Hyperfine Structure Formulae for the Configuration d^2s

Application to $5d^26s$ ^4F States of La I*

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Hfs interval factor formulae for the states of d^2s are derived by the method of Breit and Wills. The formulae for intermediate coupling are expressed in two forms: one involving the coefficients C's, the other the coefficients K's, corresponding to the representation of the functions of the states in intermediate coupling as a linear combination of the (jj) and the (LS) functions, respectively. The $(LS)\leftrightarrow(jj)$ transformation matrices as well as the interval factor formulae for (LS) and (jj) coupling are also given. As the coefficients are determined from the empirical multiplet separations, the energy matrices in both (LS) and (jj) coupling are listed. The theory is applied to the $5d^26s$ ^4F states of La I and consistently accounts for the relative hfs separations of these states. The values obtained for the interaction constants of the $6s$ and $5d$ electrons are 0.119 and 0.0039 cm^{-1}, respectively. The nuclear g-factor computed from these is $g(\mathrm{I})=0.71$. The corresponding value of the nuclear magnetic moment is 2.5 nuclear magnetons.

I. INTRODUCTION

BREIT and Wills[1] extended the relativistic theory of hyperfine structure[2] to intermediate coupling and derived the interval factor formulae for several important types of configurations. A number of other configurations, types with an unpaired s electron which usually give rise to measurable hyperfine separations, merit consideration. The configuration d^2s is frequently encountered in the spectra of the elements with partially filled d shells. As the data on the nuclear magnetic moments of these elements are meager, it is of some importance to treat d^2s by the same method; particularly since by so doing one obtains the interval factors of the states individually rather than the sum of the interval factors of all states with the same J, and is thus able to evaluate the nuclear magnetic moment when the hyperfine separations of only a few states are known. Hyperfine structure formulae for d^2s are presented in this paper and applied to the observed structures of the $5d^26s$ ^4F states of La I.

II. WAVE FUNCTIONS AND INTERVAL FACTOR FORMULAE IN INTERMEDIATE COUPLING

The configuration d^2s gives rise to sixteen states: two with $J=9/2$, three with $J=7/2$, four with $J=5/2$, four with $J=3/2$, and three with $J=1/2$. As the relativistic treatment of hyperfine structure must be made via (jj) coupling, the configuration will be considered first in this coupling. The functions representing the states in (jj) coupling can be conveniently thought of as arising from the coupling of the states of d^2 with an s electron. The states of d^2 are given in Table I. They are designated by

TABLE I. *States of* d^2.

$(j_1, j_2)=$	(5/2, 5/2),			(5/2, 3/2),				(3/2, 3/2)	
Resultant $J=$	4	2	0,	4	3	2	1,	2	0
Symbol	ψ^4	ψ^2	ψ^0	φ^4	φ^3	φ^2	φ^1	χ^2	χ^0

Greek letters with their J values indicated by superscripts.

The calculations can be made expediently by using eigenfunctions for a given magnetic quantum number. These functions of d^2 states will be represented by the state symbols with the magnetic quantum number m added as a subscript. It is necessary in the calculation to express the two-election functions in terms of the one-electron functions, which for a given j and m will be represented by symbols of the type $(j)_m$. The two-electron functions for the states with $m=J$, expressed in terms of the one-electron functions, are given here. The functions for other values of m can readily be obtained from these by the method of Gray and Wills.[3]

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[1] G. Breit and L. A. Wills, Phys. Rev. **44**, 470 (1933).

[2] G. Breit, Phys. Rev. **35**, 1447 (1930); **38**, 463 (1931); G. Racah, Zeits. f. Physik **71**, 431 (1931); E. Fermi and E. Segrè, Zeits. f. Physik **82**, 729 (1933).

[3] N. Gray and L. A. Wills, Phys. Rev. **38**, 248 (1931).

(5 2, 5/2) Sub-group.

$\psi_4{}^4=(1/\sqrt{2})[(5/2)_{5/2},\ (5/2)_{3/2}].$

$\psi_2{}^2=((5/14)^{\frac{1}{2}}1/\sqrt{2})[(5/2)_{5/2},\ (5/2)_{-\frac{1}{2}}]-((9/14)^{\frac{1}{2}}1/\sqrt{2})[(5/2)_{3/2},\ (5/2)_{\frac{1}{2}}].$

$\psi_0{}^0=(1/\sqrt{3}\cdot 1/\sqrt{2})[(5/2)_{5/2},\ (5/2)_{-5/2}]-(1/\sqrt{3}\cdot 1/\sqrt{2})[(5/2)_{3/2},\ (5/2)_{-3/2}]$
$+(1/\sqrt{3}\cdot 1/\sqrt{2})[(5/2)_{\frac{1}{2}},\ (5/2)_{-\frac{1}{2}}].$

(5/2, 3/2) Sub-group.

$\varphi_4{}^4=(1/\sqrt{2})[(5/2)_{5/2},\ (3/2)_{3/2}].$

$\varphi_3{}^3=((3/8)^{\frac{1}{2}}1/\sqrt{2})[(5/2)_{3/2},\ (3/2)_{3/2}]-((5/8)^{\frac{1}{2}}1/\sqrt{2})[(5/2)_{5/2},\ (3/2)_{\frac{1}{2}}].$

$\varphi_2{}^2=((10/21)^{\frac{1}{2}}1/\sqrt{2})[(5/2)_{5/2},\ (3/2)_{-\frac{1}{2}}]-((8/21)^{\frac{1}{2}}1/\sqrt{2})[(5/2)_{3/2},\ (3/2)_{\frac{1}{2}}]$
$+((3/21)^{\frac{1}{2}}1/\sqrt{2})[(5/2)_{\frac{1}{2}},\ (3/2)_{3/2}].$

$\varphi_1{}^1=((10/20)^{\frac{1}{2}}1/\sqrt{2})[(5/2)_{5/2},\ (3/2)_{-3/2}]-((6/20)^{\frac{1}{2}}1/\sqrt{2})[(5/2)_{3/2},\ (3/2)_{-\frac{1}{2}}]$
$+((3/20)^{\frac{1}{2}}1/\sqrt{2})[(5/2)_{\frac{1}{2}},\ (3/2)_{\frac{1}{2}}]-((1/20)^{\frac{1}{2}}1/\sqrt{2})[(5/2)_{-\frac{1}{2}},\ (3/2)_{3/2}].$

(3/2, 3/2) Sub-group.

$\chi_2{}^2=(1/\sqrt{2})[(3/2)_{3/2},\ (3/2)_{\frac{1}{2}}].$

$\chi_0{}^0=(1/\sqrt{2}\cdot 1/\sqrt{2})[(3/2)_{\frac{1}{2}},\ (3/2)_{-\frac{1}{2}}]-(1/\sqrt{2}\cdot 1/\sqrt{2})[(3/2)_{3/2},\ (3/2)_{-3/2}].$

Each square bracket in the preceding equations is an abbreviation for the determinant representation of a two-electron function as a combination of products of two one-electron functions.

