Gravitational and Electromagnetic Mass in the Born-Infeld Electrodynamics

(Received March 8, 1935)

BANESH HOFFMANN, *University of Rochester, Rochester, New York*

By postulating that infinite relativistic gravitational potentials are to be rejected in the Born-Infeld theory, it is shown that the gravitational mass of an electron becomes equal to its electromagnetic mass, and that difficulties in the usual relativistic treatment of gravitational mass are avoided. The above postulate is extended and its bearing on the alternative sets of field equations of the Born theory, and on a proposal made by the author is discussed.

§1.

THE classical concept of electromagnetic mass depended on the assumption of a definite size and structure for the electron, but was able to predict that the mass of a body would depend on its state of motion. The special theory of relativity showed, however, that the change of mass with velocity was a property of any inertial mass, whatever its origin, and in consequence the concept of electromagnetic mass has tended to fall into disuse.

Recently Born and Infeld[1] have developed a new system of electrodynamics in which the concept of electromagnetic mass takes on a new significance. Since the Born electron has no infinities in its potential, it is unnecessary to assign a definite radius to it, and its mass is a measure of the total energy of its field over all space. An electron no longer has an "interior" and an "exterior," and no need arises for arbitrarily avoiding the energy of the field "within" the electron since there is no region that can properly be characterized as within it.

In the general theory of relativity, the gravitational mass of a spherically symmetric distribution of matter arises as a constant of integration in Schwarzschild's field, and the gravitational potentials expressing the field contain an infinity at the center of the mass. To avoid this infinity it has been customary to point out that the gravitational potentials of this Schwarzschild field refer only to the region outside the matter producing the field and that, within the matter, other field equations are valid. The interior solution is then fitted to the exterior solution at the surface of the sphere, and this procedure leads to a relationship between the inertial and gravitational masses of the sphere.

In connection with the interior Schwarzschild solution, it is to be noted that an infinity at the origin would arise even here were one not to set equal to zero the integration constant that gives rise to this infinity. That is, one avoids an infinity at the origin, not because such an infinity does not exist in the most general mathematical solution of the interior field equations, but simply because one decides that such an infinity is objectionable on physical grounds.

A difficulty arises in the case of the exterior solution, when this is considered as standing alone, with no reference to a corresponding interior solution; for, since the gravitational mass arises as an arbitrary constant of integration, there is no reason why it should not take on negative values, and a negative mass is not considered desirable, outside the quantum theory. As soon as one relates the exterior solution to a corresponding interior one this difficulty seems to be removed since the gravitational mass is now identified with a quantity which closely approximates the total inertial mass of the sphere, and this will be positive if the density of the sphere is positive. Actually, however, the difficulty has merely been moved rather than removed, for, whereas in the isolated exterior solution we had to choose a positive value for a constant of integration after the general solution had been found, we now have to choose a positive value for the density as soon as the problem is set up.

In this paper we discuss the gravitational mass of an electron according to the Born-Infeld theory and show that the difficulties of the classical relativistic treatment of gravitational

[1] Born and Infeld, Proc. Roy. Soc. **A143**, 410 (1934); **A144**, 425 (1934) and **A147**, 522 (1934). We shall refer to the second of these as II.

mass may be avoided. Since, in the Born-Infeld theory, there are no longer two distinct regions, interior and exterior, to an electron, the one set of field equations must suffice throughout space. It turns out that no recourse is necessary to an interior solution, that the inertial mass of the electron can be related to its gravitational mass, and that this ensures not only that there will be no infinities in the gravitational potentials of the electron, but also that its gravitational mass will necessarily be positive.

§2.

