# A Nuclear Model

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The problem of a number of bodies which attract each other with forces proportional to the distance can be solved completely in terms of normal vibrations, and the expressions for the frequencies can be written down. From these the quantum mechanical energy levels can be written down directly. This makes a good model for the representation of a nucleus insofar as it can be considered as composed of alpha-particles, protons and neutrons, and can be treated by ordinary quantum mechanics. The assumption of constants of a reasonable order of magnitude makes possible the computation of binding energies which show the observed general trends. The instability of $Be^8$ composed of two alpha-particles is attributed to the high "zero point" energy of the system, although the system of three alpha-particles, $C^{12}$, is stable. The Pauli exclusion principle applied to the neutrons gives a reason for the limited number of observed isotopes.

THE discovery of the neutron as a nuclear constituent suggested the possibility of ignoring altogether the difficulties connected with the presence of electrons in nuclei. Thus Heisenberg has regarded nuclei as built up of protons and neutrons. Among these, alpha-particles may be regarded as particularly stable combinations, which may exist in nuclei in a way somewhat similar to that in which atoms exist in molecules. The great advantage of this point of view is that these particles can be treated by quantum mechanics even in the small volume attributed to the nucleus. Heisenberg[1] treated the energy of such a nucleus by a statistical method and was able to show some general properties of such assemblies. However, there is some advantage in having a nuclear model whose properties can be more precisely determined. Such a model can be constructed by assuming that the various nuclear particles attract each other with forces proportional to their separation.[2] This is probably a very crude representation of the true forces but because it is susceptible of exact treatment it can be used as a zero approximation to which modifications can be applied by perturbation methods.

## 1. Classical Treatment of the Problem

For simplicity the treatment will be given for three different kinds of particles, and because of the application to nuclei these will be designated as alpha-particles, protons and neutrons. The extension to more different kinds of particles is immediate and obvious.

Consider a system composed of $a$ alpha-particles, $p$ protons and $n$ neutrons. Let the coordinates of the alpha-particles be $\xi_r$, $\eta_r$, $\zeta_r$, where $0 \leqslant r \leqslant a$, of the protons be $x_r$, $y_r$, $z_r$, where $0 \leqslant r \leqslant p$, and of the neutrons be $u_r$, $v_r$, $w_r$, where $0 \leqslant r \leqslant n$. Let the mutual potential energy of two particles be as follows:

$$
\begin{aligned}
&\text{Two alpha-particles} && V_{aa} = C_{aa}\{(\xi_r-\xi_s)^2+(\eta_r-\eta_s)^2+(\zeta_r-\zeta_s)^2\} - D_{aa}\\
&\text{Alpha-particle and proton} && V_{ap} = C_{ap}\{(\xi_r-x_s)^2+(\eta_r-y_s)^2+(\zeta_r-z_s)^2\} - D_{ap}\\
&\text{Alpha-particle and neutron} && V_{an} = C_{an}\{(\xi_r-u_s)^2+(\eta_r-v_s)^2+(\zeta_r-w_s)^2\} - D_{pn}\\
&\text{Two protons} && V_{pp} = C_{pp}\{(x_r-x_s)^2+(y_r-y_s)^2+(z_r-z_s)^2\} - D_{pp}\\
&\text{Proton and neutron} && V_{pn} = C_{pn}\{(x_r-u_s)^2+(y_r-v_s)^2+(z_r-w_s)^2\} - D_{pn}\\
&\text{Two neutrons} && V_{nn} = C_{nn}\{(u_r-u_s)^2+(v_r-v_s)^2+(w_r-w_s)^2\} - D_{nn}.
\end{aligned}
\tag{1}
$$

Let $\mu$ be the mass of the alpha-particle, $M$ that of the proton and $m$ that of the neutron.

[1] W. Heisenberg, Zeits. f. Physik **80**, 587 (1933).
[2] This model was mentioned by Fowler, Nature **128**, 453 (1931) but he seems not to have worked it out in any detail.

