

whose origin is unknown but which possibly arise from the configuration $4d^8 5s 5p$. These terms are designated by numbers with the assigned j value as subscript. A few additional lines arising from the transition $4d^9 5p - 4d^9 5d$ have been identified. Among these, the line $4d^9 5p \ ^3P_1^o - 4d^9 5d \ ^1S_0$ is of interest in that it confirms Blair's $4d^9 5d \ ^1S_0$ term.

The extension of the $4d^9 ns \ ^3, ^1D$ series permits the calculation of the series limit by means of a Ritz formula. The application of this formula to the $4d^9 6s, 7s$ and $8s \ ^3D_3$ terms yields $134,110 \text{ cm}^{-1}$ as the limit of this series with respect to the $4d^9 \ ^2D_{2\frac{1}{2}}$ term of Ag III. The $4d^{10} \ ^1S_0$ term of Ag II is $39,164 \text{ cm}^{-1}$ below $4d^9 5s \ ^3D_3$. Thus the absolute value of the $4d^{10} \ ^1S_0$ term of Ag II is $173,274 \text{ cm}^{-1}$ with respect to the $4d^9 \ ^2D_{2\frac{1}{2}}$ level of the Ag III ion. This corresponds to an ionization potential of 21.4 volts. By the application of a Rydberg formula to the $4d^9 5s$ and $6s \ ^3D_3$ terms, Shenstone⁷ obtained a value of 21.9 volts for this quantity. The Ritz formula for the $4d^9 ns \ ^3D_3$ series is:

$$T_m = 134,110 - \nu \\ = RZ^2 / [m - 3.07285 - 8.047 \times 10^{-7}(T_m)]^2.$$

Of the $4d^9 ns \ ^3, ^1D$ series, the 3D_3 and 3D_2 series converge to the $4d^9 \ ^2D_{2\frac{1}{2}}$ state of Ag III and the

3D_1 and 1D_2 series converge to $4d^9 \ ^2D_{1\frac{1}{2}}$. The calculation of the limits for these four series by means of a Ritz formula should yield the separation of the $4d^9 \ ^2D$ terms of Ag III with considerable accuracy. The results are as follows:

Series	*Term value of $4d^9 5s \ ^3D_3$	Limit in Ag III
$ns \ ^3D_3$	134,110 cm^{-1}	$4d^9 \ ^2D_{2\frac{1}{2}}$
3D_2	134,111	$^2D_{2\frac{1}{2}}$
3D_1	138,716	$^2D_{1\frac{1}{2}}$
1D_2	138,718	$^2D_{1\frac{1}{2}}$

* As obtained from the series indicated in column one and referred to the limit indicated in column three.

From the average of these results, the separation of, the $4d^9 \ ^2D$ terms of Ag III should be 4607 cm^{-1} . This is in agreement with the separation reported by Gibbs and White¹¹ and with that recently found by the author in his analysis of the Ag III spectrum the results of which will soon be published. Thus the correctness of the classifications here reported is further confirmed.

The author acknowledges with pleasure the advice and criticism of Professor R. C. Gibbs during the course of this investigation. He is also indebted to Dr. C. W. Gartlein for the design of the Schüler tube and for assistance in some of the experimental work.

¹¹ Gibbs and White, Phys. Rev. **32**, 318 (1928).

Neutron-Proton Interaction

Part I. The Binding Energies of the Hydrogen and Helium Isotopes

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In the present state of nuclear theory it is reasonable to assume, for particles bound in the same nucleus, that the neutron-proton interaction operator can be represented fairly well by a potential function $J(r)$. By taking $J(r)$ to have the form $Ae^{-\alpha r^2}$, which is well adapted to the discussion of the three- and four-body problems, A and α have been determined to fit the binding energies of the deuteron and the alpha-particle. The results are roughly $A = 170 \text{ mc}^2$, $1/\alpha^{1/2} = 1.3 \times 10^{-13} \text{ mc}$, with both the Wigner and Majorana theories. The computed binding energy of H^3 has the value 12.7 mc^2 (Wigner) or 11.2 mc^2 (Majorana), not very far below the experimental value of 16 mc^2 . The two theories appear to be about equally good.