According to Born and Infeld,[2] the field equations of the new theory, when gravitation is taken into account, may be written as

$$R_{ab}-\tfrac{1}{2}g_{ab}R=-8\pi E_{ab},$$

$$\frac{\partial}{\partial x^b}\{(-g)^{\frac{1}{2}}(F^{ab}-GF^{*ab})/(1+F-G^2)^{\frac{1}{2}}\}=0, \quad (1)$$

and $$\partial F_{bc}/\partial x^a+\partial F_{ca}/\partial x^b+\partial F_{ab}/\partial x^c=0,$$

where R_{ab} is the Ricci tensor formed out of the g_{ab}, F and G are defined in II Eqs. (2.16), the energy tensor, E_{ab}, of the electromagnetic field is given by

$$E_{ab}=-g_{ab}\{1-(1+F-G^2)^{\frac{1}{2}}-G^2/(1+F-G^2)^{\frac{1}{2}}\}$$
$$-g^{cd}F_{ac}F_{bd}/(1+F-G^2)^{\frac{1}{2}}, \quad (2)$$

and the units are such that the velocity of light, the gravitational constant, and Born's natural unit of field strength, b, are all taken as unity.

These equations have been solved by the author for the general spherically symmetrical case, and it turns out that the field for this case is necessarily static.[3] The field equations of S.S. were written down with a wrong sign given to the value for the electromagnetic energy tensor E_{ab}, and the interpretation given to the field was different from that to be given in the present paper. If we pay regard to the effects of altering the sign of the electromagnetic energy tensor in S.S., it is seen[4] that the field equations will be satisfied by the line element

$$ds^2=Adt^2-A^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\, d\varphi^2) \quad (3)$$

and the radial electrostatic intensity $F_{14}=\epsilon/(r^4+\epsilon^2)^{\frac{1}{2}}$, ($\epsilon$ a constant of integration), provided that

$$e^\nu(rd\nu/dr+1)-1=-8\pi r^2E_4{}^4$$
$$=-8\pi\{(r^4+\epsilon^2)^{\frac{1}{2}}-r^2\}. \quad (4)$$

This equation may be written as

$$(d/dr)(re^\nu)=1-8\pi\{(r^4+\epsilon^2)^{\frac{1}{2}}-r^2\}$$

and gives the integral

$$A\equiv e^\nu=1-2m/r-(8\pi/r)\int_0^r\{(r^4+\epsilon^2)^{\frac{1}{2}}-r^2\}dr, \quad (5)$$

where $(-2m)$ is a constant of integration which,[5] in classical relativity, is identified, from a consideration of the trajectories of a test particle in the field, with the gravitational mass of an uncharged sphere. It can be shown to be a first approximation to the inertial mass of the sphere either from consideration of a related interior field or from a general theorem due to Einstein[6] concerning the equality of gravitational and inertial mass in *weak* fields.

The Born theory was propounded in order to remove the infinity in the potential energy of the electron that occurs in the Coulomb field of the Maxwell theory. It is therefore reasonable to make the postulate that not only the electromagnetic potentials and intensities shall contain no infinities, but also that the gravitational potentials shall be free from such singularities.

Now the term involving m in the formula (5) for A becomes infinite as r approaches zero, though the term involving the integral remains finite at the pole despite the factor $1/r$. Thus the above postulate will require that we set m equal to zero.[7] And this requires that we obtain a gravitational mass in (5) from the integral term alone. The integral is not a constant, so that we shall be unable to find an exact duplication of the role played by m.

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[2] See II, Eqs. (3.2), (3.4), (3.6) and below (4.5).
[3] Hoffmann, *On the Spherically Symmetric Field in Relativity, III*, to appear in Quarterly J. Math. (Oxford). We shall refer to this paper as S.S.
[4] S.S., Eqs. (27α), (27β) and (29); $A\equiv e^\nu$.

[5] Cf. the classical relativistic field of a charged sphere, Tolman, *Relativity, Thermodynamics and Cosmology*, Oxford University Press, §107.
[6] Tolman, reference 5, §80.
[7] Cf. Tolman, reference 5, §96, where the constant C is taken as zero on similar grounds.