The problem is already separated along the three coordinate axes so it is only necessary to consider one axis. The differential equations of motion then have the form

$$\mu\ddot{\xi}_r = -2C_{aa}\sum_{s=1}^{a}(\xi_r-\xi_s)-2C_{ap}\sum_{s=1}^{p}(\xi_r-x_s)-2C_{an}\sum_{s=1}^{r}(\xi_r-u_s),$$

$$M\ddot{x}_r = -2C_{ap}\sum_{s=1}^{a}(x_r-\xi_s)-2C_{pp}\sum_{s=1}^{p}(x_r-x_s)-2C_{pn}\sum_{s=1}^{n}(x_r-u_s), \tag{2}$$

$$m\ddot{u}_r = -2C_{an}\sum_{s=1}^{a}(u_r-\xi_s)-2C_{np}\sum_{s=1}^{p}(u_r-x_s)-2C_{nn}\sum_{s=1}^{n}(u_r-u_s).$$

To find the normal vibrations make the usual substitutions, $\xi_r=E_re^{i\omega t}$, $x_r=F_re^{i\omega t}$, $u_r=G_re^{i\omega t}$. This leads to $N$ simultaneous algebraic equations for $E$, $F$ and $G$. The condition for their compatibility is that the determinant of the coefficients be equal to zero, which gives an equation of the $N$th degree in $\omega^2$. This equation can be handled by adding and subtracting rows and columns of the determinant. The first two rows are

$$\begin{matrix}\{2(a-1)C_{aa}+2pC_{ap}+2nC_{an}-\mu\omega^2\} & -2C_{aa} & \cdots -2C_{ap}\cdots -2C_{an}\\ -2C_{aa} & \{2(a-1)C_{aa}+2pC_{ap}+2nC_{an}-\mu\omega^2\} & \cdots -2C_{ap}\cdots -2C_{an}.\end{matrix}$$

If the second row is subtracted from the first and then the first column is added to the second, one factor of the determinant is isolated in the upper left-hand corner. This gives as one root of the equation

$$\omega^2=(1/\mu)\{2aC_{aa}+2pC_{ap}+2nC_{an}\}. \tag{3}$$

This process can be carried out $(a-1)$ times to give $(a-1)$ roots identical with (3). There then comes a transition between the rows referring to alpha-particles and those referring to protons. If this row is skipped the above process of subtracting rows and adding columns can be continued to give $(p-1)$ roots of the equation whose values are

$$\omega^2=(1/M)\{2aC_{ap}+2pC_{pp}+2nC_{pn}\}. \tag{3a}$$

By skipping the next transition there can be obtained $(n-1)$ roots,

$$\omega^2=(1/m)\{2aC_{an}+2pC_{pn}+2nC_{nn}\}. \tag{3b}$$

After this process has been carried as far as possible there remains a determinant with as many rows and columns as there are different kinds of particles. One root of this is always $\omega^2=0$, and the rest must be obtained by solving the equation. For the special case considered here

$$\begin{vmatrix}(2pC_{ap}+2nC_{an}-\mu\omega^2) & -2pC_{ap} & -2nC_{an}\\ -2aC_{ap} & (2aC_{ap}+2nC_{pn}-M\omega^2) & -2nC_{pn}\\ -2aC_{an} & -2pC_{pn} & (2aC_{an}+2pC_{pn}-m\omega^2)\end{vmatrix}=0. \tag{3c}$$

By adding all three columns together the root $\omega^2=0$ can be isolated and the result is a quadratic equation whose solutions are

$$\omega^2=\left(\frac{a}{M}+\frac{p}{\mu}\right)C_{ap}+\left(\frac{a}{m}+\frac{n}{\mu}\right)C_{an}+\left(\frac{p}{m}+\frac{n}{M}\right)C_{pn}\pm\Bigg\{\left(\frac{a}{M}+\frac{p}{\mu}\right)^2C_{ap}^2$$
$$+\left(\frac{a}{m}+\frac{n}{\mu}\right)^2C_{an}^2+\left(\frac{p}{m}+\frac{n}{M}\right)C_{pn}^2-2\left[\left(\frac{a}{M}+\frac{p}{\mu}\right)\left(\frac{a}{m}+\frac{n}{\mu}\right)-2\frac{np}{\mu^2}\right]C_{ap}C_{an}$$
$$-2\left[\left(\frac{a}{M}+\frac{p}{\mu}\right)\left(\frac{p}{m}+\frac{n}{M}\right)-2\frac{an}{M^2}\right]C_{ap}C_{pn}-2\left[\left(\frac{a}{m}+\frac{n}{\mu}\right)\left(\frac{p}{m}+\frac{n}{M}\right)-2\frac{ap}{m^2}\right]C_{an}C_{pn}\Bigg\}^{\frac{1}{2}}. \tag{3d}$$

These roots give the frequencies with which the various normal coordinates vibrate. The expressions for the normal coordinates in terms of the position coordinates can be readily obtained but since they are not necessary in the determination of the quantum-mechanical energy levels, they need not be given here. The coordinate corresponding to the frequency given by $\omega^2=0$ is just the center of mass of the system.

