The Interaction Between a Neutron and a Proton and the Structure of H^3

L. H. THOMAS, *Mendenhall Laboratory, Ohio State University*
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Suppose that the interaction between a neutron and a proton depends on their distance apart so as to be negligible above a certain small distance a, and yet is responsible for the mass defect of H^2. Suppose further that the interaction between two neutrons and a proton may be compounded in the usual way from that between a neutron and a proton, the interaction between two neutrons being neglected, while there is no prohibition of a wave function symmetrical in the positions of two neutrons. Then it is shown that the mass defect of H^3 is made arbitrarily large by taking a small enough. The observed mass defect of H^3 thus provides, on the above assumptions, a lower limit for a; and in particular rules out the possibility that the interaction may be regarded as arising from a singularity in configuration space. We conclude, in effect, that: either two neutrons repel one another by an amount not negligible compared with the attraction between a neutron and a proton; or that the wave function cannot be symmetrical in their positions; or else that the interaction between a neutron and a proton is not confined within a relative distance very small compared with 10^{-13} cm[1].

THE EQUATION ASSUMED FOR A NEUTRON AND A PROTON

SUPPOSE that the positional part $\psi(\mathbf{r}_1, \mathbf{r}_2)$ of a wave function for a neutron (at $\mathbf{r}_2$) and a proton (at $\mathbf{r}_1$) satisfies the equation

$$-(h^2/8\pi^2m)\{\nabla_1^2\psi+\nabla_2^2\psi\}+V(r_{12})\psi-E\psi=0, \tag{1}$$

where the effective potential energy $V(r_{12})$, depending only on the distance apart, r_{12}, of the two particles, is such that

$$V(r)<0, \quad r<a; \qquad V(r)=0, \quad r\geq a. \tag{2}$$

Suppose that Eq. (1) has a solution of the form

$$\psi=\varphi(r_{12}), \qquad E=E_0<0, \tag{3}$$

such that (a)

$$4\pi\int_0^\infty \varphi(r)^2r^2dr=1, \tag{4}$$

so that

$$\varphi(r)=Ae^{-\lambda r}/r, \tag{5}$$

where

$$(h^2/4\pi^2m)\lambda^2=-E_0, \tag{6}$$

and

$$4\pi A^2e^{-2\lambda a}/2\lambda<1; \tag{7}$$

and (b)

$$4\pi\int_0^a(d\varphi/dr)^2r^2dr<K/a. \tag{8}$$

In the above, h is to be taken to be the usual Planck's constant, m the proton mass or the neutron mass, and E_0 the energy of the normal state of a deuteron.

Assumption (b) is probably not necessary for the result that will be obtained, but is necessary for the proof that will be given. It is true if $V(r)$ and $\varphi(r)$ are everywhere finite, and remains true, K approaching a limit, if we suppose that, E_0 remaining unaltered, a tends to zero while $V(r)$ approaches a definite form,

i.e.,

$$V(r)\sim a^{-2}f(r/a). \tag{9}$$

The modifications required in the proof when the potential energy $V(r)$ is replaced by a singularity of ψ in the configuration space as $r\rightarrow 0$ will be considered later.

[1] Such a lower limit to a cannot be deduced from the observations of neutron-proton scattering until these are known more accurately. (See E. Wigner, Zeits. f. Physik **83**, 253 (1933).)

The Equation Assumed for Two Neutrons and a Proton

Suppose that the positional part $\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ of a wave function for two neutrons (at $\mathbf{r}_1$ and $\mathbf{r}_2$) and a proton (at $\mathbf{r}_3$), symmetrical in the positions of the two neutrons, satisfies the equation

$$-(h^2/8\pi^2 m)\{\nabla_1^2\psi+\nabla_2^2\psi+\nabla_3^2\psi\}+V(r_{13})\psi+V(r_{23})\psi-E\psi=0, \tag{10}$$

where $V(r_{13})$ and $V(r_{23})$ represent the effective potential energies of interaction of the neutrons and the proton, depending only on their distances apart r_{13} and r_{23}, and are of the form 2 above: the interaction between two neutrons is neglected, but it cannot affect the result that will be obtained unless it gives rise to a repulsion of the same order of magnitude as the attraction between a neutron and a proton.