The integral can be written[8] as

$$\int_0^r r^2 E_4{}^4 dr = (1/4\pi)\int_0^r\int_0^\pi\int_0^{2\pi} E_4{}^4 r^2 \sin\theta drd\theta d\varphi$$
$$= (1/4\pi)\int_0^r\int_0^\pi\int_0^{2\pi}(E_4{}^4(-g)^{\frac{1}{2}})drd\theta d\varphi, \quad (6)$$

which measures the amount of $E_4{}^4(-g)^{\frac{1}{2}}$ in the sphere of coordinate radius r whose center is at the pole of the coordinate system. But $E_4{}^4(-g)^{\frac{1}{2}}$ is the energy density of the electrostatic field and therefore, in our present units, is equal to the mass density of this energy, i.e., to the electromagnetic mass density of the electrostatic field. Therefore, if we write

$$(1/4\pi)\int_0^r r^2 E_4{}^4 dr = m_r,$$

the quantity m_r measures the amount of electromagnetic mass contained in a sphere of coordinate radius r about the pole. The relativistic gravitational potential, A, now takes the form

$$A = 1 - 2m_r/r. \quad (7)$$

Though m_r is not a constant, it is very nearly equal to m_∞, the total mass, for values of r appreciably larger than the radius of the classical electron. Furthermore, it is easily seen that m_r is always positive, and it has already been pointed out that m_r/r does not become infinite at the pole. The fact that m_r is not a constant but measures the mass within the sphere of radius r, has a complete analog in the Newtonian theory, since it shows that at a point whose coordinate distance from the center of the electron is r, the gravitational effect is solely due to the matter within the concentric sphere passing through this point, the outer spherically symmetric distribution of mass having no gravitational effect at this point.

Thus by postulating that gravitational potentials that contain infinities are to be rejected, we have found that the electromagnetic and gravitational mass of an electron are equivalent, and this implies that all mass arising from electrons is essentially of an electromagnetic nature. At the same time, the introduction of an extra constant of integration has been avoided.[9]

The gravitational and electromagnetic masses have turned out to be identical in the case of a spherically symmetric electrostatic field, but this is due to the accident that g_{11} and g_{44} are reciprocals for this case. In general the relationship between the masses will be one of only approximate equivalence.

§3.

Born and Infeld have considered two possible sets of field equations for the electromagnetic field,[10] and have not decided which set is to be preferred. Some indication of the respective merits of the two sets of field equations can be obtained by the use of the postulate that all infinities are to be avoided.

In S.S.[11] the general spherically symmetric fields allowed by the two sets of field equations were obtained. If we make allowance for the wrong sign in the field equations of S.S. and ignore the cosmological constant λ and the integration constant m, we may write the two solutions as:

For $G\neq 0$:

$$\left.\begin{aligned} &ds^2 = Adt^2 - A^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),\\ &F_{14} = \epsilon/(r^4+\mu^2+\epsilon^2)^{\frac{1}{2}},\\ &F_{23} = \mu \sin\theta, \end{aligned}\right\} \quad (8)$$

with $A = 1 - (8\pi/r)\int_0^r\{(r^4+\mu^2+\epsilon^2)^{\frac{1}{2}} - r^2\}dr$;

For $G = 0$:

$$\left.\begin{aligned} &ds^2 = Adt^2 - A^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),\\ &F_{14} = \epsilon(r^4+\mu^2)^{\frac{1}{2}}/r^2(r^4+\epsilon^2)^{\frac{1}{2}},\\ &F_{23} = \mu \sin\theta, \end{aligned}\right\} \quad (9)$$

with $A = 1 - (8\pi/r)\int_0^r[(r^4+\mu^2)(r^4+\epsilon^2)]^{\frac{1}{2}}/r^2 - r^2\}dr$.

Each field represents the effect of a particle having electric pole strength ϵ and magnetic pole strength μ.

In S.S. it was argued that since the $G = 0$ field contains infinities at the pole unless μ is taken to

[8] Since $(-g)^{1/2} = r^2 \sin\theta$ here, because g_{11} and g_{44} in (3) are reciprocals.