SECTION I. INTRODUCTION

THE proton-neutron nuclear model^{1, 2, 3} in which the important internuclear interac-

tions are those between neutrons and protons provides a satisfactory explanation of the most striking features of nuclear structure. These are:

(a) The large ratio of binding energies of the alpha-particle and deuteron and the general linear increase of binding energy with mass

¹ Heisenberg, Zeits. f. Physik **77**, 1 (1932); **78**, 156 (1932); **80**, 587 (1933).

² E. Majorana, Zeits. f. Physik **82**, 137 (1933).

³ Wigner, Phys. Rev. **43**, 252 (1933).

number for the light and intermediate nuclei.

(b) The existence of a long series of light nuclei for which the mass number is exactly double the charge number.

(c) The general tendency, for intermediate and heavy nuclei, of the mass charge ratio to increase with increasing charge.

The first part of (a) can be understood in terms of a very large neutron-proton interaction with a range of about 10^{-13} cm. The small binding energy of the deuteron results then from the almost complete cancelation of a large kinetic energy against an only slightly larger potential energy.⁴ The second part of (a) can be understood in terms of the Majorana form of exchange interaction. Item (b) implies that the neutron-neutron interaction is relatively small so that in a hypothetical series of light nuclei of given mass the most stable arrangement is the one with the largest possible number of proton-neutron interaction terms.¹ The long range Coulomb repulsive force between the protons decreases the binding energy by a term proportional to the five-thirds power of the charge number⁵ and hence with increasing charge number tends to shift the most stable arrangement toward a neutron-proton ratio greater than unity. This is the explanation of (c).

For particles bound in the same nucleus, it is a plausible assumption that the neutron-proton interaction operator can be represented fairly well by a potential operator $J(r)P_{np}$ with $J(r)$ a function of the separation r and P_{np} an "exchange" operator which works on the coordinates of the two particles. This assumption is basic to the nuclear theories of Heisenberg, of Majorana and of Wigner. In the Heisenberg theory P_{np} interchanges both the space and spin coordinates of the particles, in Majorana's theory only the space coordinates are interchanged. In Wigner's theory P_{np} drops out (in other words the Wigner theory is a direct application of the Schrödinger wave equation). Only the latter two theories yield a stable alpha-particle. Because it is unsatisfactory in this respect the Heisenberg form of the exchange operator will not be considered further.

The Majorana theory provides that two particles with non-overlapping wave functions do not

interact thus making long range interaction forces compatible with the observed roughly linear increase of binding energy with charge number. Under the same conditions the Wigner theory yields a dependence roughly quadratic in the charge number. With the short range forces required by the physical facts both theories, in conjunction with the Pauli principle applied separately to neutrons and to protons, make the alpha-particle a particularly stable system capable of playing the part of a secondary unit in the structure of complex nuclei. It was thought best to carry through calculations for both theories even though the Wigner theory is not acceptable for heavy nuclei.

The immediate problem is the determination of $J(r)$, a problem which Wigner³ has already attacked with considerable success. In this paper the variation method of calculation is simplified and extended to the Majorana theory. A non-rigorous method of reducing the three- and four-body systems to "equivalent" two-body systems is also developed. It is believed that the results obtained are more accurate than those of Wigner.