If $\psi=\psi(\mathbf{s}_1, \mathbf{s}_2)$, where

$$\mathbf{s}_1=\mathbf{r}_1-\mathbf{r}_3, \qquad \mathbf{s}_2=\mathbf{r}_2-\mathbf{r}_3 \tag{11}$$

(10) becomes

$$-(h^2/4\pi^2 m)\{\nabla_1^2\psi+(\nabla_1\cdot\nabla_2)\psi+\nabla_2^2\psi\}+V(s_1)\psi+V(s_2)\psi-E\psi=0 \tag{12}$$

and the algebraically least value E_1 of E, if it exists, for which a continuous differentiable function $\psi(\mathbf{s}_1, \mathbf{s}_2)$ can be found to satisfy (12), must obey the inequality

$$E_1<\frac{\int\int[(h^2/4\pi^2 m)\{(\nabla_1\psi)^2+(\nabla_1\psi\cdot\nabla_2\psi)+(\nabla_2\psi)^2\}+V(s_1)\psi^2+V(s_2)\psi^2]dv_1dv_2}{\int\int\psi^2dv_1dv_2}, \tag{13}$$

where ψ is any continuous function of $\mathbf{s}_1$ and $\mathbf{s}_2$, differentiable almost everywhere, for which the integrals exist, dv_1 and dv_2 being volume elements in $\mathbf{s}_1$ and $\mathbf{s}_2$ space.

The Choice of a Function

Put $s^2=(4/3)(\mathbf{s}_1^2-(\mathbf{s}_1\cdot\mathbf{s}_2)+\mathbf{s}_2^2)$,

$$\tan\theta_1=3^{\frac{1}{2}}s_1/|\mathbf{s}_1-2\mathbf{s}_2|, \qquad \tan\theta_2=3^{\frac{1}{2}}s_2/|\mathbf{s}_2-2\mathbf{s}_1|. \tag{14}$$

Then

$$\psi=\frac{1}{s^2}K_0(\mu s)\left[\frac{\pi/2-\theta_1}{\sin\theta_1\cos\theta_1}+\frac{\pi/2-\theta_2}{\sin\theta_2\cos\theta_2}\right] \tag{15}$$

satisfies the equation

$$\nabla_1^2\psi+(\nabla_1\cdot\nabla_2)\psi+\nabla_2^2\psi=\mu^2\psi, \tag{16}$$

where K_0 is Macdonald's Bessel function of order zero.[2]

For if

$$\mathbf{s}_1=\mathbf{u}, \qquad \mathbf{s}_1+2\mathbf{s}_2=3^{\frac{1}{2}}\mathbf{v}, \tag{17}$$

then

$$\nabla_1^2\psi+(\nabla_1\cdot\nabla_2)\psi+\nabla_2^2\psi\equiv\nabla_u^2\psi+\nabla_v^2\psi,$$

and if

$$u=s\sin\theta_1, \qquad v=s\cos\theta_1$$

then

$$\frac{\partial^2\psi}{\partial u^2}+\frac{2}{u}\frac{\partial\psi}{\partial u}+\frac{\partial^2\psi}{\partial v^2}+\frac{2}{v}\frac{\partial\psi}{\partial v}\equiv\frac{\partial^2\psi}{\partial s^2}+\frac{5}{s}\frac{\partial\psi}{\partial s}+2\frac{(\cot\theta_1-\tan\theta_1)}{s^2}\frac{\partial\psi}{\partial\theta_1}+\frac{1}{s^2}\frac{\partial^2\psi}{\partial\theta_1^2},$$

while

$$2(\cot\theta-\tan\theta)\frac{d}{d\theta}\left[\frac{A+B\theta}{\sin\theta\cos\theta}\right]+\frac{d^2}{d\theta^2}\left[\frac{A+B\theta}{\sin\theta\cos\theta}\right]=\frac{4(A+B\theta)}{\sin\theta\cos\theta},$$

and

$$\frac{d^2}{ds^2}\left[\frac{K_0(\mu s)}{s^2}\right]+\frac{5}{s}\frac{d}{ds}\left[\frac{K_0(\mu s)}{s^2}\right]+\frac{4}{s^2}\left[\frac{K_0(\mu s)}{s^2}\right]=\mu^2\left[\frac{K_0(\mu s)}{s^2}\right].$$

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[2] Watson, *Theory of Bessel Functions*, p. 78.