The states of d^2s will now be considered. The functions of the s electron will be symbolized by s_m (as $j=\frac{1}{2}$ always it need not be specified). They can be combined with the functions for the d^2 states without paying attention to symmetry.

$J=9/2$. The functions representing the two states with $J=9/2$, $m=9/2$ in (jj) coupling can be written

$$\mathrm{I}=s_{\frac{1}{2}}\psi_4{}^4,\quad \mathrm{II}=s_{\frac{1}{2}}\varphi_4{}^4. \tag{1}$$

The function representing a state with $J=9/2$, $m=9/2$ in intermediate coupling, symbolically written $(9/2)_{9/2}$, then is

$$(9/2)_{9/2}=C_1\mathrm{I}+C_2\mathrm{II}, \tag{2}$$

where C_1 and C_2 are numerical coefficients whose squares sum to unity. The interval factor formula obtained from this function by the procedure of Breit and Wills[1] is

$$9/2\cdot A(J=9/2)=1/2\cdot a_s(C_1{}^2+C_2{}^2)+a'(4C_1{}^2+5/2\cdot C_2{}^2)+3/2\cdot a''C_2{}^2+4C_1C_2a'''. \tag{3}$$

a_s, a', a'', a''' represent the hyperfine structure interaction or coupling constants as in the paper of Breit and Wills. Eq. (3) is the general expression for the interval factors of the states with $J=9/2$ in intermediate coupling. The expression involves the coefficients C_1 and C_2, and there is a different set of these for each of the two states.

$J=7/2$. The functions representing the three states with $J=7/2$, $m=7/2$ in (jj) coupling are

$$\mathrm{I}=(8/9)^{\frac{1}{2}}s_{-\frac{1}{2}}\psi_4{}^4-1/\sqrt{9}\cdot s_{\frac{1}{2}}\psi_3{}^4,\quad \mathrm{II}=(8/9)^{\frac{1}{2}}s_{-\frac{1}{2}}\varphi_4{}^4-1/\sqrt{9}\cdot s_{\frac{1}{2}}\varphi_3{}^4,\quad \mathrm{III}=s_{\frac{1}{2}}\varphi_3{}^3. \tag{4}$$

In intermediate coupling the general function for a state with $J=7/2$, $m=7/2$ is

$$(7/2)_{7/2}=C_1I+C_2\mathrm{II}+C_3\mathrm{III}. \tag{5}$$

From this we obtain

$$7/2\cdot A(J=7/2)=a_s(-7/18\cdot C_1^2-7/18\cdot C_2^2+1/2\cdot C_3^2)+a'(35/9\cdot C_1^2+175/72\cdot C_2^2+17/8\cdot C_3^2$$
$$+\sqrt{15}/12\cdot C_2C_3)+a''(35/24\cdot C_2^2+7/8\cdot C_3^2-\sqrt{15}/12\cdot C_2C_3)+a'''(35/9\cdot C_1C_2+2(5/12)^{\frac{1}{2}}\cdot C_1C_3). \quad (6)$$

$J=5/2$. The functions representing the four states with $J=5/2$, $m=5/2$ in (jj) coupling are

$$\mathrm{I}=s_{\frac{1}{2}}\psi_2^2,\quad \mathrm{II}=(6/7)^{\frac{1}{2}}s_{-\frac{1}{2}}\varphi_3^3-(1/7)^{\frac{1}{2}}s_{\frac{1}{2}}\varphi_2^3,\quad \mathrm{III}=s_{\frac{1}{2}}\varphi_2^2,\quad \mathrm{IV}=s_{\frac{1}{2}}\chi_2^2. \quad (7)$$

In intermediate coupling the general function is

$$(5/2)_{5/2}=C_1\mathrm{I}+C_2\mathrm{II}+C_3\mathrm{III}+C_4\mathrm{IV}. \quad (8)$$

From this we obtain

$$5/2\cdot A(J=5/2)=a_s(\tfrac{1}{2}C_1^2-5/14\cdot C_2^2+\tfrac{1}{2}C_3^2+\tfrac{1}{2}C_4^2)+a'(2C_1^2+85/42\cdot C_2^2+11/6\cdot C_3^2$$
$$+16/21\sqrt{2}\cdot C_2C_3)+a''(5/6\cdot C_2^2+\tfrac{1}{6}C_3^2+2C_4^2-16/21\sqrt{2}\cdot C_2C_3)$$
$$+a'''(-4/7\cdot C_1C_2+4\sqrt{2}\cdot C_1C_3+8/\sqrt{21}\cdot C_2C_4-14(2/21)^{\frac{1}{2}}C_3C_4). \quad (9)$$

$J=3/2$. The functions for the four states with $J=3/2$, $m=3/2$ in (jj) coupling are:

$$\mathrm{I}=2/\sqrt{5}\cdot s_{-\frac{1}{2}}\psi_2^2-1/\sqrt{5}\cdot s_{\frac{1}{2}}\psi_1^2,\quad \mathrm{II}=2/\sqrt{5}\cdot s_{-\frac{1}{2}}\varphi_2^2-1/\sqrt{5}\cdot s_{\frac{1}{2}}\varphi_1^2,\quad \mathrm{III}=s_{\frac{1}{2}}\varphi_1^1,$$
$$\mathrm{IV}=2/\sqrt{5}\cdot s_{-\frac{1}{2}}\chi_2^2-1/\sqrt{5}\cdot s_{\frac{1}{2}}\chi_1^2. \quad (10)$$

In intermediate coupling the general function is

$$(3/2)_{3/2}=C_1\mathrm{I}+C_2\mathrm{II}+C_3\mathrm{III}+C_4\mathrm{IV}. \quad (11)$$

From this we obtain

$$3/2\cdot A(J=3/2)=a_s(-3/10\cdot C_1^2-3/10\cdot C_2^2+\tfrac{1}{2}C_3^2-3/10\cdot C_4^2)+a'(9/5\cdot C_1^2+33/20\cdot C_2^2$$
$$+7/4\cdot C_3^2-3\sqrt{7/10}\cdot C_2C_3)+a''(3/20\cdot C_2^2-\tfrac{3}{4}C_3^2+9/5\cdot C_4^2+3\sqrt{7/10}\cdot C_2C_3)$$
$$+a'''(18\sqrt{2}/5\cdot C_1C_2-2\sqrt{14/5}\cdot C_1C_3-3\sqrt{42/5}\cdot C_2C_4+\sqrt{6/5}\cdot C_3C_4). \quad (12)$$

