the function at resonance differs by $\pi/2$ from what it is far away from resonance and this is in agreement with Eq. (10).

3. Applications

By means of Eq. (22) we change Eq. (14) into

$$Y = 142.8N_1N_2l_0(L+\tfrac{1}{2}\pm\tfrac{1}{2})A^2(c/v)k^{-5}\lambda^{-3} \times |\int_0^\infty \rho\Psi\bar{G}d\rho|^2/\int\bar{G}^2d\rho, \quad (23)$$

where λ again stands for the wavelength. For $r>r_0$, $\bar{G}$ is identical with G_L and for $r<r_0$ it is the continuation of G_L in the field U. As has been shown in connection with Eq. (13) this continuation is regular at $r=r_0$ at resonance. The integral in the denominator is to be extended only

through the nucleus. The function $\overline{G}$ is the only quantity in this formula which contains the energy of the incident particle exponentially. It occurs with equal powers in the numerator and denominator of Y and as a result Y depends on the energy much less critically than the disintegration yield away from resonance. The $\Delta T/T$ is proportional in its order of magnitude to $e^{-2\pi\eta}$ while $\sigma_0/F_L{}^4\delta_L{}^2$ is the collision cross section at resonance and it behaves as $F_L{}^{-2}$ and is of the order $e^{2\pi\eta}$. At resonance the collision cross section increases in the same ratio as the width of the resonance region decreases. This property is seen to be general and not confined to radiative captures. We may, therefore, expect observable yields of nuclear reactions in cases of resonance even though the exponential factors describing the probability of penetrating a nuclear barrier should be very small.

The expected probability of radiative capture was calculated for carbon with several values of the nuclear radius and of the depth of the potential well. Only a certain range of values comes into consideration because mass defects indicate that the energy released in the capture of a proton by C^{12} is of the order of 6 or 7 MEV. It is therefore necessary to have the radius and the depth such as to allow an s level at $E \cong -6$ MEV. At the same time it is necessary to account for resonance to incident protons at about 0.5 MEV. Supposing that the process occurs as a transition $L=1 \rightarrow L=0$, only certain radii and depths will give simultaneously the correct position of the bound s level and of the virtual p level. For simplicity we have supposed that the $1s$ and $2p$ levels are responsible for the transition so as to have no nodes inside the nucleus for the wave function, either in the initial or in the final state. Similarly the potential energy $-U$ inside the nucleus was taken to be constant. This corresponds to the usual treatment of radioactive nuclei and although actually U may vary, an approximate idea about the probability of the process is obtainable with such a model. Letting $r_0 \times 10^{12} = 0.33$, 0.41, 0.52 cm we found by trial that the corresponding values of U which had to be used in order to obtain resonance to the first p level at 564 kv are 21.4, 13.4, 8.3 MEV. The energy of the s level calculated with these values of r_0 and the corresponding values of U is approximately -10, -6.5, -4.0 MEV and it is seen that $r_0 = 0.41 \times 10^{-12}$ cm, $U = 13.4$ MEV fit the requirement of correct binding energy best.

For any of these radii and depths resonance is sharp and the direct calculation of $1 - FG\delta$ and $F^2\delta$ in the vicinity of resonance is troublesome. Eq. (23) is more suitable. Since it was supposed in deriving it that Eq. (22) is at least approximately valid we have verified the latter by comparison with approximate direct calculations in which the values of F, G, were used. For the latter calculations $1 - FG\delta$ and $F^2\delta$ were computed for $T = 224$, 356, 564, 894 kv and the rate of change of $1 - FG\delta$ at 564 kv was compared with $F^2\delta$. The quantity $F^2\delta$ varies sufficiently rapidly to make its change through the resonance region appreciable but not very serious and we used the value of $F^2\delta$ at maximum resonance in estimating the resonance width. We obtain for $\Delta T/T$ by Eq. (22) 1.63×10^{-2} and by direct estimate 1.44×10^{-2} for $r_0 = 0.31 \times 10^{-12}$ cm. For $r_0 = 0.41 \times 10^{-12}$ cm the agreement is worse, Eq. (22) giving 2.5×10^{-2} and direct estimate 2.0×10^{-2}. It could be expected that for broader resonance Eq. (22) will be less accurate. In both cases the agreement is sufficiently good to warrant the application of Eq. (22) to the calculation of the yield.

Formulas (14), (23) give the yield expected in thick targets on the supposition that only particles with nearly the resonance velocity need be considered. This assumption is justified for the theoretical model used here as may be seen by comparing the collision cross section at resonance with that at other voltages. Thus for $U = 13.4$ MEV, $r_0 = 0.41 \times 10^{-12}$ cm the approximate values of σ are given below:

$T = 224$ kv	356 kv	564 kv	894 kv
$\sigma = 6.1 \times 10^{-31}$	1.8×10^{-29}	5.0×10^{-26}	9.8×10^{-29} cm^2.

