Theory of Disintegration of Nuclei by Neutrons[1]

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The large probability of nuclear disintegration by slow neutrons as well as the large cross section for the elastic scattering of slow neutrons can be explained without any new assumption. Interaction between neutron and nucleus is assumed to be only present when the neutron is inside the nucleus or very near its boundary. The rate of change of the potential energy of the neutron at the boundary of the nucleus is important for the quantitative, but not for the qualitative results; in agreement with other data, it has been assumed that the potential drops to $1/e$ in a distance $1.5.10^{-13}$ cm (range of the forces between neutron and nucleus).

The large disintegration cross sections are due to two factors. The first is elementary: the cross section is inversely proportional to the neutron velocity, because a slow neutron stays longer in the nucleus. The second factor is $1/\sin^2 \varphi_0$, where φ_0 is the phase of the neutron wave function at the nuclear boundary. This resonance factor explains the large differences between the cross sections of different elements. φ_0 cannot be predicted theoretically, but reasonable assumptions lead to agreement with experiment. The resonance factor occurs in all phenomena with slow neutrons; therefore large capture cross sections should always be accompanied by large elastic scattering. The explanation of the large neutron cross sections on the basis of ordinary wave mechanics makes one confident in the applicability of orthodox quantum theory in nuclear phenomena.

1. **Elastic scattering.** May be large for slow neutrons because of resonance. Magnitude is 5.10^{-24} cm^2 without, 10^{-22} and more with resonance. If present, large cross section persists up to neutron energies of 10,000 or 100,000 volts.

2. **Capture with emission of γ-rays.** Cross section large for slow netrons. About half the elastic scattering cross section for gas-kinetic energy. Cross section inversely proportional to neutron velocity. All capture effects observed should be due to admixtures of slow neutrons in the incident beam. Capture only possible, if unoccupied neutron level with angular momentum $l=1$ exists in the nucleus.

3. **Disintegration with emission of α-particles.** Very probable for slow neutrons if exothermic process, which is usually the case. Cross section for gas-kinetic neutrons and light nuclei 10^{-21} cm^2 without resonance, for $Z=11$ same cross sections as for fast neutrons (10^{-25} cm^2). May be increased by resonance which may occur as well for neutrons as for α-particle. Cross section inversely proportional to neutron velocity up to neutron energies of some 100,000 volts, then cross section increases again because faster α-particles penetrate more easily through nuclear potential barrier. Disintegration by slow neutrons should stop at $Z\approx 16$, by fast ones at $Z\approx 27$, the latter in agreement with experiments.

4. **Disintegration with emission of protons.** Always endothermic, therefore impossible with slow neutrons. With fast neutrons, cross section $\approx 10^{-25}$ cm^2 up to $Z\approx 20$. Weak effects should be observable up to $Z=40$, in case of resonance even to 60.

5. **Excitation of nucleus without capture of neutron or emission of particles.** Should have cross section of the order 10^{-25} cm^2 independent of atomic number. Possible only for nuclei with suitable excited states and for fast neutrons.

1. INTRODUCTION

FERMI and his collaborators[2] have shown that neutrons, especially slow ones, are very effective in disintegrating nuclei. It is generally accepted that for the heavier nuclei the disintegration process consists in a simple capture of the neutron by the nucleus with the emission of a γ-ray. Indeed, it seems inconceivable that a heavy nucleus could emit a charged particle (proton or α-particle) if bombarded with neutrons, since the particle could never escape from the nucleus because of the potential barrier surrounding it (cf. paragraph 5 of this paper).

The cross sections for the capture of the neutron are surprisingly high, some of them amounting to 10^{-22} cm^2 and more for slow neutrons. On the basis of naïve considerations, one would expect cross sections of only about 10^{-28} cm^2. For, if the neutron hits the nucleus, the probability of its radiating is about 1 in 10,000 according to ordinary radiation theory; on the other hand, the (geometric) cross section

[1] This paper in its essential parts has been presented at the February Meeting of the American Physical Society, in place of a previous paper printed in the Abstracts of that meeting (Phys. Rev. **47**, 640 (1935)), which I had realized at the time of the meeting to be an unsuccessful attempt. Subsequently I received from Professor Fermi a manuscript of a paper submitted to the Royal Society, containing the same explanation of the large cross sections as the present paper.
Added in proof: Perrin and Elsasser (Comptes rendus **200**, 450 (1935)) came to the same conclusions concerning the large cross sections of slow neutrons; and an attempt in the same direction has been made by Beck and Horsley (Phys. Rev. **47**, 510 (1935)).