$J=1/2$. The functions for the three states with $J=\frac{1}{2}$, $m=\frac{1}{2}$ in (jj) coupling are

$$\mathrm{I}=s_{\frac{1}{2}}\psi_0^0,\quad \mathrm{II}=(2/3)^{\frac{1}{2}}s_{-\frac{1}{2}}\varphi_1^1-1/\sqrt{3}\cdot s_{\frac{1}{2}}\varphi_0^1,\quad \mathrm{III}=s_{\frac{1}{2}}\chi_0^0. \quad (13)$$

The general function in intermediate coupling is

$$(1/2)_{\frac{1}{2}}=C_1\mathrm{I}+C_2\mathrm{II}+C_3\mathrm{III}. \quad (14)$$

From this we obtain

$$\tfrac{1}{2}A(J=\tfrac{1}{2})=a_s(\tfrac{1}{2}C_1^2-\tfrac{1}{6}C_2^2+\tfrac{1}{2}C_3^2)+7/6\cdot C_2^2a'-\tfrac{1}{2}C_2^2a''+a'''(20/3\sqrt{5}\cdot C_1C_2+2(10/3)^{\frac{1}{2}}\cdot C_2C_3). \quad (15)$$

III. Interval Factor Formulae in (jj) and (LS) Coupling, and the $(jj)\leftrightarrow(LS)$ Transformation Matrices

The interval factors of the states in (jj) coupling can be obtained directly from the preceding formulae. Consider the states with $J=9/2$. It is obvious that for one of these states in (jj) coupling $C_1=1$, $C_2=0$; and for the other $C_1=0$, $C_2=1$. The substitution of these values in Eq. (3) gives the following expressions for the (jj) interval factors.

$J=9/2$. Sub-group.

$$(5/2,\,5/2,\,\tfrac{1}{2}):\quad A(J=9/2)=1/9\cdot a_s+8/9\cdot a'.$$
$$(5/2,\,3/2,\,\tfrac{1}{2}):\quad A(J=9/2)=1/9\cdot a_s+5/9\cdot a'+\tfrac{1}{3}\cdot a''. \quad (16)$$

Similarly for the other states we get:

J=7/2. Sub-group.

$$(5/2,\,5/2,\,\tfrac{1}{2}):\quad A(J=7/2)=-1/9\cdot a_s+10/9\cdot a'.$$
$$(5/2,\,3/2;\,j=4;\,\tfrac{1}{2}):\quad A(J=7/2)=-1/9\cdot a_s+25/36\cdot a'+5/12\cdot a''. \tag{17}$$
$$(5/2,\,3/2;\,j=3;\,\tfrac{1}{2}):\quad A(J=7/2)=1/7\cdot a_s+17/28\cdot a'+\tfrac{1}{4}a''.$$

J=5/2. Sub-group.

$$(5/2,\,5/2,\,\tfrac{1}{2}):\quad A(J=5/2)=\tfrac{1}{5}a_s+4/5\cdot a'.$$
$$(5/2,\,3/2;\,j=3;\,1/2):\quad A(J=5/2)=-1/7\cdot a_s+17/21\cdot a'+1/3\cdot a''.$$
$$(5/2,\,3/2;\,j=2;\,\tfrac{1}{2}):\quad A(J=5/2)=\tfrac{1}{5}a_s+11/15\cdot a'+1/15\cdot a''. \tag{18}$$
$$(3/2,\,3/2,\,\tfrac{1}{2}):\quad A(J=5/2)=\tfrac{1}{5}a_s+4/5\cdot a''.$$

J=3/2. Sub-group.

$$(5/2,\,5/2,\,\tfrac{1}{2}):\quad A(J=3/2)=-\tfrac{1}{5}a_s+6/5\cdot a'.$$
$$(5/2,\,3/2;\,j=2;\,\tfrac{1}{2}):\quad A(J=3/2)=-\tfrac{1}{5}a_s+11/10\cdot a'+1/10\cdot a''.$$
$$(5/2,\,3/2;\,j=1;\,\tfrac{1}{2}):\quad A(J=3/2)=\tfrac{1}{3}a_s+7/6\cdot a'-\tfrac{1}{2}a''. \tag{19}$$
$$(3/2,\,3/2,\,\tfrac{1}{2}):\quad A(J=3/2)=-\tfrac{1}{5}a_s+6/5\cdot a''.$$

J=1/2. Sub-group.

$$(5/2,\,5/2,\,\tfrac{1}{2}):\quad A(J=\tfrac{1}{2})=a_s.$$
$$(5/2,\,3/2,\,\tfrac{1}{2}):\quad A(J=\tfrac{1}{2})=-\tfrac{1}{3}a_s+7/3\cdot a'-a''. \tag{20}$$
$$(3/2,\,3/2,\,\tfrac{1}{2}):\quad A(J=\tfrac{1}{2})=a_s.$$

TABLE II. *Transformation matrices, (jj)⟷(LS).*

$J=9/2.$	$s\psi^4$	$s\varphi^4$
$^4F_{9/2}$	$2/\sqrt{5}$	$1/\sqrt{5}$
$^2G_{9/2}$	$-1/\sqrt{5}$	$2/\sqrt{5}$

$J=7/2.$	$s\psi^4$	$s\varphi^4$	$s\varphi^3$
$^4F_{7/2}$	$1/\sqrt{5}$	$\frac{1}{2}/\sqrt{5}$	$\sqrt{3}/2$
$^2G_{7/2}$	$1/\sqrt{5}$	$-2/\sqrt{5}$	0
$^2F_{7/2}$	$3/\sqrt{15}$	$3/2\sqrt{15}$	$-\frac{1}{2}$

$J=5/2.$	$s\psi^2$	$s\varphi^3$	$s\varphi^2$	$s\chi^2$
$^4F_{5/2}$	$2/5\sqrt{5}$	$-5/3\sqrt{5}$	$-8\sqrt{2}/15\sqrt{5}$	$-4\sqrt{21}/15\sqrt{5}$
$^2F_{5/2}$	$1/5$	$\frac{2}{3}$	$-4\sqrt{2}/15$	$-2\sqrt{21}/15$
$^2D_{5/2}$	$6/5\sqrt{3}$	0	$-\sqrt{6}/5$	$\sqrt{21}/5\sqrt{3}$
$^4P_{5/2}$	$4\sqrt{7}/5\sqrt{10}$	0	$3\sqrt{7}/5\sqrt{5}$	$-\sqrt{6}/5\sqrt{5}$