From the function given by (15) we build up one continuous, and differentiable almost everywhere, as follows: Let $l \gg a$.

In region (i), $s>l$, $s \sin\theta_1>a$, $s\sin\theta_2>a$, let

$$\psi(\mathbf{s}_1, \mathbf{s}_2)=\frac{1}{s^2}K_0(\mu s)\left[\frac{\pi/2-\theta_1}{\sin\theta_1\cos\theta_1}+\frac{\pi/2-\theta_2}{\sin\theta_2\cos\theta_2}\right]; \tag{18}$$

in region (ii) $s>l$, $s\sin\theta_1<a$, $s\sin\theta_2>a$, let

$$\psi(\mathbf{s}_1, \mathbf{s}_2)=\frac{1}{s^2}K_0(\mu s)\left[C(s)\varphi(s\sin\theta_1)+\frac{\pi/2-\theta_2}{\sin\theta_2\cos\theta_2}-\frac{2\pi}{3^{\frac{3}{2}}}\right]; \tag{19}$$

in region (iii), $s>l$, $s\sin\theta_1>a$, $s\sin\theta_2<a$, let

$$\psi(\mathbf{s}_1, \mathbf{s}_2)=\frac{1}{s^2}K_0(\mu s)\left[\frac{\pi/2-\theta_1}{\sin\theta_1\cos\theta_1}-\frac{2\pi}{3^{\frac{3}{2}}}+C(s)\varphi(s\sin\theta_2)\right]; \tag{20}$$

where
$$C(s)A\frac{e^{-\lambda a}}{a}=\frac{\pi/2-\sin^{-1}(a/s)}{a/s(1-a^2/s^2)^{\frac{1}{2}}}+\frac{2\pi}{3^{\frac{3}{2}}}=\frac{\pi s}{2a}+\frac{2\pi}{3^{\frac{3}{2}}}-1+0(a/s), \quad a\ll s, \tag{21}$$

and φ is given by (3); in region (iv), $s<l$, let

$$\psi(\mathbf{s}_1, \mathbf{s}_2)=\frac{s}{l}\psi\left(\frac{l}{s}\mathbf{s}_1, \frac{l}{s}\mathbf{s}_2\right). \tag{22}$$

In the sequel ψ will stand for the function given by (18), (19), (20) and (22).

The Bound for the Energy

As $a/l\rightarrow 0$ and $\mu l\rightarrow 0$,

$$\int\int\psi^2dv_1dv_2\rightarrow\int\int\left\{\frac{1}{s^2}K_0(\mu s)\left[\frac{\pi/2-\theta_1}{\sin\theta_1\cos\theta_1}+\frac{\pi/2-\theta_2}{\sin\theta_2\cos\theta_2}\right]\right\}^2dv_1dv_2=G, \quad \text{say}, \tag{23}$$

a convergent integral of which the value lies between twice and four times

$$\int\int\left\{\frac{1}{s^2}K_0(\mu s)\frac{\pi/2-\theta_1}{\sin\theta_1\cos\theta_1}\right\}^2dv_1dv_2=\frac{3^{\frac{3}{2}}}{2^3}16\pi^2\int_0^\infty s\{K_0(\mu s)\}^2ds\int_0^{\pi/2}\left(\frac{\pi}{2}-\theta\right)^2d\theta=\frac{3^{\frac{3}{2}}}{2^3}16\pi^2\frac{1}{2\mu^2}\frac{\pi^2}{24}, \tag{24}$$

i.e., between $3^{\frac{1}{2}}\pi^4/4\mu^2$ and $3^{\frac{1}{2}}\pi^4/2\mu^2$. Let

$$I=\int\int\left[\{(\nabla_1\psi)^2+(\nabla_1\psi\cdot\nabla_2\psi)+(\nabla_2\psi)^2\}+\frac{4\pi^2m}{h^2}\{V(s_1)\psi^2+V(s_2)\psi^2\}\right]dv_1dv_2$$
$$=I_{(\text{i})}+I_{(\text{ii})}+I_{(\text{iii})}+I_{(\text{iv})}, \tag{25}$$

where the integration is extended through region (i) in $I_{(\text{i})}$, and so on.