[2] E. Fermi, E. Amaldi, O. D'Agostino, F. Rasetti and E. Segrè, Proc. Roy. Soc. **A146**, 483 (1934).

of the nucleus is of the order 10^{-24} cm^2. The naïve theory thus gives a cross section about a million times too small.

We want to show in this paper that a straightforward application of wave mechanics leads to cross sections of just the right magnitude. Furthermore, long distance forces between neutron and nucleus are not required, it being assumed that the interaction is appreciable only when the neutron is inside the nucleus.

2. Wave Function of Slow Neutrons

Since the most interesting phenomena seem to occur with slow neutrons, we shall consider primarily neutrons with kinetic energies well below a million volts. For these neutrons, the de Broglie wavelength $2\pi\lambda$, *viz.*

$$\lambda=\frac{\hbar}{(2ME)^{\frac{1}{2}}}=\frac{4.54\cdot 10^{-10}}{E^{\frac{1}{2}}}\text{ cm.} \tag{1}$$

(E measured in volts) is large compared with the radius of the nucleus. The radius of a nucleus with medium atomic weight (about 100) being about 7.10^{-13} cm, our condition $\lambda > r_0$ is fulfilled for neutron energies below half a million volts.

Neutrons of a given angular momentum l pass the nucleus in general at a distance of the order $l\lambda$. For slow neutrons of, say, 1000 volts energy or less, there is therefore no chance of getting into the nucleus, if they have an angular momentum different from zero, since their distance of closest approach would be at least 30 times larger than the nuclear radius. Even if there is a resonance level for the neutron with $l=1$, it has no appreciable effect on the scattering or capture of slow neutrons (see Appendix 1). Therefore we need only consider the spherically symmetrical part of the wave function of the incident neutron, i.e., the part corresponding to $l=0$. The slow neutrons reaching the nucleus are consequently in an "s-state."

The wave function of the neutron at large distances from the nucleus can be written in the familiar form

$$\psi=(1/v^{\frac{1}{2}})e^{ikz}+\text{scattered wave,}$$

ψ represents a stream of neutrons traveling in the z direction and is normalized per unit current, and

$$k=1/\lambda=Mv/\hbar=(2ME)^{\frac{1}{2}}/\hbar. \tag{2}$$

ψ can be expanded in spherical harmonics:

$$\psi=\frac{1}{v^{\frac{1}{2}}}\sum_{l=0}^{\infty}\frac{\psi_l(r)}{r}P_l(\vartheta), \tag{3}$$

r being the distance between nucleus and neutron. For $l\neq 0$, ψ_l has practically the same form far from the nucleus as if the neutron was entirely free, i.e., not influenced by the nucleus, for the reason given above. For $l=0$, the radial wave function satisfies the Schrödinger equation

$$d^2\psi_0/dr^2+2M/\hbar^2(E-V)\psi_0=0. \tag{4}$$

V is the potential energy of the neutron in the field of the nucleus.

Outside the nucleus, ($V=0$), the solution of (4) is

$$\psi_0=\sin(kr-\delta)/k. \tag{5}$$

It is properly normalized so as to fulfill (3). The phase shift δ determines the cross section for elastic scattering

$$\Phi_{\text{el}}=4\pi\sin^2\delta/k^2, \tag{6}$$

if, as we have supposed, only the partial wave $l=0$ contributes to the scattering.

Inside the nucleus, the neutron has a large negative potential energy V, of the order of some million volts. Its magnitude may be estimated from the binding energy of neutrons in the nucleus (cf. Appendix 2). From the number of neutrons present in the nucleus, one should expect the existence of 3 or 4 bound s-states for a neutron inside a medium-weight nucleus like, e.g., Ag. Correspondingly, the wave function of a neutron with small positive energy will oscillate several times inside the nucleus (3 to 5 complete oscillations for Ag). Therefore the wave function can be represented by the WKB (Wentzel-Kramers-Brillouin) solution:

$$\psi_0(r)=C(y(r))^{-\frac{1}{2}}\cos\varphi(r) \tag{7}$$

for $r<r_0$ (r_0 = radius of the nucleus). Here

$$y(r)=[2M(E-V(r))]^{\frac{1}{2}}/\hbar$$
$$\varphi(r)=\int^r y(\rho)d\rho. \tag{8}$$

At the boundary r_0 of the nucleus the potential V is known to drop rather suddenly to zero. We shall assume an *infinitely* sudden potential change from the value $-V_0$ inside the nucleus to zero. The influence of a gradual change of V is treated in Appendix 3. It does not change the results of the theory, but only the meaning of the constant λ_0 occurring in (11) and (14).