$J=3/2.$	$s\psi^2$	$s\varphi^2$	$s\varphi^1$	$s\chi^2$
$^4F_{3/2}$	$3/5\sqrt{5}$	$-4\sqrt{2}/5\sqrt{5}$	0	$-2\sqrt{21}/5\sqrt{5}$
$^2D_{3/2}$	$2\sqrt{3}/5$	$-\sqrt{6}/5$	0	$\sqrt{7}/5$
$^4P_{3/2}$	$2\sqrt{21}/15\sqrt{5}$	$\sqrt{21}/5\sqrt{10}$	$-5\sqrt{3}/3\sqrt{10}$	$-1/5\sqrt{5}$
$^2P_{3/2}$	$2\sqrt{7}/5\sqrt{3}$	$\sqrt{21}/5\sqrt{2}$	$1/\sqrt{6}$	$-1/5$

$J=1/2.$	$s\psi^0$	$s\varphi^1$	$s\chi^0$
$^4P_{1/2}$	$2/\sqrt{15}$	$1/\sqrt{3}$	$\sqrt{2}/\sqrt{5}$
$^2P_{1/2}$	$\sqrt{2}/\sqrt{15}$	$-\sqrt{2}/\sqrt{3}$	$1/\sqrt{5}$
$^2S_{1/2}$	$\sqrt{3}/\sqrt{5}$	0	$-\sqrt{2}/\sqrt{5}$

The interval factors of the states in (*LS*) coupling can be obtained from Eqs. (3, 6, 9, 12, 15) by using the (*jj*)⟷(*LS*) transformation matrices. These matrices are given in Table II.

The interval factors of the states in (*LS*) coupling, obtained by substituting the appropriate coefficients from the transformation matrices in Eqs. (3, 6, 9, 12, 15), are

$$A(^4F_{9/2})=1/9\cdot a_s+37/45\cdot a'+1/15\cdot a''+16/45\cdot a'''\cong 1/9\cdot a_s+40/63\cdot a_d.$$
$$A(^2G_{9/2})=1/9\cdot a_s+28/45\cdot a'+4/15\cdot a''-16/45\cdot a'''\cong 1/9\cdot a_s+8/9\cdot a_d.$$
$$A(^4F_{7/2})=5/63\cdot a_s+46/63\cdot a'+4/21\cdot a''+16/63\cdot a'''\cong 5/63\cdot a_s+344/441\cdot a_d.$$
$$A(^2G_{7/2})=-1/9\cdot a_s+7/9\cdot a'+\tfrac{1}{3}\cdot a''-4/9\cdot a'''\cong -1/9\cdot a_s+10/9\cdot a_d. \tag{21*}$$
$$A(^2F_{7/2})=-1/21\cdot a_s+19/21\cdot a'+1/7\cdot a''+4/21\cdot a'''\cong -1/21\cdot a_s+122/147\cdot a_d.$$
$$A(^4F_{5/2})=1/105\cdot a_s+1609/2625\cdot a'+991/2625\cdot a''-368/2625\cdot a'''\cong 1/105\cdot a_s+1272/1225\cdot a_d.$$

* Eqs. (21) continued on next page.

$A(^2F_{5/2})=1/21\cdot a_s+232/525\cdot a'+268/525\cdot a''-464/525\cdot a'''\cong 1/21\cdot a_s+296/245\cdot a_d.$

$A(^2D_{5/2})=\frac{1}{5}a_s+14/25\cdot a'+6/25\cdot a''-8/25\cdot a'''\cong\frac{1}{5}a_s+4/5\cdot a_d.$

$A(^4P_{5/2})=\frac{1}{5}a_s+91/125\cdot a'+9/125\cdot a''+168/125\cdot a'''\cong\frac{1}{5}a_s+12/25\cdot a_d.$

$A(^4F_{3/2})=-\frac{1}{5}a_s+46/125\cdot a'+104/125\cdot a''-192/125\cdot a'''\cong-\frac{1}{5}a_s+304/175\cdot a_d.$

$A(^2D_{3/2})=-\frac{1}{5}a_s+42/50\cdot a'+18/50\cdot a''-24/50\cdot a'''\cong-\frac{1}{5}a_s+6/5\cdot a_d.$

$A(^4P_{3/2})=11/45\cdot a_s+1456/1125\cdot a'-606/1125\cdot a''+688/1125\cdot a'''\cong 11/45\cdot a_s-8/225\cdot a_d.$

$A(^2P_{3/2})=-1/9\cdot a_s+434/450\cdot a'+66/450\cdot a''+632/450\cdot a'''\cong-1/9\cdot a_s+34/45\cdot a_d.$

$A(^4P_{\frac{1}{2}})=5/9\cdot a_s+7/9\cdot a'-\frac{1}{3}a''+40/9\cdot a'''\cong 5/9\cdot a_s-4/9\cdot a_d.$

$A(^2P_{\frac{1}{2}})=1/9\cdot a_s+14/9\cdot a'-\frac{2}{3}a''-40/9\cdot a'''\cong 1/9\cdot a_s+4/9\cdot a_d.$

$A(^2S_{\frac{1}{2}})=a_s.$

The relativistic corrections for a d type electron are usually quite small and frequently can be neglected. In this nonrelativistic approximation a', a'', a''' can be expressed in terms of a_d as follows:

$$a'=24/35\cdot a_d,\quad a''=8/5\cdot a_d,\quad a'''=-1/10\cdot a_d, \tag{22}$$

where $a_d=2g\mu_0^2(\overline{r^{-3}})$. The second simplified expression given above for each interval factor in (LS) coupling is obtained by using these nonrelativistic approximations for the a's.

IV. Interval Factors in Intermediate Coupling Expressed in Terms of the Coefficients, K's, of the Transformation Relating the Intermediate to the (LS) States

When the coupling in a configuration tends towards (LS) it is convenient for the application of the theory to have the interval factors expressed in terms of the coefficients, K's, that relate the intermediate states to the (LS) states. The interval factors can be readily converted into this form by making use of the $(jj)\rightarrow(LS)$ transformation matrices. In terms of the K's the interval factors are:

$$9/2\cdot A(J=9/2)=\tfrac{1}{2}a_s(K_1^2+K_2^2)+a'(37/10\cdot K_1^2+14/5\cdot K_2^2-6/5\cdot K_1K_2)$$
$$+a''(3/10\cdot K_1^2+6/5\cdot K_2^2+6/5\cdot K_1K_2)+4a'''(\tfrac{2}{5}K_1^2-\tfrac{2}{5}K_2^2+\tfrac{3}{5}K_1K_2). \tag{23}$$