## 2. Quantum-Mechanical Energy Levels

Since each of the normal vibrations exists along each of the three coordinate axes, the state of minimum energy will be obtained by multiplying each frequency by $3h/2$. If any of the vibrations are excited the energy can be expressed in terms of a quantum number for each different frequency. Hence the energy for a system of three kinds of particles is

$$E=(a-1+2m_1/3)(3h/4\pi)\{(2aC_{aa}+2pC_{ap}+2nC_{an})/\mu\}^{\frac{1}{2}}+(p-1+2m_2/3)(3h/4\pi)$$
$$\times\{(2aC_{ap}+2pC_{pp}+2nC_{pn})/M\}^{\frac{1}{2}}+(n-1+2m_3/3)(3h/4\pi)\{(2aC_{an}+2pC_{pn}+2nC_{nn})/m\}^{\frac{1}{2}}$$
$$+(1+2m_4/3)(3h\omega_1/4\pi)+(1+2m_5/3)(3h\omega_2/4\pi)-a(a-1)D_{aa}/2-apD_{ap}-anD_{an}$$
$$-p(p-1)D_{pp}/2-pnD_{pn}-n(n-1)D_{nn}/2. \quad (4)$$

$\omega_1$ and $\omega_2$ are from Eq. (3d). The ground state is obtained by putting all the quantum numbers, $m_1\cdots m_5$ equal to zero. There is clearly a great deal of degeneracy to this problem since each $m_i$, except $m_4$ and $m_5$, is the sum of the quantum numbers belonging to all the normal vibrations with the given frequency. Some of this would be removed by the application of perturbations correcting the law of force between the particles and many of the finer details of nuclear behavior are perhaps to be understood in this way.

## 3. Application to Nuclei

It is certain from the fact of the stability of nuclei that there are effectively attractive forces between some of the nuclear constituents. Attractive forces given by potential energies of the form treated above are certainly very crude approximations but it still seems that they can be used to describe some of the characteristic features of nuclear behavior.

### Nuclei containing alpha-particles only

It has often been assumed that nuclei of even atomic number contain all of their protons combined in the form of alpha-particles. This point of view has recently been emphasized by Landé.[3] Some information on the force between two alpha-particles is available from scattering experiments. The analysis of these by Taylor[4] indicates that the force is repulsive and coulomb at distances greater than about $3\times10^{-13}$ cm; below this the force is attractive and such that at $r=0$ the potential energy has a large negative value. In accord with these results we may assume that the energy is given by (1) from $r=0$ to such a point that it becomes equal to the coulomb repulsive potential. From there on it may be considered as coulomb. The values of $C_{aa}$ and $D_{aa}$ can then be chosen so as to fit the known energies as well as possible. If the coulomb part of the field is entirely neglected, the energy can be written down at once.

For a system containing alpha-particles only Eq. (4) becomes

$$E=(a-1+2m_1/3)(3h/4\pi)(2aC_{aa}/\mu)^{\frac{1}{2}}$$
$$-a(a-1)D_{aa}/2. \quad (4a)$$

When $a$ becomes large this energy decreases with $a^2$. This is faster than the observed decrease but it is possible to select the constants so that the general trend is represented. In order to give attention to the light nuclei we may use $C^{12}$ to determine these constants. The binding energy is known, and if the excited level at about $8.0\times10^{-6}$ erg is taken as due to $m_1=1$, the constants

[3] A. Landé, Phys. Rev. **43**, 620 (1933).
[4] H. M. Taylor, Proc. Roy. Soc. **A136**, 605 (1932).

can be determined. In round numbers these are

$$C_{aa}=64\times10^{18}\text{ erg/cm},\qquad D_{aa}=9\times10^{-6}\text{ erg}.$$

The binding energies computed on this basis, together with the observed values, are given in Table I.