The Hamiltonian operator in the Wigner theory is

$$H = -\frac{1}{2}\sum\Delta_i - \sum J(r_{p_i n_i}), \quad (1a)$$

and in the Majorana theory

$$H = -\frac{1}{2}\sum\Delta_i - \sum J(r_{p_i n_j})P_{p_i n_j}. \quad (1b)$$

The units are $m_e c^2 = 510,000$ e.v. for energy and $(\hbar^2/4\pi^2 m_e m_p)^{\frac{1}{2}} = 8.97 \times 10^{-13}$ cm for length. The operator $P_{p_i n_j}$ changes a function $\psi(\dots x_{p_i} \dots x_{n_j} \dots)$ into $\psi(\dots x_{n_j} \dots x_{p_i} \dots)$ and thus merely multiplies by ± 1 any function which is symmetric or anti-symmetric with respect to interchange of the space coordinates of the particles p_i and n_j . The energy is given by the equation⁶

$$E^0 = \int \dots \int \psi H \psi d\tau \quad (2)$$

in terms of the Hamiltonian, H , and the normalized wave function ψ .

The experimental facts to be correlated and explained are summarized in Table I. In computing the energies, the mass of the neutron is

⁴ Wigner, *Zeits. f. Physik* **83**, 253 (1933).

⁵ Gamow, *Atomic Nuclei and Radioactivity*, p. 19.

⁶ The superscript 0 will be used to designate energies computed by the variation method.

TABLE I. *Masses and binding energies of hydrogen and helium isotopes.*

Atom	Mass	Binding energy*
H ²	2.0136 ± 0.0001 ⁷	4.0 ± 1.1
H ³	3.0151 ± (?) ⁸	16.0 ± 1.7 (?)
He ³	3.0163 ± 0.0004 ⁸	13.4 ± 1.7
He ⁴	4.00216 ± 0.0002	54.0 ± 2.2

* With respect to free neutron and hydrogen atoms.

taken as 1.0080 ± 0.0005^9 and of the hydrogen atom as 1.0078.

SECTION II. THE TWO-BODY EQUATION

We write $J(r)$ in the form $Af(\alpha^{1/2}r)$. It will be necessary to make a definite assumption about the form of f , but, for the moment, all that need be stated is that f is a positive valued function which vanishes rapidly for large values of the argument. The positive valued parameters A and α are to be determined to fit the binding energies of the deuteron and the alpha-particle. Evidently A and $1/\alpha^{1/2}$ are directly proportional to the depth and breadth, respectively, of the potential well. We are looking for a solution of the equation

$$\{d^2/dr^2 + E + Af(\alpha^{1/2}r)\} \varphi = 0 \quad (3)$$

which vanishes at both $r=0$ and $r=\infty$ and does not vanish anywhere else. In our problem we know the value of E ($E = -4.0$) and will use the differential equation to obtain a relation between A and α ; thus $A = A(\alpha)$ or, more conveniently $A/\alpha = g(\alpha)$. Anywhere on this line in A, α space the normal state solution of (3) has the eigenvalue $E = -4$. This procedure reduces the two-parameter potential to a one-parameter function.

It is convenient to insert here the proof of a theorem which will be needed later. Suppose that the relation $A/\alpha = g(\alpha)$ is known. Then it can be used to compute the normal state eigenvalue for arbitrary values of A and α which do not satisfy the relation: say $A = B, \alpha = \beta$. For let $r = m^{1/2}s$ with m a positive constant. The differential equation becomes

$$\{d^2/ds^2 + mE + mBf((m\beta)^{1/2}s)\} \varphi = 0. \quad (4)$$

Now if $mE = -4$, the quantities mB and $m\beta$ are related by the equation $mB = A(m\beta)$, or $mB/m\beta = B/\beta = g(m\beta)$. Hence $m\beta$ is known and from it the required value of m . The desired eigenvalue is just $-4/m$.

Considerations of simplicity and elegance lead to the assumption that $J(r)$ belongs to the two parameter family of functions $Ae^{-\alpha r^2}$. With $J = Ae^{-\alpha r^2}$ the relation $A = A(\alpha)$ was determined by a number of numerical integrations. Table II

TABLE II. $A(\alpha)$ for $-E(H^2) = 4.0$.