$$7/2\cdot A(J=7/2)=a_s(5/18\cdot K_1^2-7/18\cdot K_2^2-\tfrac{1}{6}K_3^2-8\sqrt{3}/18\cdot K_1K_3)+a'(23/9\cdot K_1^2+49/18\cdot K_2^2$$
$$+57/18\cdot K_3^2+\tfrac{1}{3}K_1K_2+7/3\sqrt{3}\cdot K_1K_3+2/\sqrt{3}\cdot K_2K_3)+a''(\tfrac{2}{3}K_1^2+7/6\cdot K_2^2$$
$$+\tfrac{1}{2}K_3^2-\tfrac{1}{3}K_1K_2-1/\sqrt{3}\cdot K_1K_3-2/\sqrt{3}\cdot K_2K_3)+a'''(8/9\cdot K_1^2-14/9\cdot K_2^2$$
$$+\tfrac{2}{3}K_3^2-\tfrac{2}{3}K_1K_2+10/3\sqrt{3}\cdot K_1K_3-4/\sqrt{3}\cdot K_2K_3). \tag{24}$$

$$5/2\cdot A(J=5/2)=a_s(1/42\cdot K_1^2+5/42\cdot K_2^2+\tfrac{1}{2}K_3^2+\tfrac{1}{2}K_4^2+40/21\sqrt{5}\cdot K_1K_2)$$
$$+a'(1609/1050\cdot K_1^2+116/105\cdot K_2^2+7/5\cdot K_3^2+91/50\cdot K_4^2-46/15\sqrt{5}\cdot K_1K_2$$
$$+176/35\sqrt{15}\cdot K_1K_3-64\sqrt{7}/175\sqrt{2}\cdot K_1K_4+64\sqrt{3}/105\cdot K_2K_3$$
$$-4\sqrt{14}/35\sqrt{5}\cdot K_2K_4+\sqrt{42}/5\sqrt{5}\cdot K_3K_4)+a''(991/1050\cdot K_1^2+134/105\cdot K_2^2$$
$$+\tfrac{3}{5}K_3^2+9/50\cdot K_4^2+122/105\sqrt{5}\cdot K_1K_2-176\sqrt{3}/105\sqrt{5}\cdot K_1K_3$$
$$+64\sqrt{7}/175\sqrt{2}\cdot K_1K_4-64\sqrt{3}/105\cdot K_2K_3+4\sqrt{14}/35\sqrt{5}\cdot K_2K_4-\sqrt{42}/5\sqrt{5}\cdot K_3K_4)$$
$$+a'''(-184/525\cdot K_1^2-232/105\cdot K_2^2-4/5\cdot K_3^2+84/25\cdot K_4^2$$
$$-556/105\sqrt{5}\cdot K_1K_2-352\sqrt{3}/105\sqrt{5}\cdot K_1K_3+64\sqrt{14}/175\cdot K_1K_4$$
$$-128\sqrt{3}/105\cdot K_2K_3+8\sqrt{14}/35\sqrt{5}\cdot K_2K_4-2\sqrt{42}/5\sqrt{5}\cdot K_3K_4). \tag{25}$$

$$3/2\cdot A(J=3/2)=a_s(-3/10\cdot K_1^2-3/10\cdot K_2^2+11/30\cdot K_3^2-\tfrac{1}{6}K_4^2-4/3\sqrt{5}\cdot K_3K_4)$$
$$+a'(69/125\cdot K_1^2+63/50\cdot K_2^2+728/375\cdot K_3^2+217/150\cdot K_4^2$$
$$+48\sqrt{3}/25\sqrt{5}\cdot K_1K_2-16\sqrt{21}/125\cdot K_1K_3-4\sqrt{21}/25\sqrt{5}\cdot K_1K_4$$
$$-3\sqrt{7}/25\sqrt{5}\cdot K_2K_3+6\sqrt{7}/25\cdot K_2K_4+49/75\sqrt{5}\cdot K_3K_4)+a''(156/125\cdot K_1^2$$
$$+27/50\cdot K_2^2-101/125\cdot K_3^2+11/50\cdot K_4^2-48\sqrt{3}/25\sqrt{5}\cdot K_1K_2$$
$$+16\sqrt{21}/125\cdot K_1K_3+4\sqrt{21}/25\sqrt{5}\cdot K_1K_4+3\sqrt{7}/25\sqrt{5}\cdot K_2K_3$$
$$-6\sqrt{7}/25\cdot K_2K_4+17/25\sqrt{5}\cdot K_3K_4)+a'''(-288/125\cdot K_1^2-18/25\cdot K_2^2$$
$$+344/375\cdot K_3^2+316/150\cdot K_4^2-96\sqrt{3}/25\sqrt{5}\cdot K_1K_2+32\sqrt{21}/125\cdot K_1K_3$$
$$+8\sqrt{21}/25\sqrt{5}\cdot K_1K_4+6\sqrt{7}/25\sqrt{5}\cdot K_2K_3-12\sqrt{7}/25\cdot K_2K_4+502/75\sqrt{5}\cdot K_3K_4). \quad (26)$$

$$\tfrac{1}{2}A(J=\tfrac{1}{2})=a_s(5/18\cdot K_1^2+1/18\cdot K_2^2+\tfrac{1}{2}K_3^2+4\sqrt{2}/9\cdot K_1K_2)+a'(7/18\cdot K_1^2+7/9\cdot K_2^2$$
$$-7\sqrt{2}/9\cdot K_1K_2)+a''(-\tfrac{1}{6}K_1^2-\tfrac{1}{3}K_2^2+\sqrt{2}/3\cdot K_1K_2)$$
$$+a'''(20/9\cdot K_1^2-20/9\cdot K_2^2-10\sqrt{2}/9\cdot K_1K_2). \quad (27)$$

V. Energy Matrices in (*LS*) and (*jj*) Coupling

The coefficients, C's, K's, that appear in the interval factor formulae are obtainable from the empirical energies of the states of the configuration. To evaluate the coefficients from the energies the energy matrices are required, preferably for both (*LS*) and (*jj*) coupling to correspond to the two forms of the interval factor formulae. Condon and Shortley[4] have worked out the electrostatic energies of the d^2s states in (*LS*) coupling. The magnetic energies can readily be worked out in (*jj*) coupling. The complete energy matrices in both (*LS*) and (*jj*) coupling can be formed from these by using the $(jj)\leftarrow(LS)$ transformation matrices. The energy matrices in (*LS*) coupling are given in Table III. The $^4F_{9/2}$ state was chosen as the energy datum level. The (*jj*) energy matrices are given in Table IV. The energies here are referred to $s\psi^4(J=9/2)$ as the datum level.

The energy matrices of d^2s are somewhat too complicated to give explicit formulae for the determination of the coefficients. They are obtained more easily from the matrices by successive approximations to the secular equations.

The coefficients determined from the empirical energies can be checked by computing the Landé g factors from the coefficients and comparing them with the empirical g's. The g's can be computed most easily *via* (*LS*) coupling since the matrices for the g's of the (*LS*) states are diagonal. Thus the g of a state with a given J in intermediate coupling is given by $\sum_i g_iK_i^2$ summed over all the (*LS*) states with the given J. The g_i's are the g factors of the (*LS*) states and the K_i's are the coefficients in the linear combination that expresses the function of the intermediate state in terms of the functions of the (*LS*) states.