In region (i), $V(s_1)=0$ and $V(s_2)=0$ by (2), and, in virtue of (16),

$$\begin{aligned}I_{(\text{i})}&=\int\{(\mathbf{l}_1\cdot\{\nabla_1\psi+\tfrac{1}{2}\nabla_2\psi\})+(\mathbf{l}_2\cdot\{\tfrac{1}{2}\nabla_1\psi+\nabla_2\psi\})\}dS-\mu^2_{(\text{i})}\int\int\psi^2dv_1dv_2\\&=J_{(\text{ii})}+J_{(\text{iii})}+J_{(\text{iv})}-\mu^2_{(\text{i})}\int\int\psi^2dv_1dv_2,\end{aligned} \tag{26}$$

where the surface integral is extended over the boundary between (i) and (ii) in $J_{(ii)}$, between (i) and (iii) in $J_{(iii)}$, and between (i) and (iv) in $J_{(iv)}$. $\mathbf{l}_1$ and $\mathbf{l}_2$ are the direction cosines in the directions $\mathbf{s}_1$ and $\mathbf{s}_2$ of the normal to the five-dimensional boundary region outward from (i).

The volume integral in (26) gives in the limit the term $-\mu^2 G$.

The surface integral $J_{(ii)}$ reduces to

$$-\int\int s_1(\mathbf{s}_1\cdot\psi\{\nabla_1\psi+\tfrac{1}{2}\nabla_2\psi\})d\omega_1 dv_2',$$

where the inner integral is taken over all solid angle in $\mathbf{s}_1$ space for $s_1=a$ and $\mathbf{s}_2=\mathbf{s}_2'+\frac{1}{2}\mathbf{s}_1$, and then the outer integration over $\mathbf{s}_2'$ space for $s_2'^2>\frac{3}{4}(l^2-a^2)$. Now as $s_1/s_2\to 0$, so that $s_1\sim s\theta_1$,

$$\psi=(1/s^2)K_0(\mu s)[\pi/2\theta_1-1+2\pi/3^{\frac{3}{2}}+0(\theta_1)] \text{ and } (\mathbf{s}_1\cdot\{\nabla_1\psi+\tfrac{1}{2}\nabla_2\psi\})=-(1/s^2)K_0(ms)[\pi/2\theta_1+0(\theta_1)],$$

so that the inner integral becomes

$$4\pi K_0^2(\mu s)[\pi^2/4s^2a+\pi/2\{2\pi/3^{\frac{3}{2}}-1\}1/s^3+0(a/s^4)]$$

(terms of order $1/s^3$ in the parenthesis arising from the difference between $\mathbf{s}_2$ and $\mathbf{s}_2'$ vanishing on integration) and

$$J_{(ii)}=-\int 4\pi K_0^2(\mu s)[\pi^2/4s^2a+\pi/2\{2\pi/3^{\frac{3}{2}}-1\}1/s^3+0(a/s^4)]dv_2, \quad (27)$$

integrated over $\mathbf{s}_2$ space for $s=2s_2/3^{\frac{1}{2}}>l$.

The volume-integral $I_{(ii)}$ reduces, in like manner, first to

$$\int\int[(\nabla_1\varphi(s_1))^2+(4\pi^2m/h^2)V(s_1)\varphi^2(s_1)]dv_1\{(1/s^2)K_0(\mu s)C(s)\}^2[1+0(a^2/s^2)]dv_2,$$

where the inner integral is over $\mathbf{s}_1$ space for $s_1<a$, and the outer over $\mathbf{s}_2$ space for $s=2s_2/3^{\frac{1}{2}}>l$; ($V(s_2)$ vanishing in this region, and terms arising from $(\nabla_1\psi\cdot\nabla_2\psi)$ and $\nabla_2^2\psi$, as well as from the term $(\pi/2-\theta_2)/\sin\theta_2\cos\theta_2-(2\pi/3^{\frac{3}{2}})(\to 0)$ in the expression for ψ, of order a/s compared with terms retained, vanishing on integration over $\mathbf{s}_1$ space, leaving only terms of order a^2/s^2). Now, from (1), (3) and (5),