We have then merely to join the functions (5) and (7) together at the point r_0. The wave function and its derivative being continuous at r_0, we have

$$\begin{aligned} Cy(r_0)^{-\frac{1}{2}} \cos \varphi(r_0) &= k^{-1} \sin (kr_0 - \delta), \\ -Cy(r_0)^{+\frac{1}{2}} \sin \varphi(r_0) &= \cos (kr_0 - \delta). \end{aligned} \quad (9)$$

Dividing the upper equation by the lower, we obtain

$$\operatorname{tg} (kr_0 - \delta) = -(k/y(r_0)) \operatorname{ctg} \varphi(r_0). \quad (10)$$

From (2) and (8) we see that for $k \ll y(r_0)$, i.e., slow neutrons, the right-hand side of (10) is small except for exceedingly large values of the ctg, consequently

$$\delta = k(\operatorname{ctg} \varphi(r_0)/y(r_0) + r_0) = k(\lambda_0 \operatorname{ctg} \varphi_0 + r_0) \quad (11)$$

with the abbreviations

$$\lambda_0 = 1/y(r_0) = \hbar/(2MV_0)^{\frac{1}{2}}; \quad V_0 = |V(r_0)|; \quad \varphi_0 = \varphi(r_0). \quad (12)$$

In the special case when ctg φ_0 is very large, we have

$$\sin \delta = k(\lambda_0 \operatorname{ctg} \varphi_0 + r_0)/(1 + (k\lambda_0 \operatorname{ctg} \varphi_0)^2)^{\frac{1}{2}} \quad (11a)$$

when higher powers of kr_0 than the first are neglected.

The amplitude C of the wave function inside the nucleus can easily be obtained from (9), we have

$$1/C^2 = y(r_0) \sin^2 \varphi_0 + (k^2/y(r_0)) \cos^2 \varphi_0. \quad (13)$$

Since $k \ll y(r_0)$, the second term is negligible unless $\sin \varphi_0$ is very small, in that case $\cos \varphi_0 = 1$ and

$$1/C^2 = (1/\lambda_0)(\sin^2 \varphi_0 + E/V_0). \quad (14)$$

The wave function (5) is not normalized. The wave function normalized per unit energy[5] is

$$\psi_E = Ak^{\frac{1}{2}}\psi_0/r, \quad (15)$$

where

$$A = (\tfrac{1}{2}M)^{\frac{1}{2}}/\pi\hbar \quad (15a)$$

does not depend on the energy. Inside the nucleus,

$$\psi_E = ACk^{\frac{1}{2}}y^{-\frac{1}{2}} \cos \varphi/r. \quad (16)$$

It is convenient to separate the factors depending considerably on the energy, *viz.*, $Ck^{\frac{1}{2}}$, from the rest of the function by putting $\psi_E = Ck^{\frac{1}{2}}\psi_W$. Then

$$\psi_W = Ay^{-\frac{1}{2}} \cos \varphi/r \quad (17)$$

and

$$\psi_0 = rC\psi_W/A. \quad (18)$$

ψ_W changes only very slowly with the energy which makes it suitable for investigating the dependence of cross sections on E. Moreover, for large energies ψ_W becomes identical with the normalized wave function ψ_E, since for $E > V_0$ we have $k_0 = (2M)^{\frac{1}{2}}(E - V_0)^{\frac{1}{2}}/\hbar \approx k$ and therefore, according to (13), $C = 1/k^{\frac{1}{2}}$.

3. The Elastic Cross Section

Inserting (11) into (6), we obtain for the elastic cross section

$$\Phi_{el} = 4\pi(\lambda_0 \operatorname{ctg} \varphi_0 + r_0)^2 \quad (19)$$

with the abbreviations explained in (12).

If we had an accurate theory of the nucleus, we could determine λ_0 and φ_0 accurately. At present, we can only estimate them, which is fairly easy in the case of λ_0. As is shown in Appendix 3, λ_0 is connected with the range of the forces between nucleus and neutron, rather than with the potential energy inside the nucleus. Wigner has shown[6] that the forces between *proton* and neutron extend over a region of about 10^{-13} cm; this value follows from the observed mass defects of He^4 and H^2. The forces between a heavier nucleus and a neutron probably extend over a slightly larger distance, because the nucleus will become polarized under the influence of the neutron. We believe, therefore, that $l = 1.5 \cdot 10^{-13}$ cm is a fair estimate; this figure agrees well with the "apparent radius of the neutron" as derived by Rabi from the scat-

[5] Definition: $\int d\tau \psi_E \int dE' \psi_{E'} = 1$.