[4] E. U. Condon and G. H. Shortley, Phys. Rev. **37**, 1025 (1931).

VI. Application to $5d^26s\ ^4F$ of La I

Anderson[5] has measured the hyperfine structures of the $5d^26s\ ^4F$ states of La I, and from their separations has evaluated the magnetic moment of the lanthanum nucleus. There are two limitations in his computation: First, he considers that the coupling is strictly (*LS*), and second, he neglects the interactions of the spins of the d electrons with the magnetic nucleus. He points out that his treatment is only approximate and that it does not give a very consistent explanation of the relative magnitudes of the hyperfine separations of the ^4F states. It is evident that his experimental data should be treated more rigorously before one can rely on the nuclear magnetic moment derived from them. The preceding theory is applied in this section to Anderson's data.

We determine the coefficients from the em-

[5] O. E. Anderson, Phys. Rev. **46**, 473 (1934).

TABLE III. *Energy matrices, (LS) coupling.*

States	$^4F_{9/2}$	$^2G_{9/2}$	$^4F_{7/2}$	$^2G_{7/2}$	$^2F_{7/2}$	$^4F_{5/2}$	$^2F_{5/2}$	$^2D_{5/2}$	$^4P_{5/2}$	$^4F_{3/2}$	$^2D_{3/2}$	$^4P_{3/2}$	$^2P_{3/2}$	$^4P_{1/2}$	$^2P_{1/2}$	$^2S_{1/2}$
$^4F_{9/2}$	0	$-\bar{a}$														
$^2G_{9/2}$	$-\bar{a}$	$12F_2+10F_4+G_2-3\bar{a}/2$														
$^4F_{7/2}$			$-3\bar{a}/2$	$\frac{1}{2}\bar{a}$	$\bar{a}\sqrt{3}/2$											
$^2G_{7/2}$			$\frac{1}{2}\bar{a}$	$12F_2+10F_4+G_2-3\bar{a}/2$	$\bar{a}\sqrt{3}/2$											
$^2F_{7/2}$			$\bar{a}\sqrt{3}/2$	$\bar{a}\sqrt{3}/2$	$3G_2-\frac{1}{2}\bar{a}$											
$^4F_{5/2}$						$-8\bar{a}/3$	$-\bar{a}\sqrt{5/3}$	$4\bar{a}/\sqrt{15}$	0							
$^2F_{5/2}$						$-\bar{a}\sqrt{5/3}$	$3G_2-17\bar{a}/6$	$2\bar{a}/\sqrt{3}$	0							
$^2D_{5/2}$						$4\bar{a}/\sqrt{15}$	$2\bar{a}/\sqrt{3}$	$5F_2+45F_4+G_2-3\bar{a}/2$	$\bar{a}\sqrt{21}/\sqrt{10}$							
$^4P_{5/2}$						0	0	$\bar{a}\sqrt{21}/\sqrt{10}$	$15F_2-75F_4-\bar{a}$							
$^4F_{3/2}$										$-7\bar{a}/2$	$2\bar{a}\sqrt{3}/\sqrt{5}$	0	0			
$^2D_{3/2}$										$2\bar{a}\sqrt{3}/\sqrt{5}$	$5F_2+45F_4+G_2-3\bar{a}/2$	$\bar{a}\sqrt{7}/2\sqrt{5}$	$\bar{a}\sqrt{7}/2$			
$^4P_{3/2}$										0	$\bar{a}\sqrt{7}/2\sqrt{5}$	$15F_2-75F_4-11\bar{a}/6$	$\bar{a}\sqrt{5}/6$			
$^2P_{3/2}$										0	$\bar{a}\sqrt{7}/2$	$\bar{a}\sqrt{5}/6$	$15F_2-75F_4+3G_2-7\bar{a}/6$			
$^4P_{1/2}$														$15F_2-75F_4-7\bar{a}/3$	$-\bar{a}\sqrt{2}/6$	$2\bar{a}$
$^2P_{1/2}$														$-\bar{a}\sqrt{2}/6$	$15F_2-75F_4+3G_2-13\bar{a}/6$	$\sqrt{2}\bar{a}$
$^2S_{1/2}$														$2\bar{a}$	$\sqrt{2}\bar{a}$	$22F_2+135F_4+G_2-3\bar{a}/2$

pirical energies. Russell and Meggers[6] have identified all the states of $5d^26s$ of La I except ^2S. Their term values relative to $^4F_{9/2}$ are given in Table V. As the coupling in this configuration approaches (LS) we use the (LS) energy matrices and determine the coefficients by successive approximations to the secular equations. The first step is the evaluation of the parameters, F_2, F_4, G_2, $\bar{a}$, from the empirical term values. The sum of the diagonal terms of the energy matrix for the states with a given J is equal in any coupling to the sum of the roots (energy values) of the secular equation for these states. This fact is made use of. For each J matrix the mean of the diagonal terms is taken and equated to the mean of the empirical term values of the states involved. The resulting relation for each J matrix is then subtracted from the corresponding relation for each of the other matrices. In general this procedure gives a sufficient number of equations to evaluate the parameters.

In the present case, however, some judgment must be used in applying this procedure. As $^2S_{\frac{1}{2}}$ has not been identified, the mean for the states with $J=\frac{1}{2}$ cannot be used. Further there is a possibility that the two groups of states with $J=5/2$ and $3/2$ are appreciably perturbed by the $^2D_{3/2,\,5/2}$ of the adjacent $5d6s^2$ configuration. Thus one hesitates to use the differences obtained by subtracting the mean of either of these two groups from the means of the unperturbed groups. However, according to theory[7] one expects that the perturbation by $5d6s^2$ will displace the means of the $J=5/2$ states and $J=3/2$ states about equally in the same direction. Thus the relation obtained by taking the difference between the means of these two groups should be fairly reliable. Actually in forming this difference the $5d6s^2\ ^2D$ states were included and five diagonal terms used in the $J=5/2$ and $J=3/2$ matrices. The electrostatic components in the diagonal terms of the $5d6s^2\ ^2D$ states cancel out when the difference is taken; the magnetic parts do not, but as $\bar{a}$ is assumed the same for both configurations no additional parameters are introduced. The inclusion of the $5d6s^2\ ^2D$ states gives values of the parameters

[6] H. N. Russell and W. F. Meggers, Bur. Standards J. Research **9**, 625 (1932).
[7] C. W. Ufford, Phys. Rev. **44**, 732 (1933).