At the resonant voltage the effective collision cross section is 4.2×10^{-26} cm^2 and is appreciably higher than the other values listed above. In the above tabulation changes in $\int r^2 R\bar{u} dr$ with voltage were neglected, being relatively unimportant because $\bar{u}$ is primarily determined by the depth and width of the potential well rather than the kinetic energy of the incident proton.

The values of F, G, $FG\delta$, $F^2\delta$ and $\int \rho \Psi \bar{u} d\rho$ for different r_0 and U are listed in Table I for different voltages. The latter as well as the values

TABLE I. *γ-ray yield per proton as a function of energy of incident particle and radius of bombarded nucleus.*

T	F'/F	F	G	$FG\delta$	$F^2\delta$	$\int \rho\Psi\bar{u}d\rho$	Y per proton
	$r_0=0.33\times10^{-12}$ cm			$U=21.4$ MEV,			
894	3.525	.02664	6.727	1.036	0.00410		
564	4.494	.00962	14.43	.994	.000663	0.267	7.3×10^{-7}
356	5.699	.002876	37.82	.971	.000074		
224	7.212	.0006735	127.0	.956			
	$r_0=0.41\times10^{-12}$ cm			$U=13.4$ MEV			
894	2.837	.04465	5.03	1.073	0.010		
564	3.640	.01626	10.48	1.002	.00155	0.371	2.66×10^{-7}
356	4.640	.00488	27.12	.967	.000182		
224	5.880	.001147	90.40	.942			
	$r_0=0.52\times10^{-12}$; $U=8.3$ MEV						
894	2.281	.0754	3.81	1.135	0.0225		
564	2.952	.02778	7.56	1.006	.00369	0.525	0.85×10^{-7}
356	3.784	.00840	19.18	.944	.00041		
224	4.818	.00198	63.1	.907			

of r_0 were chosen so as to be able to use the tables for F and G without interpolation. The values of Y also listed in Table I were obtained with $l_0=6.96\times10^{-4}$ cm for 564 kv while 1.136×10^{23} was used as the number of carbon atoms per cm^3. The values of $\Delta T/T$ used for Y were 1.036×10^{-2} for $r_0=0.33\times10^{-12}$ cm, 1.44×10^{-2} for $r_0=0.41\times10^{-12}$ cm, 1.90×10^{-2} for $r_0=0.52\times10^{-12}$ cm. The values of Y for these three radii correspond to 4.6×10^6, 1.7×10^6, 0.54×10^6 captures per microampere. The decrease of Y with increase of nuclear radius is due to a large extent to an increase in the wavelength of the emitted γ-ray. The energy of the s level increases as r_0 and so does the wavelength of the γ-ray. The wavelengths used for the three radii were 1.21×10^{-11} cm, 1.89×10^{-11} cm, 3.07×10^{-11} cm.

According to Hafstad and Tuve[7] and to Crane and Lauritsen[8] the observed yield is due to resonance which takes place at about 425 kv. The observations of Cockcroft, Gilbert and Walton[9] are consistent with the existence of resonance but speak for a somewhat higher value of the resonance energy. The difference between the 564 kv used in the calculations and the true experimental energy is not certain and cannot be very significant because according to formula (23) the yield does not vary critically with the barrier. This is also indicated by the fact that our calculated yields decrease with the radius.

The observed yield as estimated by Hafstad and Tuve is one in 10^{10} incident protons. Among the radii which we tried $r_0=0.41\times10^{-12}$ cm seems to be the most acceptable because the corresponding binding energy corresponds most closely to the expected 6 or 7 MEV. The theoretical yield for this radius is 2700 times greater than the observed yield and it is thus quite possible to explain the formation of radioactive N^{13} as due to radiative capture of the proton. The discrepancy between theory and experiment indicates that the model used is only a poor approximation to reality and this could have been expected because both the assumption that the field acting on the proton is central and that this field has the particular shape used by us are open to question. As in all intensity calculations the transition probability is proportional to the square of the matrix element for the displacement and is therefore sensitive to the degree of overlapping of the initial and final state. By increasing U at small r one can contract the wave function of the bound s state towards small r without affecting the p function of the incident state nearly as much. Almost any desired result can be obtained by this means and the theoretical value can be decreased to the experimental one. It is questionable, however, whether such forcing of theory to experiment has much meaning because the assumption of the central nature of the field is also questionable.

According to Massey and Mohr[10] who follow a discussion of Taylor and Mott[11] one may expect dipole radiation to be absent on the Heisenberg Majorana theory of nuclear binding forces. According to Bethe and Peierls,[12] however, dipole radiation is expected on either the Heisenberg Majorana or the Wigner theory. It is clear without calculation that a dipole moment exists in our special problem and a more detailed calculation which we omit here shows that the exchange interaction does not essentially modify the possibility of dipole radiation.