[6] E. Wigner, Phys. Rev. **43**, 252 (1932).

tering of fast neutrons.[7] Then, according to (80),

$$\lambda_0 = \tfrac{1}{2}\pi l = 2.4 \cdot 10^{-13} \text{ cm}. \quad (20)$$

The corresponding V_0, calculated from (12), would be $V_0 = 3.4$ MV. We believe that λ_0 is not in error more than by a factor $\sqrt{2}$, and therefore the error introduced into the cross section is not more than a factor 2 one way or the other. λ_0 is only about one-third of the nuclear radius r_0.

The phase φ_0 cannot be estimated from pure theory. As we have already pointed out (above Eq. (7)), the wave function makes several oscillations inside the nucleus, so that φ_0 is of the order 2π. It is therefore impossible at present to predict φ_0 with an accuracy of the order $\pi/20$, which would be necessary to give a significant estimate for ctg φ_0. All we can say is that φ_0 is a constant characteristic for a given element, which is practically independent of the neutron energy. A change of about 150,000 volts in E is necessary to produce a change of one degree in φ_0 (cf. Appendix 2). The change of φ_0 with the number of protons and neutrons in the nucleus, may possibly be just regular enough so that large values of ctg φ_0 would be found preferably with neighboring elements. There seems indeed to be a slight indication of such an effect in the experiments (Rh, Ag and Cd all have large cross sections!). But φ_0 is probably not regular enough to allow *predictions* of large cross sections from the behavior of neighboring elements. Nor is it probable that the isotopes of the same element *all* have large cross sections, if one of them has: A change of only 2 percent would change φ_0 from $n\pi$ (n an integer) to $(n+1/25)\pi$, ctg φ_0 from ∞ to $25/\pi \approx 8$, and the cross section from infinity to the moderate value $50 \cdot 10^{-24}$ cm^2. The addition of a neutron to the nucleus will probably change φ_0 by more than 2 percent.

Therefore the only practicable procedure is to make a statistical study of all the experimental cross sections observed for various nuclei, and to compare the result with reasonable theoretical expectations. From the definition of φ_0 it is clear that any value of φ_0 between $-\pi/2$ and $+\pi/2$ (an additional integer multiple of π has no effect on the cross section) is *a priori* equally probable. Starting from this fact, we can easily calculate the "probable" distribution of cross sections.

Since the nuclear radius $r_0 = 7 \cdot 10^{-13}$ cm is about three times larger than our accepted value for λ_0 (Eq. (20)), the preponderant term in the cross section (19) is r_0 for the larger part of the possible values φ_0, *viz.*, for φ_0 between $-90°$ and $-18°$ and between $18°$ and $90°$ (ctg $18° = r_0/\lambda_0$). In both of these regions the cross section is of the order of magnitude $4\pi r_0^2 = 6 \cdot 10^{-24}$ cm^2, slightly smaller in the first, larger in the second region mentioned. If φ_0 is in the immediate neighborhood of $-18°$, the cross section is considerably smaller than the "hard sphere value" $4\pi r_0^2$, whereas for φ_0 near zero exceedingly large cross sections are obtained.

Table I gives the probabilities for various magnitudes of the cross section. With the assumed values of λ_0 and r_0, we have

$$\Phi_{el} = 0.7 \cdot 10^{-24} (\text{ctg } \varphi_0 + 3)^2. \quad (21)$$

If we want to know, e.g., the probability for a cross section between 10 and 20×10^{-24} cm^2, we calculate first the values φ_0 corresponding to $\Phi = 10 \cdot 10^{-24}$, *viz.*, $+52.0°$ and $-8.4°$, and to $\Phi = 20 \cdot 10^{-24}$, *viz.*, $+23.0°$ and $-6.9°$. Hence we find that two intervals of φ_0, of $29.0°$ and $1.5°$ length, respectively, lead to the desired magnitude of Φ_{el}. The sum of the two intervals, $30.5°$, corresponds to 17 percent of the total range available for φ_0, *viz.*, $180°$. If we could study the cross section of each isotope of each element separately, we should find that 17 percent of all nuclei have cross sections for slow neutrons between 10 and 20×10^{-24} cm^2. Actually, almost all elements consist of several isotopes. If one of these has a large cross section, the other a small one, the large cross section will be measured. Therefore with natural elements the large cross sections should be more frequent, the small ones less frequent than for pure isotopes. The probable distribution of cross sections has been calculated (a) for elements containing two isotopes, in equal abundance, (b) for 4 isotopes. Case (a) is realized in most elements with odd atomic number, case (b) corresponds to the average even-numbered element. It has been assumed that the φ_0's of several isotopes have no relation to each other. It can be seen that the difference between 2-isotope and 4-isotope ele-

[7] Rabi, Phys. Rev. **43**, 838 (1933).