A/α	3.45	3.66	3.88	4.11	4.35	4.60	4.85	5.10	5.36
$1/\alpha^{1/2}$	0.12	0.15	0.18	0.21	0.24	0.27	0.30	0.33	0.36

gives A/α as a function of $1/\alpha^{1/2}$. The relation is almost linear.

With the approximate wave function

$$\varphi = (v/\pi)^{3/4} r e^{-vr^2/2}$$

the variation Eq. (2) has the explicit form

$$E^0 = (3/2)\alpha\sigma - A(\sigma/(\sigma+1))^{3/2}, \quad \sigma = v\alpha. \quad (5)$$

Eq. (5) is needed for comparison with later results.

SECTION III. THE THREE- AND FOUR-BODY PROBLEMS

A suitable and fairly flexible wave function for use in (2) is obtained if we approximate to ψ by a product of functions each depending only on the distance between one pair of particles: thus

$$\psi(1, 2, 3) = u(12)u(13)v(23) \quad (6)$$

for H³ and He³ and

$$\psi(1, 2, 3, 4) = u(13)u(14)u(23)u(24)v(12)v(34) \quad (7)$$

for the alpha-particle. The u 's tie together unlike particles while the v 's serve to increase the average potential field in which the particles move by holding like particles together. It turns out that the v 's are important in both theories and absolutely essential in the Majorana theory. With these wave functions (2) can be put in the form

$$E^0(H^3) = X(H^3) + Y(H^3) - Z(H^3), \quad (8)$$

$$E^0(He^4) = -Q(H^3) + X(He^4) + Y(He^4) - Z(He^4), \quad (9)$$

⁷ Bainbridge, Phys. Rev. **44**, 56 (1933).

⁸ Oliphant, Hartek and Rutherford, Nature **133**, 413 (1934); Proc. Roy. Soc. **A144**, 692 (1934); and Dee, Proc. Roy. Soc. **A148**, 623 (1935).

⁹ Chadwick and Goldhaber, Nature **134**, 237 (1934).

in which

$$\begin{aligned}
 X(H^3) &= \frac{1}{2} \int \int \int \{ 3 |\nabla_1 u(13)|^2 - u(13) \Delta_1 u(13) \} \int \int u^2(12) v^2(23) d\tau_2 d\tau_{13}, \\
 Y(H^3) &= - \int \int \int v(23) \Delta_2 v(23) \int \int u^2(12) u^2(13) d\tau_1 d\tau_{23}, \\
 Z(H^3) &= 2 \int \int \int u^2(13) J(13) \int \int [u^2(12) v^2(23)] d\tau_2 d\tau_{13}, \\
 Q(He^4) &= \int \dots \int u^2(23) \nabla_1 u^2(13) \cdot \nabla_1 v^2(12) \int \dots \int u^2(24) u^2(14) v^2(34) d\tau_3 d\tau_4 d\tau_{12}, \\
 X(He^4) &= 2 \int \int \int \{ |\nabla_1 u(13)|^2 - u(13) \Delta_1 u(13) \} \int \dots \int u^2(14) u^2(23) u^2(24) v^2(12) v^2(34) d\tau_2 d\tau_4 d\tau_{13}, \\
 Y(He^4) &= -2 \int \int \int v(12) \Delta_1 v(12) \int \dots \int u^2(13) u^2(14) u^2(23) u^2(24) v^2(34) d\tau_3 d\tau_4 d\tau_{12}, \\
 Z(He^4) &= 4 \int \int \int u^2(13) J(13) \int \dots \int u^2(24) [u^2(23) u^2(14) v^2(12) v^2(34)] d\tau_2 d\tau_4 d\tau_{13}.
 \end{aligned} \tag{10}$$

Eq. (10) is correct for the Wigner operator and with the exception of $Z(H^3)$ and $Z(He^4)$ also for that of Majorana; the correct Z 's in the latter theory are obtained if in the square brackets the functions $u^2(a1)$, $u^2(b3)$ are replaced by $u(a1) \cdot u(a3)$ and $u(b3) u(b1)$, respectively, and the same substitution applied to $v^2(a1)$, $v^2(b3)$. The symbols (W) and (M) will be used to refer to the two theories.