TABLE IV. *Energy matrices, (jj) coupling.*

$J=9/2$.	$\mathrm{I}=s\psi^4$	$\mathrm{II}=s\varphi^4$		
I	0	$-24F_2/5-4F_4-2G_2/5$		
II	$-24F_2/5-4F_4-2G_2/5$,	$36F_2/5+6F_4+3G_2/5-5\bar{a}/2$		
$J=7/2$.	$I=s\psi^4$	$\mathrm{II}=s\varphi^4$	$\mathrm{III}=s\varphi^3$	
I	$9G_2/5$,	$-24F_2/5-4F_4+G_2/2$,	$-3G_2\sqrt{3/2}\sqrt{5}$	
II	$-24F_2/5-4F_4+G_2/2$,	$36F_2/5+6F_4+21G_2/20-5\bar{a}/2$,	$-3G_2\sqrt{3/4}\sqrt{5}$	
III	$-3G_2\sqrt{3/2}\sqrt{5}$,	$-3G_2\sqrt{3/4}\sqrt{5}$,	$-12F_2/5-2F_4+11G_2/20-5\bar{a}/2$	
$J=5/2$.	$\mathrm{I}=s\psi^2$	$\mathrm{II}=s\varphi^3$	$\mathrm{III}=s\varphi^2$	$\mathrm{IV}=s\chi^2$
I	$168F_2/25-14F_4+2G_2/5$,	$2G_2/5$,	$\sqrt{2}(96F_2/25-36F_4-2G_2/5)$,	$\sqrt{21}(-2F_2/25+6F_4)$
II	$2G_2/5$,	$-12F_2/5-2F_4+17G_2/15-5\bar{a}/2$,	$-8G_2\sqrt{2/15}$,	$-4G_2\sqrt{21/15}$
III	$\sqrt{2}(96F_2/25-36F_4-2G_2/5)$,	$-8G_2\sqrt{2/15}$,	$159F_2/25-29F_4+7G_2/15-5\bar{a}/2$,	$\sqrt{42}(-14F_2/25+G_2/15)$
IV	$\sqrt{21}(-2F_2/25+6F_4)$,	$-4G_2\sqrt{21/15}$,	$\sqrt{42}(-14F_2/25+G_2/15)$,	$-7F_2/25+7F_4+6G_2/5-5\bar{a}$
$J=3/2$.	$\mathrm{I}=s\psi^2$	$\mathrm{II}=s\varphi^2$	$\mathrm{III}=s\varphi^1$	$\mathrm{IV}=s\chi^2$
I	$168F_2/25-14F_4+7G_2/5$,	$\sqrt{2}(96F_2/25-36F_4+3G_2/5)$,	$G_2\sqrt{14/5}$,	$\sqrt{21}(-2F_2/25+6F_4)$
II	$\sqrt{2}(96F_2/25-36F_4+3G_2/5)$,	$159F_2/25-29F_4+13G_2/10-5\bar{a}/2$,	$3G_2\sqrt{7/10}$,	$\sqrt{42}(-14F_2/25-G_2/10)$
III	$G_2\sqrt{14/5}$,	$3G_2\sqrt{7/10}$,	$63F_2/5-77F_4+3G_2/10-5\bar{a}/2$,	$-G_2\sqrt{3/5}\sqrt{2}$
IV	$\sqrt{21}(-2F_2/25+6F_4)$,	$\sqrt{42}(-14F_2/25-G_2/10)$,	$-G_2\sqrt{3/5}\sqrt{2}$,	$-7F_2/25+7F_4+G_2/5-5\bar{a}$
$J=1/2$.	$\mathrm{I}=s\psi^0$	$\mathrm{II}=s\varphi^1$	$\mathrm{III}=s\chi^0$	
I	$84F_2/5+49F_4+4G_2/5$,	$-2G_2/\sqrt{5}$,	$-7F_2\sqrt{6/5}-42F_4\sqrt{6}$	
II	$-2G_2/\sqrt{5}$,	$63F_2/5-77F_4+9G_2/5-5\bar{a}/2$,	$-G_2(6/5)^{\frac{1}{2}}$	
III	$-7F_2\sqrt{6/5}-42F_4\sqrt{6}$,	$-G_2(6/5)^{\frac{1}{2}}$,	$91F_2/5+91F_4+6G_2/5-5\bar{a}$	

only slightly different from those obtained when these states are not included. A second relation can be obtained from the difference of the means of the $J=9/2$ and $J=7/2$ groups; and a third from the secular equation for the $J=9/2$ states. A fourth relation is needed for the complete determination of the four parameters. This could be obtained from the secular equation for the states with $J=7/2$. But it is equally good to express F_2, F_4, G_2 in terms of $\bar{a}$ by means of the three relations discussed above, then by trial find the value of $\bar{a}$ that gives the best agreement between the theoretically predicted and the observed term values.

We obtain by the above procedure the following expressions for the three parameters in terms of $\bar{a}$:

$$6G_2=4\bar{a}-1/5798\cdot\bar{a}^2+6692.$$

$$70F_4=4375-7/6\cdot\bar{a}-5/34788\cdot\bar{a}^2. \qquad (28)$$

$$42F_2=14202+7/2\cdot\bar{a}-5/11596\cdot\bar{a}^2.$$

The term values predicted by putting $\bar{a}=400$ cm^{-1}, which gives $G_2=1377$, $F_2=370$, $F_4=56$, are in good agreement with the empirical term values except for the ^4P states which are much higher than predicted (see Table V). The poor agreement for the ^4P states cannot be attributed to perturbation by $5d6s^2$, since according to Ufford[7] $^2D_{3/2,\,5/2}$ are the only states of $5d^26s$ affected by the interaction between the two configurations. A perturbation of the ^2D states would indirectly affect $^4P_{3/2,\,5/2}$, but the effect should be a second order one. The fact that $^4P_{\frac{1}{2}}$ also is much higher than predicted further indicates that perturbation by $5d6s^2$ is not primarily responsible for this discrepancy. Similar anomalies for ^4P states of d^2s configurations of other spectra have been found.[4] The $5d^26s$ term values, except for the ^4P states, then are consistent with the theory of multiplet structure. Thus, as there are no nondiagonal elements between the ^4F and the ^4P states in the (LS) energy matrices, one expects the preceding theory to give a consistent interpretation of the hyperfine structures of the ^4F states when the values of the parameters given above are used in the energy matrices.