TABLE I. *Data on cross section.*

Range of cross section (units 10^{-24} cm^2)	Corresponding ranges of φ_0		Probability for elements containing 1	2 isotopes	4	Ave. of 2 and 4	No. of elements among 27 investigated: expected	observed
0–2.5	-11.5 to $-42.1°$		17.0	6.3	1.1	3.7	1.0	$2\frac{1}{2}$†
2.5–5	-10.0 to -11.5	-42.1 to $-71.8°$	17.4	18.1	11.4	14.7	4.0	$3\frac{1}{2}$
5–10	-8.4 to -10.0	52.0 to 108.2*	32.1	34.7	35.2	35.0	9.5	9
10–20	-6.9 to -8.4	23.0 to 52.0	16.9	19.3	22.7	21.0	5.7	4
20–40	-5.4 to -6.9	12.4 to 23.0	6.7	8.0	11.3	9.7	2.6	3
40–100	-3.8 to -5.4	6.3 to 12.4	4.3	5.7	7.4	6.6	1.8	1
100–300	-2.4 to -3.8	3.2 to 6.3	2.5	3.4	4.7	4.0	1.1	2
over 300	-2.4 to $+3.2°$		3.1	4.4	6.2	5.3	1.5	2

* $-71.8°$ is equivalent to $+108.2°$.
† If an experimental cross section is just at the limit of two intervals, half an element has been attributed to each interval.

ments is not very great, only very small cross sections are more likely to be found with odd-number elements. Taking the average probability for even and odd elements, we have calculated how many of the 27 cross sections measured thus far should lie in each group. Most of the "observed" cross sections have been taken from recent experiments of Dunning and Pegram, whom we wish to thank for the communication of their data before publication.[8] Only elements with $Z>10$ have been included, because the lighter ones are apt to emit α-particles when bombarded with slow neutrons and have then much larger (inelastic) cross sections[9] (cf. paragraph 5). The agreement between experiment and theory is satisfactory, although there seem to be slightly more large cross sections. Part of this difference may be due to the fact that the experimental values include the capture cross section which might be of the same order of magnitude as the elastic one (paragraph 4).

We shall now discuss the physical meaning of the large cross sections: According to (19) the cross section is determined by the phase φ_0 with which the neutron wave ψ_0 leaves the nucleus; small phases φ_0 lead to large cross sections. For an "ordinary" value of φ_0 (not near zero), the wave function increases linearly outside the nucleus up to $r\approx\lambda$ when ψ_0 begins to oscillate. ψ_0 is therefore much larger at great distances r than inside the nucleus, and since its value outside is fixed by normalization, it is small inside. However, if φ_0 is zero or nearly so, the wave function leaves the nucleus with horizontal tangent, it does not increase outside, and is therefore comparatively very large in the interior of the nucleus. The neutron remains much longer inside the nucleus if $\varphi_0\approx 0$, and therefore all phenomena due to the nucleus—such as scattering, capture, disintegration—are much more intense. $\varphi_0=0$ means that the neutron has a virtual quantum level (more accurately: an s-level) in the nucleus with an energy near zero. The large cross sections may thus be called a resonance effect, but the "resonance" is very unsharp, since φ_0 changes only by 1° if the energy changes by about 150,000 volts. This makes the "resonance" such a frequent phenomenon.

If the cross section is large for small neutron energy, it remains fairly constant when the energy increases, until $4\pi\lambda^2$ ($2\pi\lambda=$wavelength) becomes smaller than (19). The maximum energy up to which the cross section remains unchanged, is therefore

$$E_{\max}=\frac{\hbar^2}{2M\lambda^2_{\text{crit}}}=\frac{\hbar^2}{2M}\frac{4\pi}{\Phi_{\text{el}}}=\frac{2\cdot 6}{\Phi_{\text{el}}}\text{ MV} \quad (22)$$

if Φ_{el} is measured in units 10^{-24} cm^2. For higher energy, the cross section decreases as $4\pi\lambda^2$, i.e., proportional to $1/E$, until λ becomes of the order $r_0(E\approx\frac{1}{2}$ to 1 MV) when higher angular momenta begin to play a role.