According to this model the observed instability of $Be^8$ is due to the large "zero point" energy in the ground state. The lifetime in this state, estimated according to Gamow's formula, is very small, $\sim10^{-19}$ sec. The observed energies of carbon and oxygen are given very nicely, and the calculated value of the first excited level in $O^{16}$ is $2/(3)^{\frac{1}{2}}$ as high above the ground level as in $C^{12}$. For the heavier atoms it is clear that the kinetic energy of the model does not increase as rapidly as necessary to overcome the rapid decrease of the potential energy. It is possible to select the constants so as to give better agreement for the heavier nuclei, but this disturbs the fit with the lighter. The first effect of including the outer part of the field would be to decrease the total energy, but it would certainly be small in the case of the stable systems.

According to Eq. (4a) the capture of an alpha-particle always represents a decrease in energy. This is in agreement with the observed fact that, in the nuclei containing alpha-particles only, the mass defect increases steadily with the atomic number.

### Nuclei containing alpha-particles and neutrons

In applying the model to these cases it is simplest to include the assumption that the neutrons do not influence each other. Thus $C_{nn}=D_{nn}=0$. The energy for this case is given by

$$\begin{aligned}E=&(a-1+2m_1/3)(3h/4\pi)\{(2aC_{aa}+2nC_{an})/\mu\}^{\frac{1}{2}}\\&+(n-1+2m_3/3)(3h/4\pi)(2aC_{an}/m)^{\frac{1}{2}}\\&+(1+2m_4/3)(3h/4\pi)\{(a/m+n/\mu)2C_{an}\}^{\frac{1}{2}}\\&-a(a-1)D_{aa}/2-anD_{an}.\end{aligned}\qquad(4b)$$

The values of $C_{an}$ and $D_{an}$ can then be chosen to approximate the observed nuclear energies. The values

$$C_{an}=64\times10^{18}\text{ ergs/cm}^2\text{ and }D_{an}=11.8\times10^{-6}\text{ erg}$$

serve fairly well. They give $He^5$ composed of one alpha-particle and one neutron a positive energy of $3.6\times10^{-6}$ erg. It would thus be nonexistent since there is probably almost no barrier against the escape of a neutron if there is sufficient energy. The energies of some other cases are shown in Table I. $Be^9$ has just about zero energy while $Be^{10}$ is more stable than $Be^9$ but less so than $B^{10}$. Eq. (4b) shows that the energy will in general decrease with an increase in the number of neutrons and that the rate of this decrease will increase with the number of alpha-particles. It is also necessary, however, to take account of the exclusion principle for neutrons. If it is assumed that the wave functions must be antisymmetrical in the coordinates of the neutrons, the quantum number $m_3$ cannot be equal to zero except for the cases of one or two neutrons. For three or four it has the minimum value one, etc. This effect is shown in the energy of $C^{15}$ which is much less stable than $C^{14}$, and $O^{19}$ which is less stable than

TABLE I. *Computed and observed nuclear energies.* The energies are expressed in units of $10^{-6}$ ergs, and the numbers in parentheses are the observed nuclear energies.

*Nuclei of even atomic number composed of alpha-particles and neutrons*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $He^4$ | 0.00 | $He^5$ | +3.6 | | | | |
| $Be^8$ | +0.73 | $Be^9$ | −0.04 (0.0) | $Be^{10}$ | −1.23 | $Be^{11}$ | +6.8 |
| $C^{12}$ | −3.16 (−4.3) | $C^{13}$ | −10.2 (−17.3) | $C^{12}$ | −17.4 | $C^{15}$ | −9.3 |
| $O^{16}$ | −12.8 (−12.9) | $O^{17}$ | −26.7 (−24.8) | $O^{18}$ | −41.5 (−36.7) | $O^{19}$ | −37.9 |
| $Ne^{20}$ | −28.5 (−21.0) | $Ne^{21}$ | −50.0 | $Ne^{22}$ | −72.2 (−47.8) | $Ne^{23}$ | −74.1 |
| $Mg^{24}$ | −50.7 | $Mg^{25}$ | −80.4 | $Mg^{26}$ | −110.5 | $Mg^{27}$ | −118.6 |
| $Si^{28}$ | −79.8 (−49.6) | $Si^{29}$ | −117.8 | $Si^{30}$ | −156.5 | $Si^{31}$ | −170.9 |

*Nuclei composed of protons and neutrons*

| | | | | | |
|---|---|---|---|---|---|
| $H^2$ | −3.34 (−3.17) | $H^3$ | −7.8 (−8.1) | $H^4$ | −4.0 |
| $He^3$ | −7.8 (−6.8) | $He^4$ | −32.4 (−43.5) | $He^5$ | −22.8 |