The coefficients, K's, for the ^4F states can be determined from the energy matrices with the values of the parameters inserted by successive

TABLE V. *Comparison of predicted and empirical term values of $5d^26s$ states.*

STATE	POSITION re. $\cdot\,^4F_{9/2}$.	POSITION calc. $\bar{a}=400$ cm^{-1}	STATE	POSITION re. $\cdot\,^4F_{9/2}$	POSITION calc. $\bar{a}=400$ cm^{-1}
$^4F_{3/2}$	-1453 cm^{-1}	-1459 cm^{-1}	$^4P_{5/2}$	3558 cm^{-1}	904 cm^{-1}
$^4F_{5/2}$	-1112	-1117	$^2D_{3/2}$	4325	4600
$^4F_{7/2}$	-627	-630	$^2D_{5/2}$	5062	5322
$^4F_{9/2}$	0	0	$^2P_{1/2}$	4923	4627
$^2F_{5/2}$	2890	2922	$^2P_{3/2}$	5598	5650
$^2F_{7/2}$	3930	3893	$^2G_{9/2}$	5798	5798
$^4P_{1/2}$	3110	410	$^2G_{7/2}$	5839	5842
$^4P_{3/2}$	3369	650	$^2S_{1/2}$	——	16414

approximations to the secular equations. They are

$^4F_{9/2}$: $K_1=0.9976$, $K_1^2=0.9952$;
$K_2=0.0693$, $K_2^2=0.0048$.
$^4F_{7/2}$: $K_1=0.9966$, $K_1^2=0.9932$;
$K_2=-0.0313$, $K_2^2=0.00098$;
$K_3=-0.0762$, $K_3^2=0.0058$.
$^4F_{5/2}$: $K_1=0.9951$, $K_1^2=0.9903$;
$K_2=0.0730$, $K_2^2=0.00533$;
$K_3=-0.0664$, $K_3^2=0.00441$; $K_4=0$.
$^4F_{3/2}$: $K_1=0.9955$, $K_1^2=0.9911$;
$K_2=-0.0946$, $K_2^2=0.00894$;
$K_3=K_4=0$.

The relativistic corrections for a $5d$ electron of La I, for which the effective nuclear charge is certainly less than 50, are very small and can be neglected. Thus for this application a', a'', a''' can be expressed in terms of a_d by Eq. (22). When these substitutions are made and the values of the K's listed above are inserted in Eqs. (23, 24, 25, 26) the following formulae for the interval factors of the ^4F states are obtained. They are equated on the right to Anderson's experimentally determined interval factors.

$$\begin{aligned} A(^4F_{9/2})&=0.1111a_s+0.6508a_d \\ &=0.01571\text{ cm}^{-1}, \\ A(^4F_{7/2})&=0.09514a_s+0.7864a_d \\ &=0.01464\text{ cm}^{-1}, \\ A(^4F_{5/2})&=0.03531a_s+1.0705a_d \\ &=0.00900\text{ cm}^{-1}, \\ A(^4F_{3/2})&=-0.2000a_s+1.7988a_d \\ &=-0.01667\text{ cm}^{-1}. \end{aligned} \tag{29}$$

There are four equations in a_s and a_d, so we can solve for each and test the solution for consistency. Solving for a_s and a_d from the first and last equations of (29), which are the least sensitive to coupling changes, we obtain $a_s=0.1185$ cm^{-1}, $a_d=0.00391$ cm^{-1}. Substituting these values in the second and third equations of (29) we calculate $A(^4F_{7/2})=0.0143$, observed 0.0146; $A(^4F_{5/2})=0.0084$, observed 0.0090. Anderson gives the probable error of his most reliable interval factor as 5 percent. Thus we see that the preceding theory gives a consistent interpretation of the hyperfine separations of the ^4F states.

It is instructive to contrast the formulae for the interval factors of the ^4F states in strict (LS) coupling with the formulae (29) which take into account the actual coupling conditions. The interval factors in strict (LS) coupling are by Eqs. (21),

$$\begin{aligned} A(^4F_{9/2})&=0.1111a_s+0.6349a_d, \\ A(^4F_{7/2})&=0.07937a_s+0.7800a_d \\ A(^4F_{5/2})&=0.00952a_s+1.0384a_d, \\ A(^4F_{3/2})&=-0.2000a_s+1.7371a_d. \end{aligned} \tag{30}$$

Comparing (29) and (30), it is evident that $A(^4F_{9/2})$ and $A(^4F_{3/2})$ are not overly sensitive to departure from strict (LS) coupling; $A(^4F_{7/2})$ is more sensitive, and $A(^4F_{5/2})$ is very sensitive. This comparison shows that one should be very careful in using (LS) interval factor formulae for states dependent on coupling even though the multiplet structure indicates that the coupling is close to (LS).

The nuclear magnetic moment of lanthanum can be calculated from the values of a_s and a_d. The nuclear g factor is computed from the interaction constant of an s electron by the formula[8]

$$g(I)=\frac{3}{8}\frac{a_s}{R\alpha^2}\frac{n^3_{\text{eff.}}}{Z_iZ_0^2}\frac{1838}{K(\frac{1}{2},Z_i)}.$$

For our case $a_s=0.119$ cm^{-1}, $n_{\text{eff.}}=1.60$, $Z_i=57$, $Z_0=1$, $R\alpha^2=5.82$, $K(\frac{1}{2}, Z_i)=1.43$. The substitution of these values in the above equation gives $g(\text{I})=0.71$. A reliable value of $g(\text{I})$ cannot be readily obtained from a_d since first, a_d is very small, and second, it is difficult to estimate the value of Z_i that should be used for a $5d$ electron in the $5d^26s$ configuration. However, the substitution of $g(I)=0.71$, $a_d=0.0039$ cm^{-1}, $\Delta\nu=1000$ cm^{-1} in the formula for a non-s electron[8] gives, on solving for Z_i, $Z_i=40$. This appears to be a reasonable value, indicating that $g(\text{I})=0.71$ is consistent with $a_d=0.0039$. The nuclear spin

[8] S. Goudsmit, Phys. Rev. **43**, 636 (1933); E. Fermi and E. Segrè, Zeits. f. Physik **82**, 729 (1933).

of lanthanum is 7/2, hence the nuclear magnetic moment as determined by this analysis is 2.5 nuclear magnetons. This is in fair agreement with the value 2.8 nuclear magnetons determined from La III hyperfine structures by the writer and N. S. Grace.[9]

[9] M. F. Crawford and N. S. Grace, Phys. Rev. **47**, 536 (1935).

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Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

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In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

1.

Any serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves.

In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) "Is the theory correct?" and (2) "Is the description given by the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concepts of the theory may be said to be satisfactory. The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience. This experience, which alone enables us to make inferences about reality, in physics takes the form of experiment and measurement. It is the second question that we wish to consider here, as applied to quantum mechanics.

Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counterpart in the physical theory.* We shall call this the condition of completeness. The second question is thus easily answered, as soon as we are able to decide what are the elements of the physical reality.

The elements of the physical reality cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements. A comprehensive definition of reality is, however, unnecessary for our purpose. We shall be satisfied with the following criterion, which we regard as reasonable. *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.* It seems to us that this criterion, while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one