4. Capture with Emission of γ-Rays

Since the incident slow neutrons which reach the nucleus are in an s-state (have angular momentum $l=0$), they can only make an optical

[8] Dunning, Pegram, Fink and Mitchell, Phys. Rev. **47**, 416 (1935).
[9] The elastic cross sections are smaller for light elements because of the smaller nuclear radii.

transition to a p-state. The capture of the neutron with emission of a γ-ray is therefore only possible if there is an unoccupied p-level for the neutron inside the nucleus. A nucleus which contains no such level cannot capture the neutron but only scatter it. This offers an explanation for the fact that some substances scatter slow neutrons considerably without absorbing them: e.g., slow neutrons penetrate through a paraffin sphere of 12 cm radius without being diminished measurably in intensity;[10] they have to travel about 3 meters in order to reach the sphere's surface, the mean free path for elastic scattering being about $\frac{1}{2}$ cm;[10] therefore the "mean free path" for absorption must be at least 1000 times larger than that for scattering, whereas for other substances the two cross sections are (for slow neutrons) of the same order of magnitude.[11] The obvious conclusion is that the nonabsorbing substances (in our case, H and C) have no neutron p-level.

[It should be noted, however, that it is probably a bad approximation to speak of the quantum states of the single particles in the nucleus, and that one should rather speak of the angular momentum of the nucleus as a whole. Then our selection rule reads: A neutron capture by a nucleus of angular momentum L is possible, if and only if the nucleus produced by the capture has a state with angular momentum L or $L\pm 1$, whose energy is lower than that of the original nucleus plus a free neutron at rest. This selection rule is much less strict than the original rule for the neutron. A further restriction is, however, that the transition $L=0\rightarrow L'=0$ is, of course, forbidden: This is very important because many nuclei have no angular momentum. We shall therefore expect

(1) Capture is possible for many, but not for all nuclei.
(2) Impossibility of capture is most likely for nuclei having no angular momentum.]

The cross section for the capture of any particle with emission of radiation is[12]

$$\Phi_c=(4/3)(\omega^3/\hbar c^3)\,|\textstyle\int\psi_n{}^*\mathbf{R}\psi d\tau|^2. \tag{23}$$

ψ = wave function of incident particle (cf. (3)).
ψ_n = wave function of final (bound) state of particle.
$\omega = 2\pi$ times frequency of emitted radiation $=(E-E_n)/\hbar$
$\mathbf{R}$ = electric moment associated with particle in the system of reference, where the center of gravity of the captured particle and the capturing system is at rest.

If we treat the nucleus as rigid, as we have always done so far, the coordinate of the nucleus with respect to the center of gravity of nucleus and neutron is $-\mathbf{r}/(M+1)$, where M is the atomic weight of the nucleus and $\mathbf{r}$ the neutron coordinate with respect to the nucleus. Therefore the electric moment of the whole system is

$$\mathbf{R}=-Ze\mathbf{r}/(M+1)=e'\mathbf{r}, \tag{24}$$

where $e'=-eZ/(M+1)(=0.43e$ for Ag$)$ (25)

can conveniently be called the effective charge of the neutron.

We express ψ in (23) in terms of ψ_W (17) with the help of (3) and (18):

$$\Phi_c=\frac{4}{3}\frac{\omega^3e'^2}{\hbar c^3}\frac{C^2}{A^2}\frac{1}{v}\left|\int\psi_n{}^*\mathbf{r}\psi_W d\tau\right|^2 \tag{26}$$

and introduce the oscillator strength[13] corresponding to the transition from the state ψ_W to *one* of the p-states in the level n, namely, that one which is symmetrical round the z axis:

$$f_W=(2M/\hbar)\omega\,|\textstyle\int\psi_{nz}{}^*z\psi_W d\tau|^2. \tag{27}$$

Then, inserting C and A from (14) and (15a) and considering that transitions to each of the three p-states are possible:

$$\Phi_c=4\frac{e'^2}{\hbar c}\frac{\hbar^3\omega^2\pi^2}{M^2c^2v}\frac{\lambda_0}{\sin^2\varphi_0+E/V_0}f_W,$$

$$\Phi_c=4\pi^2\frac{e'^2}{\hbar c}\frac{\hbar\omega}{Mc^2}\frac{\lambda\lambda_0}{\sin^2\varphi_0+\lambda_0{}^2/\lambda^2}\hbar\omega f_W. \tag{28}$$

Here $2\pi\lambda$ is the de Broglie wavelength of the neutron, λ_0 and φ_0 the characteristic quantities of the nuclear field defined in (12) and discussed in paragraph 3. Furthermore, according to (25),

$$\hbar c/e'^2=137/0.43^2=740. \tag{29}$$

$\hbar\omega=-E_n$ is the binding energy of the neutron,

[10] All this according to experiments of Dunning and Pegram.
[11] Theoretically, see below.
[12] See, e.g. Handb. d. Physik **24** (1), p. 430, form. (38.11).
[13] Cf. e.g. Handb. d. Physik **24** (1), p. 431, form. (38.14).