One result we get immediately from (10) without further calculation: In the $H^3(W)$ problem, if v is replaced by a constant, Y vanishes and X and Z each reduce to twice the corresponding terms in the two-body variation problem. Hence for a constant v , the u which minimizes the three-body energy integral is the solution of the two-body equation.³ Consequently

$$-E(H^3) > -2E(H^2) = 8.0, \quad (W). \tag{11}$$

There is another more general argument which gives (11) again and a corresponding lower bound for the alpha-particle. The ratio of the numbers of potential and kinetic energy terms in the energy operator increases in passing from H^2 to H^3 and again in passing from H^3 to He^4 . Moreover, the wave functions overlap, causing the potentials to overlap also, so that the magnitude of the effective potential in which each particle moves increases along the series H^2 , H^3 , He^4 . Also the wave functions in all three problems have no nodes. For these reasons, and most important of all, because both the depth and breadth of the potential hole are changed together in such a manner that the binding energy of the two-body problem remains fixed at the experimental value, the exact binding energies of the three- and four-body problems must increase

with decreasing effective width of the potential hole and more rapidly for the alpha-particle than for the three-body nucleus. With a hole of infinite width the kinetic energy vanishes and the potential energy is just the number of particles times the limiting value of A which is 4. This gives (11) again and the following lower bound for the binding energy of the alpha-particle:

$$-E(He^4) > -4E(H^2) = 16.0, \quad (W). \tag{12}$$

To make clear the difficulties which must be overcome in applying the variation method to the three- and four-body problems a brief discussion of the two-body variation problem is inserted here. As the breadth of the potential well is narrowed the average correct kinetic and potential energies grow rapidly larger. This is associated with a constant value of the difference which, except for very broad holes, is a small fraction of either energy term alone. It is evident that an approximate wave function must meet more and more rigorous conditions as the hole is narrowed if the variation method is to give uniformly good results. If the wave function is not continually improved the computed binding energy will fall more and more below the correct value and for a sufficiently narrow hole will become negative. This conclusion is brought out clearly in Table III which is based on Eq. (5). At $\alpha = 20$ the com-

TABLE III. The variation method applied to H^2 .

α	K.E.*	-P.E.*	$-E^0$
20	18.4	19.4	1.4
30	22.6	22.7	0.1
50	27.2	24.7	-2.5

* K.E.—kinetic energy.
P.E.—potential energy.

puted energy is only thirty-five percent of the correct value. However, a comparison of the correct and approximate wave functions reveals that they agree quite well up to a distance of separation which includes eighty percent of the probability density. If $-E^0(H^2)$ is plotted against α the resulting curve is a straight line with an intercept at $\alpha=0$ having exactly the correct value, $-E(H^2)=4$. *Evidently for a sufficiently broad potential hole the variation method gives excellent results.* From Table III we draw the conclusion that if the binding energy of the three-body problem increases quite slowly with increasing α , a simple variation method is likely to yield values

for $-E^0(H^3)$ which decrease with increasing α .

Returning to the many-body problem it is evident that in general the inside integrations in (10) present formidable difficulties. Fortunately there does exist a functional form for which the integration is elementary and the result simple. This form is the Gaussian error function which has already been taken for the potential. Setting

$$u = Ne^{-\nu r^2/2}, \quad v = Ne^{-\mu r^2/2} \quad (13)$$

we find after an elementary calculation the following explicit expressions for the energies with the Wigner Hamiltonian:

$$E^0(H^3) = 3/2 \{ (2+3n+n^2)/(1+2n) \} \alpha\sigma - 2A(\sigma/\sigma+1)^{3/2}, \quad n = \mu/\nu, \quad \alpha\sigma = \nu(1+2n)/(1+n), \quad (14)$$

$$E^0(He^4) = 3/4 \{ (6+5n+n^2)/(1+n) \} \alpha\sigma - 4A(\sigma/(\sigma+1))^{3/2}, \quad n = \mu/\nu, \quad \alpha\sigma = 4\nu(1+n)/(3+n). \quad (15)$$