$$_{s_1<a}\int[(\nabla_1\varphi(s_1))^2+(4\pi^2m/h^2)\{V(s_1)-E_0\}\varphi(s_1)^2]dv_1={}_{s_1=a}\int(\mathbf{s}_1\cdot\nabla_1\varphi(s_1))\varphi(s_1)s_1d\omega_1$$
$$=-4\pi A^2\{1/a+\lambda\}e^{-2\lambda a}.$$

The term involving E_0, $\lambda^2{}_{s_1<a}\int\varphi(s_1)^2dv_1$, tends to zero by (4) and (6), and, with the expression for $C(s)$, (21),

$$I_{(ii)}=\int 4\pi K_0^2(\mu s)[\pi^2/4s^2a+(\pi^2/4s^2)\lambda+2(\pi/2)(2\pi/3^{\frac{3}{2}}-1)+0(a/s^4)]dv_2, \quad (28)$$

integrated over $\mathbf{s}_2$ space for $s=2s_2/3^{\frac{1}{2}}>l$.

From (27) and (28),

$$I_{(ii)}+J_{(ii)}=-{}_{s>l}\int 4\pi K_0^2(\mu s)\{(\pi^2/4s^2)\lambda+\pi/2(2\pi/3^{\frac{3}{2}}-1)(1/s^3)+0(a/s^4)\}dv_2$$
$$=-16\pi^2(3^{\frac{3}{2}}/2^3)\int_l^\infty K_0^2(\mu s)\{\pi^2\lambda+\pi/2s(2\pi/3^{\frac{3}{2}}-1)+0(a/s^4)\}ds$$
$$=-16\pi^2(3^{\frac{3}{2}}/2^3)(\pi/2)(2\pi/3^{\frac{3}{2}}-1)|\log\mu l|^3\{1+0(1/|\log\mu l|)+0(a/l)\} \quad (29)$$

as $a/l\to 0$ and $\mu l\to 0$, since

$$K_0(\mu s)=-\{\log\tfrac{1}{2}\mu s+\gamma\}\{1+0(\mu s)\} \quad \text{as} \quad \mu s\to 0. \quad (30)$$

$$I_{(iii)}+J_{(iii)}=I_{(ii)}+J_{(ii)}. \quad (31)$$

The surface integral $J_{(iv)}$ over part of the five-dimensional surface $s=l$ gives in the limit a similar integral to that for G (23), $-s^5(1/s^2)K_0(\mu s)(\partial/\partial s)(1/s^2)K_0(\mu s)$ replacing $\int_0^\infty sK_0(\mu s)ds$, so that

$$J_{(iv)}=0(|\log \mu l|^2) \quad \text{as} \quad \mu l\to 0. \tag{32}$$

The volume integral $I_{(iv)}$ consists of terms in $V(s_1)$ and $V(s_2)$ that are always negative and the volume integral throughout the six-dimensional region inside the surface $s=l$ of (in terms of the coordinates $\mathbf{u}$ and $\mathbf{v}$ of 17), $(\nabla_u(s/l)\psi_{(s=l)})^2+(\nabla_v(s/l)\psi_{(s=l)})^2$, which reduces to the surface integral over $s=l$ of

$$l/6\{\psi^2/l^2+(\nabla_u\psi)^2+(\nabla_v\psi)^2-(\partial\psi/\partial s)^2\},$$

which, in virtue of (8) and (30), is of order $K(l/a)|\log \mu l|^2$. Thus

$$I_{(iv)}<0(K(l/a)|\log \mu l|^2). \tag{33}$$

Thus, combining (25), (26), (29), (31), (32) and (33),

$$I=I_{(i)}+I_{(ii)}+I_{(iii)}+I_{(iv)}<(J_{(ii)}+I_{(ii)})+(J_{(iii)}+I_{(iii)})+J_{(iv)}+I_{(iv)}$$
$$<-32\pi^2(3^{\frac{3}{2}}\pi/2^32)(2\pi/3^{\frac{3}{2}}-1)|\log \mu l|^3\{1+0(K/(a/l)|\log \mu l|)+0(a/l)\} \tag{34}$$

and from (34), (13), (6) and (24),

$$\frac{E_1}{E_0}>\frac{24(2\pi/3^{\frac{3}{2}}-1)}{\pi}\frac{\mu^2}{\lambda^2}|\log \mu l|^3\left\{1+0\left(\frac{K}{(a/l)|\log \mu l|}\right)+0(a/l)\right\}. \tag{35}$$