*Nuclei of odd atomic number composed of alpha-particles, one proton and neutrons*

| | | | | | |
|---|---|---|---|---|---|
| $Li^6$ | −5.6 (−5.1) | $Li^7$ | −17.0 (−16.7) | $Li^8$ | −13.2 |
| $B^{10}$ | −15.9 (−9.8) | $B^{11}$ | −32.4 (−25.3) | $B^{12}$ | −33.5 |
| $N^{14}$ | −33.2 (−21.1) | $N^{15}$ | −57.1 (−40.8) | $N^{16}$ | −63.0 |
| $F^{18}$ | −57.6 | $F^{19}$ | −83.5 (−48.0) | $F^{20}$ | −97.4 |
| $Na^{22}$ | −89.2 | $Na^{23}$ | −128.9 | $Na^{24}$ | −146.6 |
| $Al^{26}$ | −128.3 | $Al^{27}$ | −176.2 | $Al^{28}$ | −200. |

*Values of the constants used in compiling this table*

| | | | |
|---|---|---|---|
| $C_{aa}=64\times10^{18}$ | $C_{an}=64\times10^{18}$ | $C_{ap}=64\times10^{18}$ | $C_{pn}=64\times10^{18}$ |
| $D_{aa}=9\times10^{-6}$ | $D_{an}=11.8\times10^{-6}$ | $D_{ap}=11.8\times10^{-6}$ | $D_{pn}=22.7\times10^{-6}$ |
| $C_{nn}=0$ | $C_{pp}=0$ | $D_{nn}=0$ | $D_{pp}=0$ |

$O^{18}$. For the heavier nuclei the constants used do not actually show a definite instability but this property of the model does show why the number of isotopes is not unlimited, and indicates why the maximum number of neutrons is dependent on the number of alpha-particles.

**Nuclei containing alpha-particles, protons and neutrons**

For the nuclei of odd atomic number the whole Eq. (4) must be used. The constants listed in Table I seem to give the general trend correctly. According to these a proton and a neutron act in the same way with an alpha-particle. Also $H^2$ and $He^3$ show the same binding energy. This is, of course, not correct but it is not a bad approximation. For these odd elements the number of neutrons is limited by their turning into protons and forming alpha-particles. Thus $B^{12}$ is less stable than $C^{12}$, when the energy of the alpha-particle is considered.

## 4. Discussion of the Results

It is to be emphasized that no great significance is to be attached to the exact numerical results nor to the exact values of the constants used. It is surely not correct to use the same force constant between all kinds of particles, but the determination of the differences would hardly be justified by the accuracy of the model. The constants are of a reasonable order of magnitude and the results show the correct general trends, although the actual values of the energy deviate widely for the higher atomic numbers because the constants have been selected to fit the lighter nuclei, as well as because of the crudeness of the model.

The assumption of a strong force between protons and neutrons and no force between protons gives, as indicated by Wigner,[5] the right general trend in $H^2$, $H^3$, $He^3$, $He^4$. Together with the exclusion principle it also makes $He^5$ less stable than $He^4$. The amount is different from that obtained by treating $He^5$ as an alpha-particle and a neutron, but this difference is not surprising in view of the rough determination of the constants. The increase in mass defect with the increase in the number of alpha-particles and with the increase in the number of neutrons is in accord with the observations. The application of the exclusion principle to the neutrons leads to a limitation on the number of neutrons which can be added with a decrease in total energy.

Two outstanding characteristics of nuclei which are not apparent in the model are the alpha-particle radioactivity of some heavy nuclei, and the large number of close-lying excited energy levels which are observed. Heisenberg attributes the radioactivity to the repulsive force between protons. This exists also between alpha-particles. Since this has not been included here it is not surprising that the instability does not appear. On the other hand the excited levels of the model are required to be rather far apart because of the values of the constants, and it seems difficult to reduce sufficiently the constants and at the same time preserve the necessary binding energy relationships. Nevertheless a relatively small change in the form of the force might remove enough of the degeneracy to provide the necessary number of close excited levels.

In spite of the extreme simplification this model is free from the objections to the consideration of one nuclear particle moving in an average field due to all the rest, and gives a description of the trends of the binding energies

[5] E. Wigner, Phys. Rev. **43**, 252 (1933).