The substitution into (14) and (15) of the value of n for which the energies are as small as possible leaves expressions for the energies containing only one variable parameter:

$$E^0(H^3) = 2.7991\alpha\sigma - 2A(\sigma/(\sigma+1))^{3/2}, \quad 2n = \sqrt{3}-1, \quad \nu = 0.7887\alpha\sigma, \quad (16)$$

$$E^0(He^4) = 4.3713\alpha\sigma - 4A(\sigma/(\sigma+1))^{3/2}, \quad n = \sqrt{2}-1, \quad \nu = 0.6036\alpha\sigma. \quad (17)$$

The corresponding results for the Majorana Hamiltonian are

$$E^0(H^3) = 6 \{ (2+n)/(5+n) \} \alpha\sigma - 16A \{ (1+2n)/(5+6n+n^2) \}^{3/2} (\sigma/(\sigma+1))^{3/2}, \\ n = \mu/\nu, \quad 4\alpha\sigma = \nu(5+n), \quad (18)$$

$$E^0(He^4) = 6 \{ (2+n)/(3+n) \} \alpha\sigma - 64 \times 2^{3/2} A \{ (n+1)^{3/2}/(n+3)^{3/2} \} (\sigma/(\sigma+1))^{3/2}, \\ n = \mu/\nu, \quad 2\alpha\sigma = \nu(3+n). \quad (19)$$

In this case the best choice of n depends on the value given to σ , but the dependence is so slight that it may be ignored. It was found by trial that $n=0.5$ and $n=0.7$ give about the best results in (18) and (19), respectively. With these values of n the expressions for the energies are

$$E^0(H^3) = 2.7273\alpha\sigma - 1.9098A(\sigma/(\sigma+1))^{3/2}, \quad n = 0.5, \quad \nu = 0.7273\alpha\sigma, \quad (20)$$

$$E^0(He^4) = 4.3784\alpha\sigma - 3.9606A(\sigma/(\sigma+1))^{3/2}, \quad n = 0.7, \quad \nu = 0.5405\alpha\sigma. \quad (21)$$

The coulomb energy corrections ΔE for He^3 and He^4 due to repulsion between the protons are computed by a first order perturbation calculation and prove to be

$$\Delta E(He^3) = (\nu+2\mu)^{1/2}/4, \quad \Delta E(He^4) = (2\nu+2\mu)^{1/2}/4. \quad (22)$$

Numerical results obtained from (16), (17), (20), (21), (22) in conjunction with Table II are collected in the Tables IV and V.

TABLE IV. *The variation method applied to He⁴.*

α	Effective radius		A	Eq. (17) (W)			Eq. (21) (M)	
	$\bar{\alpha}^{1/2} \times 10^{13}$ cm			K.E.	-P.E.	$-E^0$	$-E^0$	C.E.*
20	0.224	2.00	84.4	90.2	122.1	31.9	30.6	1.5
30	0.183	1.64	117.1	121.7	156.4	34.7		1.7
40	0.158	1.42	148.8	151.2	187.9	36.7		1.9
50	0.141	1.27	180.0	179.6	218.1	38.5	36.1	2.1
75	0.116	1.04	256.4	247.2	289.0	41.8	38.6	2.5

* C.E.—Coulomb energy.

TABLE V. *The variation method applied to H³, He³.*

α	Eq. (16) (W)			Eq. (20) (M)		He ³ only C.E.*
	K.E.	-P.E.	$-E_0$	$-E_0$		
20	39.5	44.9	5.4	4.5	1.1	
50	67.9	67.2	-0.7	-2.0	1.4	
75	83.7	78.0	-5.7	-7.0	1.6	

* C.E.—Coulomb energy.