Now put $l=a/\epsilon$, $\mu=(1/a)\epsilon e^{-1/K\epsilon^2}$, where ϵ is chosen so that ϵ and $K\epsilon^2$ are small, and we obtain

$$E_1/E_0>(1\cdot 6/a^2\lambda^2)e^{-2/K\epsilon^2}/K^3\epsilon^4\{1+0(\epsilon)\}. \tag{36}$$

In order to obtain a numerical result from this it would be necessary first to take a value for K. If $V(r)$ is of the form given by (9) and (2), this can be done. Secondly it would be necessary to know how small ϵ must be taken, which would involve estimates of a number of complicated integrals. For the observed ratio $E_1/E_0\approx 4$, a lower bound of a as a definite fraction of $1/\lambda$, $=h/2\pi(mE_0)^{\frac{1}{2}}$ $\approx 6\cdot 10^{-13}$ cm would be obtained. The value so obtained would be a rather poor one, for there is no reason why the function ψ chosen should (if $a\lambda$ is not very small) be at all a good approximation to the solution of (12); moreover the introduction of K into (36) came from $I_{(iv)}$ in which negative terms in $V(r_{13})$ and $V(r_{23})$ were left out; if these could be taken into account, and if ψ were chosen suitably in region (iv), this could probably be avoided.

However, it is seen that a cannot be very small compared with 10^{-13} cm. A closer limit could perhaps best be obtained by numerical approximation to E_1 for particular forms of $V(r)$.

The Assumptions When the Potential Energy is Replaced by a Singularity[3]

Suppose that the positional part $\psi(\mathbf{r}_1, \mathbf{r}_2)$ of a wave function for a neutron (at $\mathbf{r}_2$) and a proton (at $\mathbf{r}_1$) satisfies the equation

$$-(h^2/8\pi^2m)\{\nabla_1^2\psi+\nabla_2^2\psi\}-E\psi=0 \tag{37}$$

but is not continuous at $r_{12}=0$, satisfying, however, the boundary condition

$$\psi(\mathbf{r}_1, \mathbf{r}_2)-A(1/r_{12}-\lambda)\to 0 \quad \text{as} \quad r_{12}\to 0. \tag{38}$$

(37) has a solution of the form

$$\psi=\varphi(r_{12}), \qquad E=E_0<0, \tag{39}$$

[3] I owe to Professor E. Wigner the remark that this case does not immediately follow from the above, as the integral I does not converge.

such that
$$4\pi\int_0^\infty \varphi(r)^2 r^2 dr = 1, \tag{40}$$
viz.
$$\varphi(r) = Ae^{-\lambda r}/r, \tag{41}$$
where
$$(h^2/4\pi^2 m)\lambda^2 = -E_0 \tag{42}$$
and
$$4\pi A^2 = 2\lambda. \tag{43}$$

Suppose that the positional part $\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ of a wave function for two neutrons (at $\mathbf{r}_1$ and $\mathbf{r}_2$) and a proton at $(\mathbf{r}_3)$, symmetrical in the positions of the two neutrons, satisfies the equation

$$-(h^2/8\pi^2 m)\{\nabla_1^2\psi + \nabla_2^2\psi + \nabla_3^2\psi\} - E\psi = 0 \tag{44}$$

but is not continuous as $r_{12}\to 0$ or $r_{23}\to 0$, although functions $f(\mathbf{r}_1, \mathbf{r}_3)$ and $f(\mathbf{r}_2, \mathbf{r}_3)$ exist such that

$$\left.\begin{aligned} &\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) - f(\mathbf{r}_1, \mathbf{r}_3)\{1/r_{23} - \lambda\} \to 0 \quad \text{as} \quad r_{23}\to 0 \quad \text{for fixed } \mathbf{r}_1 \\ \text{and} \quad &\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) - f(\mathbf{r}_2, \mathbf{r}_3)\{1/r_{13} - \lambda\} \to 0 \quad \text{as} \quad r_{13}\to 0 \quad \text{for fixed } \mathbf{r}_2 \end{aligned}\right\} \tag{45}$$