which is, according to the observed mass defects for elements of medium atom weight, about 8 MV,[14] so that

$$Mc^2/\hbar\omega \approx 120. \qquad (30)$$

Finally, f_W is a quantity of the dimension of a reciprocal energy. Moreover it is known that the total oscillator strength of all transitions starting from one level, say the level nz, is unity, and that the strong spectral lines ending at this level, extend over a frequency range of the order ω. Therefore we estimate[15]

$$\hbar\omega f_W = 1, \qquad (31)$$

so that

$$\Phi_c \approx (\lambda\lambda_0/2300)[1/(\sin^2\varphi_0 + \lambda_0^2/\lambda^2)]. \qquad (32)$$

For slow neutrons, unless φ_0 is unreasonably small, λ_0/λ may be neglected in comparison with $\sin\varphi_0$. Then the capture cross section is seen to increase with decreasing velocity as λ, i.e., as $1/v$. The slowest neutrons have the largest capture cross sections. Now the smallest velocity obtainable is gas-kinetic velocity, i.e.,

$$E = kT = 1/40 \text{ volt at room temperature,}$$
$$\lambda = 2.9\cdot 10^{-9} \text{ cm.} \qquad (33)$$

With the value (20) for λ_0, we have then

$$\Phi_c = 0.3\cdot 10^{-24}/\sin^2\varphi_0. \qquad (34)$$

The capture cross section becomes large if φ_0 is small, just as the elastic cross section. The reason for this "phase effect" has been explained at the end of paragraph 3. If φ_0 is small, we find from (34) and (21):

$$\Phi_{el}/\Phi_c \approx 2 \qquad (35)$$

independent of φ_0. The ratio of the two cross sections should thus be independent[16] of their absolute magnitude, provided the cross sections are large, i.e., φ_0 small.

If the phase φ_0 is not small,

$$\Phi_{el}/\Phi_c = 2(\cos\varphi_0 + 3\sin\varphi_0)^2, \qquad (36)$$

which is larger than (35). Therefore, if the elastic cross section is of the order of magnitude of the hard sphere value, the capture is less probable, even relative to the elastic scattering. The maximum of (36) is 10 times 2.

The capture cross section (32) decreases rapidly with increasing neutron velocity. Since the theory seems to be in agreement with experiment, if gas-kinetic energies are assumed for the neutrons, it seems reasonable actually to make this assumption. The effects observed with apparently fast neutrons should then be ascribed to a small admixture of slow neutrons which may be present in the neutron beam. We expect that the heavy elements cannot be made radioactive if the slow neutrons are carefully kept away. For 1 MV neutrons formula (32) would give a cross section of the order 5.10^{-29} cm^2.

5. Neutron-Produced Disintegrations with Emission of Charged Particles

Two kinds of disintegrations under this heading have been observed:

$$Z^M + n^1 \rightarrow (Z-1)^M + H^1, \qquad (I)$$

$$Z^M + n^1 \rightarrow (Z-2)^{M-3} + He^4, \qquad (II)$$

where Z^M denotes a nucleus with mass M and charge Z. Reaction (I) is found to occur, with fast neutrons ($E \sim 7$ MV), up to $Z = 30$, whereas process (II) stops at $Z = 27$.

a. Energy balance

Process (I) is always endothermic. For of the two isobars Z^M and $(Z-1)^M$, the former is known to be stable, so that the latter must disintegrate with β-emission

$$(Z-1)^M \rightarrow Z^M + \beta. \qquad (III)$$

Since the masses of neutron and hydrogen atom are nearly equal, the energy absorbed in process (I) is just equal to that emitted in (III), i.e., to the energy liberated in the β-disintegration which is usually between 1 and 4 MV. Since very slow protons cannot escape from the nucleus, the minimum neutron energy required for the

[14] Here it has been assumed that the neutron is captured in the ground state, otherwise the capture cross section should be smaller.

[15] The actual value of f_W in a given case may easily be wrong by a factor 5 one way or the other. This uncertainty applies to all the formulae below.

[16] It should however be noted that there are irregular variations of the ratio Φ_c/Φ_{el} owing to different values of the oscillator strength f_W.

"proton-type disintegration" (I) is between 1 and 5 MV.