The rapid increase of $-E^0(\text{He}^4)$ with α indicates that the variation method works quite well when applied to the four-body problem. It yields values for $E^0(\text{He}^4)$ lying between 3/5 and 4/5 of the experimental binding energy, the largest value occurring for the smallest effective radius in the range 10^{-13} cm to 2×10^{-13} cm. However the method is unsatisfactory in the three-body problem as Table V shows. The striking feature of Table V is that $-E^0(\text{H}^3)$ decreases with decreasing effective radius and actually becomes negative for sufficiently small values of $1/\alpha^{1/2}$. But this result we had reason to anticipate. It is proper to compare $2E^0(\text{H}^2)$ with $E^0(\text{H}^3)$ (because there are two potential terms in the three-body Hamiltonian and only one in the equation for the deuteron. The tables show that the kinetic energy of H^2 is less than half that of H^3). From the tables it is seen that $2E^0(\text{H}^2) - E^0(\text{H}^3)$ increases as the effective radius is made smaller showing that the variation method is less unsatisfactory for H^3 than for H^2 .

The energy intercept at $\alpha=0$ is equal to the limiting value of the coefficient of $(\sigma/(\sigma+1))^3$ in the energy formula. The minimum property of the variation method and the fact that the actual binding energy $-E(\text{H}^3)$ must increase with α yields (11) again and a new result

$$-E(\text{H}^3) > 7.64, \quad (M). \quad (23)$$

The energies given by the Majorana theory lie

above those of the Wigner theory as required by a general theorem due to Eckart¹⁰ on the relation between the two forms of the interaction operator.

SECTION IV. THE METHOD OF THE "EQUIVALENT" TWO-BODY EQUATION

The results of the preceding section indicate that the effective radius of interaction falls somewhere in the range 10^{-13} to 2×10^{-13} cm. To get a more definite result we might attempt to improve the variation method by using better wave functions. But any simple change in the wave function immediately complicates the calculation many-fold and is not likely, in the case of H^3 , to yield better results than we already have in (11) and (23). We must help ourselves in some other way. The method adopted by the writer is based on analogy and considerations of plausibility although it appears that a rigorous justification may be possible.

We observe that the variation Eqs. (5), (16), (17), (20) and (21) all have the form

$$E^0 = 3\beta' \sigma / 4m - B(\sigma/(\sigma+1))^3 \quad (24)$$

with suitable choice of m , β' and B . But in the case of the general two-body problem with reduced mass m (24), is obtained from the equation

$$\{d^2/ds^2 + E' + Be^{-\beta s^2}\} \varphi = 0 \quad (25)$$

in which $\beta = \beta'/2m$ and $s = (2m)^{1/2}r$ (cf. fine print in Section II). We therefore associate with each many-body problem an equation of the form (25) in which B and β have the following values fixed by direct comparison of (24) with (16), (17), (20) and (21):

$$\left. \begin{aligned} B &= 4A \\ \beta &= 2.9142\alpha \end{aligned} \right\} \text{He}^4(W),$$

$$\left. \begin{aligned} B &= 2A \\ \beta &= 1.8660\alpha \end{aligned} \right\} \text{H}^3, \text{He}^3, (W),$$

$$\left. \begin{aligned} B &= 3.9606A \\ \beta &= 2.9189\alpha \end{aligned} \right\} \text{He}^4, (M),$$

$$\left. \begin{aligned} B &= 1.9098A \\ \beta &= 1.8182\alpha \end{aligned} \right\} \text{H}^3, \text{He}^3, (M). \quad (26)$$

¹⁰ Eckart, Phys. Rev. 44, 109 (1933).

Note that in each case B and β are uniquely determined without regard to the value assumed for m . Now by the minimum property of the variation method, the lowest eigenvalue of (25), say E' , lies below the corresponding E^0 .