(while if $A(\mathbf{r}_3)$ exists such that

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) - A(\mathbf{r}_3)(1/r_{23} - \lambda)(1/r_{13} - \lambda) \to 0 \quad \text{as} \quad (r_{13}, r_{23}) \to (0, 0), \tag{46}$$

no boundary condition at $(r_{13}, r_{23}) = (0, 0)$ would be violated: another requirement at $(r_{13}, r_{23}) = (0, 0)$ would correspond to the limit of an equation of type (10) with potential energy $V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ not of the form $V(r_{12}) + V(r_{13})$); as before, interaction between the two neutrons is neglected.

If $\psi = \psi(\mathbf{s}_1, \mathbf{s}_2)$, where
$$\mathbf{s}_1 = \mathbf{r}_1 - \mathbf{r}_3, \qquad \mathbf{s}_2 = \mathbf{r}_2 - \mathbf{r}_3 \tag{47}$$
(44) becomes
$$-(h^2/4\pi^2 m)\{\nabla_1^2\psi + (\nabla_1\cdot\nabla_2)\psi + \nabla_2^2\psi\} - E\psi = 0, \tag{48}$$

subject to the condition that $f(\mathbf{s}_1)$ exists such that

$$\left.\begin{aligned} &\psi(\mathbf{s}_1, \mathbf{s}_2) - f(\mathbf{s}_1)(1/s_2 - \lambda) \to 0 \quad \text{as} \quad s_2 \to 0, \\ &\psi(\mathbf{s}_1, \mathbf{s}_2) - f(\mathbf{s}_2)(1/s_1 - \lambda) \to 0 \quad \text{as} \quad s_1 \to 0 \end{aligned}\right\} \tag{49}$$

(while if A exists such that

$$\psi(\mathbf{s}_1, \mathbf{s}_2) - A(1/s_2 - \lambda)(1/s_1 - \lambda) \to 0 \quad \text{as} \quad (s_1, s_2) \to (0, 0),$$

no boundary condition at $(s_1, s_2) = (0, 0)$ would be violated).

The algebraically least value E_1 of E, if it exists, for which a continuous twice differentiable function $\psi(\mathbf{s}_1, \mathbf{s}_2)$ can be found to satisfy (48) and (49) must obey the inequality

$$E_1 < (h^2/4\pi^2 m)\int\int\{\nabla_1^2\psi + (\nabla_1\cdot\nabla_2)\psi + \nabla_2^2\psi\}\psi dv_1 dv_2 / \int\int \psi^2 dv_1 dv_2 \tag{50}$$

where ψ is any continuous differentiable function of $\mathbf{s}_1$ and $\mathbf{s}_2$, twice differentiable almost everywhere, except as $s_1\to 0$ and $s_2\to 0$, where it must satisfy (49). A form like (13) cannot be used here because the integral does not converge.

Under the Singularity Assumption There is No Algebraically Least Energy

As we wish to use the same function $\psi(\mathbf{s}_1, \mathbf{s}_2)$ as above, except that $\varphi(r)$ is now to be given by (41) everywhere, which is not differentiable at $s = l$, $r_1 = a$, and $r_2 = a$, (50) must be modified to take account of this: to the volume integral in the numerator must be added the difference of surface integrals

$$\Big|_1^2 \int\{(\mathbf{l}_1\cdot\{\nabla_1\psi + \tfrac{1}{2}\nabla_2\psi\}) + (\mathbf{l}_2\cdot\{\tfrac{1}{2}\nabla_1\psi + \nabla_2\psi\})\}\psi dS\Big| \tag{51}$$

over all five-dimensional surfaces of discontinuity of the first derivatives of ψ, $\mathbf{l}_1$ and $\mathbf{l}_2$ being the direction cosines in the directions $\mathbf{s}_1$ and $\mathbf{s}_2$ of the normal to the five-dimensional boundary region from region 1 to region 2. A function with continuous first derivatives and near ψ will have a value for (50) nearly equal to that for ψ with this addition.