The revised masses of Bethe[13] give 1.0032 for the difference between the masses of C^{13} and C^{12} instead of the previously supposed 1.0003 which

[7] L. R. Hafstad and M. A. Tuve, Phys. Rev. **47**, 506, 507 (1935).

[8] C. C. Lauritsen and H. R. Crane, private communication.

[9] J. D. Cockcroft, C. W. Gilbert and E. T. S. Walton, Proc. Roy. Soc. **A148**, 225 (1935).

[10] H. S. W. Massey and C. B. O. Mohr, Proc. Roy. Soc. **A148**, 225 (1935).

[11] H. M. Taylor and N. F. Mott, Proc. Roy. Soc. **A138**, 665 (1932).

[12] H. Bethe and R. Peierls, Proc. Roy. Soc. **A148**, 146 (1935).

[13] H. Bethe, Phys. Rev. **47**, 633 (1935).

followed from Aston's value 12.0036 for C^{12} and King and Birge's value 13.0039 for C^{13}. This change when used for an estimate of the mass of N^{13} lowers the estimated frequency of the γ-ray to about 3 MEV with 0.0014 for the kinetic energy of the positrons in $N^{13} \rightarrow C^{13} + e^+$ and neglecting the mass of the neutrino. The nuclear radius has to be increased and the depth of the well must be decreased to fit this γ-ray wavelength. The estimated yield is decreased by approximately a factor of 3.

According to Hafstad and Tuve there is a strong indication of a fine structure of the resonance of carbon to protons. If this were due to the proton magnetic moment we would expect the fine structure to have the same order of magnitude as though the proton obeyed Dirac's equation. The discontinuity in the potential energy will be expected to give rise to a doublet splitting of the order of

$$\Delta W = \int_{R-0}^{R+0} H' G^2 dr$$

where $H' = (\mu_0/1840)^2(2L+1)dU/rdr$ which amounts to $(2L+1)(v^2/4c^2)G^2U/\int_0^\infty G^2 d\rho$. Multiplying the result by 3 in order to take into account the fact that the magnetic moment of the proton is greater than would be expected on Dirac's theory one still has an energy difference of the order of only 700 volts which is much less than the observed structure of about 70 kv. The structure may however be accounted for by the coupling between movable parts of the C^{12} nucleus or else by the simultaneous presence of two or more "wells" for the incident proton. As is well known, the interaction of two equal wells in molecules gives rise to a doublet structure and a similar phenomenon may be imagined to take place here.

Application to $Be^9 + H^1$

Applying the above formulas to the γ-rays emitted in the bombardment of Be^9 by protons we obtain the following values of $(c/v)(F^2/k^2)/(1-FG\delta)^2$ for $r_0 = 0.318 \times 10^{-12}$ cm and $U = 9$ MEV:

$T=$	99.8	158.2	250.7	397.3	629.7	998.0	1582 kv
$(c/v)(F^2/k^2)/(1-FG\delta)^2=$	2.3×10^{-29}	2.1×10^{-28}	1.21×10^{-27}	4.8×10^{-27}	1.43×10^{-26}	3.5×10^{-26}	7.6×10^{-26} cm^2.
$\bar{\sigma}/\sigma=$			.28	.33	.38	.43	

The above numbers are nearly proportional to σ because the part of $\bar{u}$ which matters is in the "well" and is nearly independent of T. Plotting the logs of above values against $T^{-\frac{1}{2}}$ one obtains nearly a straight line. Approximating the graph by a series of straight lines one determines for each T the coefficient b for the approximation $\sigma = e^{-b/\sqrt{T}}$. Using the 3/2 power law and this approximation we obtain by integration the ratio $\bar{\sigma}/\sigma$ where $\bar{\sigma}$ is the effective cross section for thick targets defined by $\bar{\sigma}\cdot x = \int_0^x \sigma(x)dx$ where x is the distance inside the target measured from the face. The approximate values of $\bar{\sigma}/\sigma$ are also tabulated above. The values of

$$S = (c/v)F^2k^{-2}(1-FG\delta)^{-2}(\bar{\sigma}/\sigma)(T/397.3)^{\frac{3}{2}}$$

should be proportional to the observed yield and are compared below with the observed γ-ray yield on an arbitrary scale. The observations are due to Tuve and Hafstad and were kindly made available to us before publication. The

T	600	700	800	900	920	1000 kv
$10^{25}S$	.9	1.7	2.8	4.3	—	6.0
Experimental ray yield	.3	.5	1.2	5.0	6.7	—

experimental yield increases much faster from 700 to 900 kv than the theoretical. This suggests a deeper well and perhaps the approach of resonance around 1 MEV which, however, has not been found so far.