For the deuteron E' is the correct eigenvalue. In the case of the alpha-particle the depth and breadth of the "equivalent" potential are so large that (24) is able to yield fairly good results and hence is closely correlated with the differential Eq. (25). Since (25) is the correct deuteron equation and is closely related to the He^4 problem, the correlation between (24) and (25) should also be reasonably close for H^3 which falls between the deuteron and the alpha-particle and possesses the same type of nodeless wave function. We are led to suppose that E' should lie fairly close to the eigenvalue of the associated many-body problem. It is then obviously possible, although not necessary, to interpret (25) in terms of a physical model: for H^3 , a neutron interacting with a deuteron, the neutron spins being parallel; in the case of the alpha-particle, the interaction of two deuterons with antiparallel spins or a neutron interacting with He^3 (neutron spins antiparallel). Values of E' computed in part directly by numerical integration and partly by the method described in Section II are presented in Table VI which should be compared with Tables

IV and V based on the cruder method of Section III. It is seen that $-E'(\text{He}^4)$ crosses the experimental binding energy in the neighborhood of $\alpha = 50$. Here the computed coulomb energy for He^3 has the value 1.4. If this coulomb correction is subtracted from the *experimental* difference $E(\text{He}^3) - E(\text{H}^3) = 2.6$ there is left a remainder of 1.2 mc^2 which must be ascribed primarily to small *attractive* forces acting between the neutrons in H^3 . But the approximate wave functions are too large for small separations and too small for large separations. Also the second order coulomb correction is negative. Hence this coulomb correction is probably excessive. Furthermore it is likely that for small separations the actual proton-proton potential lies far below the coulomb value and even changes sign.¹¹ All these considerations indicate that the proton-proton interaction in He^3 may be somewhat smaller than 1.4 mc^2 with the consequence that the neutron-neutron interaction energy in H^3 may be larger than 1.2 mc^2 . In any event the sum of the two corrections should be small enough to permit a direct comparison of $-E(\text{He}^4)$ with the experimental value of 54 mc^2 .

We then find with the Wigner theory that $-E(\text{He}^4) = 54$ at the point $\alpha = 45(1/\alpha^3 = 0.149$ or $1.34 \times 10^{-13} \text{ cm})$, $A = 165$. The corresponding point for the Majorana theory is $\alpha = 48(1/\alpha^3 = 0.144$ or $1.29 \times 10^{-13} \text{ cm})$, $A = 174$. At these points the computed binding energy of H^3 is greater than $-E'(\text{H}^3) + 1.2 = 12.7$ (W) or $-E'(\text{H}^3) + 1.2 = 11.2$ (M). The agreement with the experimental value of 16.0 ± 1.7 is good enough to serve as a justification for the original assumption that the neutron-proton interaction operator can be represented fairly well by a potential function in the case of binding. The uncertainty in the method of calculation is too great to permit any conclusion about the relative merits of the two theories for the light nuclei. Incidentally in all cases the normal state eigenvalue of the "equivalent" equation is the only energy level in the discrete spectrum.

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TABLE VI. The "equivalent" two-body method.

	β	α	$-E'$		β	α	$-E'$
$\text{He}^4(W)$	80.0	27.5	43.7	$\text{He}^4(M)$	105.1	36.0	47.2
	104.9	36.0	49.1		157.6	54.0	57.1
	120.0	41.2	51.9		200.3	68.6	64.0
	157.4	54.0	59.3				
	200.0	68.6	66.8				
$\text{He}^3(W)$	37.3	20.0	10.2	$\text{H}^3(M)$	36.4	20.0	9.0
	93.3	50.0	11.7		90.9	50.0	10.1
	140.0	75.0	12.8		136.4	75.0	10.7

IV and V based on the cruder method of Section III. It is seen that $-E'(\text{He}^4)$ crosses the experimental binding energy in the neighborhood of $\alpha = 50$. Here the computed coulomb energy for He^3 has the value 1.4. If this coulomb correction

¹¹ White, Phys. Rev. 47, 573 (1935).