We can now write (50) in the form

$$E_1 < (h^2/4\pi^2 m)\{I_{(i)}+I_{(ii)}+I_{(iii)}+I_{(iv)}+J_{(ii)}+J_{(iii)}+J_{(iv)}\}/\int\int\psi^2 dv_1 dv_2 \qquad (52)$$

where $I_{(i)}$, $I_{(ii)}$, $I_{(iii)}$, and $I_{(iv)}$, are the values of the volume integral extended over the regions (i), (ii), (iii) and (iv), and $J_{(ii)}$, $J_{(iii)}$ and $J_{(iv)}$, are the values of the surface integral (51) over the regions $s_1=a$, $s_2=a$, and $s=l$.

By (16),

$$I_{(i)}=0. \qquad (53)$$

$I_{(ii)}$ can be expressed as in (28) as an integration over $\mathbf{s}_1$ space followed by one over $\mathbf{s}_2$ space. Terms of order $1/s_1^3$ and $1/s_1^4$ which would make the inner integral diverge do not occur, on account of (41), and the terms that remain give rise to terms of order a/l in the result compared to the principal terms from $J_{(ii)}$.

The two integrals in $J_{(ii)}$ now combine like (27) and (28) to give (as before)

$$I_{(ii)}+J_{(ii)}=-16\pi^2(3^{\frac{3}{2}}\pi/2^3 2)(2\pi/3^{\frac{3}{2}}-1)|\log \mu l|^3\{1+0(1/|\log \mu l|)+0(a/l)\}, \qquad (54)$$

while

$$I_{(iii)}+J_{(iii)}=I_{(ii)}+J_{(ii)}. \qquad (55)$$

The integral over the side $s>l$ of $s=l$ in $J_{(iv)}$ is as before, (32). The integral over the side $s<l$ of $s=l$ reduces to the integral of ψ^2/l over $s=l$, and so gives a result of order $|\log \mu l|^2$.

The volume integral $I_{(iv)}$ reduces (in terms of $\mathbf{u}$ and $\mathbf{v}$ of (17)) to the surface integral over $s=l$ of

$$\psi l\{\tfrac{2}{5}(\psi/l^2-\partial\psi/l\partial s)+\tfrac{1}{6}(\partial\psi/l\partial s+\{\nabla_u^2\psi+\nabla_v^2\psi-\partial^2\psi/\partial s^2\})\}. \qquad (56)$$

The terms in (33) which required (8) now do not occur, terms in $\nabla_u^2\psi+\nabla_v^2\psi$ that would make the integral diverge not occurring on account of (41), and no term gives a result larger than of order $|\log \mu l|^2$: i.e.,

$$I_{(iv)}+J_{(iv)}=0(|\log \mu l|^2). \qquad (57)$$

Thus, combining (53), (54), (55) and (57), with (50), (42) and (24),

$$\frac{E_1}{E_0}>\frac{24(2\pi/3^{\frac{3}{2}}-1)}{\pi}\frac{\mu^2}{\lambda^2}|\log \mu l|^3\left\{1+0\left(\frac{1}{|\log \mu l|}\right)+0\left(\frac{a}{l}\right)\right\} \qquad (58)$$

where we may let μ have any value, $\mu l\to 0$, and $a/l\to 0$, so that E_1 cannot exist.

My thanks are due to Professor Uhlenbeck for suggesting this problem to me and to Professor Wigner for helpful discussion.[4]

Note added in proof: No change is made in the above argument if the force assumed between a neutron and a proton is of Majorana[5] type, for the wave function used is symmetrical in their positions wherever the potential energy differs from zero.

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[4] Some remarks on the H³ problem may be found in E. Wigner, Phys. Rev. **43**, 252 (1933).
[5] Majorana, Zeits. f. Physik **82**, 137 (1932).