[9] In classical theories of electromagnetic mass, the possibility of a body having a large mass and zero charge is explained by the fact that charges and masses add algebraically, but the masses associated with positive and negative charges are both positive. In the Born theory, even when gravitation is neglected, the field equations are not linear so that the addition of charges and masses is probably only approximate.

[10] Cf. II, p. 431, Eq. (2.15) and p. 432, Eq. (2.28), the latter being obtained from the former by ignoring the quantity G whenever it appears in the field equations.

[11] S.S. Eq. (30) for the case $G\neq 0$; Eq. (30′) for the case $G = 0$.

be zero, while the $G\neq 0$ field is free from such singularities, the $G\neq 0$ field is preferable if we allow the existence of isolated magnetic poles; but that the relative values of the two fields cannot be determined from these fields if we decide that isolated magnetic poles have no physical existence.

However, in terms of the postulate of the present paper, the situation is reversed. For, since when $\mu\neq 0$, the $G=0$ field involves infinities that are absent from the $G\neq 0$ field, we may argue that the $G=0$ equations require that the constant of integration μ be taken as zero in order that the infinities be avoided. The $G\neq 0$ equations give no reason for rejecting the possibility of a nonvanishing μ. Thus, if isolated magnetic poles are held to be physically nonexistent, the $G=0$ field equations are to be preferred since they require that $\mu=0$. On the other hand, if, in accordance with Dirac's theory,[12] isolated magnetic poles are assumed to exist, the $G\neq 0$ equations have the preference.

When μ is taken to be zero, the two fields become identical and we are back in the situation discussed in the preceding sections.

§4.

So far, we have considered the relationship between gravitational and electromagnetic mass in the Born-Infeld theory for the spherically symmetric case. However, it is to be noted that the particular form given by Born and Infeld for the electromagnetic energy tensor is not necessary to the argument. It will suffice merely that the energy tensor give an electrostatic energy that contains no infinities, is everywhere positive, and approaches zero sufficiently rapidly as r increases. The author has discussed[13] a modified set of field equations that avoids some of the difficulties of the equations proposed by Born and Infeld. However, it encounters other difficulties that are not to be found in the Born theory; the field equations are of the fourth order instead of the second, the spherically symmetric field of a charged particle has not been obtained, and it is shown that this field would differ from the corresponding Born field because of the influence of its own gravitational field, there even being a danger that this influence might introduce an infinity and thus spoil the whole purpose of the theory.

These difficulties lose most of their force when considered in the light of the argument used in the present paper. The fact that the field equations are of the fourth order implies that several unwanted constants of integration will probably arise in the spherically symmetric field; but it is possible that many of these will be removed by the postulate that potentials involving infinities must be avoided. And then, if the gravitational field contains no infinities the formula[14]

$$d\varphi/dr = ke^{\frac{1}{2}(\lambda+\nu)}(1+R)^{\frac{1}{2}}/(2k^2+r^4)^{\frac{1}{2}}$$

that gives the electrostatic intensity, will not be spoiled by infinities in the gravitational potentials e^{λ} and e^{ν}, and there is now considerable likelihood that the electrostatic potential itself will remain finite; which means that the requirement that all infinities, including those in the electrostatic potential, be avoided, will not cause k, essentially the electric charge, to be taken as zero.

I wish to thank Mr. A. G. Hill for the benefit of stimulating discussion.

Note added in proof, May 3, 1935: The criterion used in the present paper for determining infinities in the field is not an invariant one, and some steps in the argument therefore require modification. It is hoped to make this modification the subject of a further paper.

[12] P. A. M. Dirac, Proc. Roy. Soc. **A133**, 60 (1931).
[13] B. Hoffmann, Proc. Roy. Soc. **A148**, 353 (1935).
[14] Reference 13, p. 358, Eq. (30). We are using the notation of that paper here.