Process (II) is in general exothermic, because the neutron has a very high "internal" energy, which is set free in the process. For atoms of medium atomic weight, the packing fraction is known to change very little from element to element[17] and to be of the order $-1/1000$. Therefore the deviations of the masses of the particles occurring in process (II) from integral numbers in the O^{16}-scale are (in thousandths of mass units)[17a]

for the neutron	$+8.5$	α-particle	$+3.3$
nucleus Z_M	$-M$	nucleus Z_{M-3}	$-(M-3)$
initial particles	$8.5-M$	final part.	$6.3-M$

This leaves an energy balance of 2.2 thousandths of a mass unit, corresponding to about 2 MV. The "α-type" disintegration (II) can therefore be produced by *slow* neutrons, having practically no kinetic energy, at least with light nuclei. In fact, Chadwick, Taylor and Goldhaber[18] as well as Fermi and his collaborators have observed this type of disintegration with Li^6 and B^{10} and have in both cases found very large cross sections (10^{-21} cm^2).

b. Cross section; general

The cross section for the disintegration of a nucleus by a neutron with emission of an α-particle[19] is given by the well-known formula (Born theory)

$$\Phi_{\text{dis}}=(2\pi/\hbar)\left|\int\psi_{\text{neut}}\psi_{\alpha E}{}^*\times u(Z)u^*(Z-2)u^*(\alpha)\,Vd\tau\right|^2, \quad (37)$$

where ψ_{neut} is the wave function (3) of the incident neutron, normalized per unit current, $\psi_{\alpha E}$ that of the α-particle, normalized per unit energy, $u(Z)$ is the wave function describing the internal motion of the particles in the original nucleus, $u(Z-2)$ and $u(\alpha)$ the same for final nucleus and α-particle, V is the total potential energy between all particles. We express ψ_{neut} by ψ_W (cf. (17), (3), (18)), and we introduce a function $\psi_{\alpha W}$ for the α-particle, which is exactly analogous to ψ_W for the neutron: Its value inside the nucleus varies slowly with the energy and it goes over into $\psi_{\alpha E}$ for high energy. It can easily be shown that

$$\psi_{\alpha E}=\psi_{\alpha W}/(4e^{2G}\sin^2\varphi_\alpha+\tfrac{1}{4}e^{-2G}\cos^2\varphi_\alpha)^{\frac{1}{2}}, \quad (38)$$

where e^{-2G} is the well-known penetrability of the potential barrier round the nucleus, *viz.*

$$G=((2M)^{\frac{1}{2}}/\hbar)\int(V-E)^{\frac{1}{2}}dr, \quad (39)$$

the integral extending over the region where the radicand is positive. φ_α is the phase of the α-particle wave function at the boundary of the nucleus.

Inserting (38) and (3), (18), (14) into (37), we obtain

$$\Phi_{\text{dis}}=\frac{2\pi}{\hbar}\frac{2\pi^2\hbar^2}{Mv}\frac{\lambda_0}{\sin^2\varphi_0+E/E_0}\times\frac{1}{4e^{2G}\sin^2\varphi_\alpha+\frac{1}{4}e^{-2G}\cos^2\varphi_\alpha}\times\left|\int\psi_W\psi_{\alpha W}{}^*u(Z)u^*(Z-2)u^*(\alpha)\,Vd\tau\right|^2 \quad (40)$$

or, if φ_0 and φ_α are not too small (not exact resonance),

$$\psi_{\text{dis}}=(\pi^3\lambda\lambda_0e^{-2G}/\sin^2\varphi_0\sin^2\varphi_\alpha)I \quad \text{(neutron and } \alpha \text{ slow)} \quad (41)$$

where I stands for the integral in (40).

I is dimensionless, ψ_W and $\psi_{\alpha W}$ being normalized per unit energy. I should be rather smaller than unity, because the formation of the α-particle in the nucleus requires a rather serious rearrangement. The order of magnitude of I is determined by experiments with fast neutrons: if the neutron energy is a few million volts, $\lambda\lambda_0/\sin^2\varphi_0$ has to be replaced by $r_0{}^2$.[20] If the α-particle is fast enough to go over the top of the potential barrier, the factor $e^{-2G}/4\sin^2\varphi_\alpha$ has to be left out, so that for high energy of neutron and α-particle[21]

[17] Aston, *Mass Spectra and Isotopes*, 1933, pp. 106, 167.
[17a] H. A. Bethe, Phys. Rev. **47**, 633 (1935).
[18] J. Chadwick and Goldhaber, Nature **135**, 65 (1935); Taylor and Goldhaber, Nature **135**, 341 (1935).
[19] We speak for definiteness of α-particles, the theory is exactly similar for protons.

[20] The factor C^2 (cf. 18) has no longer the value $\lambda_0/\sin^2\varphi_0$ but is approximately λ (cf. 13). Besides, one has to consider that now neutron angular momenta up to $l=r_0/\lambda$ are effective, which introduces a factor $(r_0/\lambda)^2$.
[21] It should be noted, however, that the wave functions inside I change also, though slowly, with the energy of the particles. I in (42) (high energy) is therefore different from

$$\Phi_{\text{dis}} = 4\pi^3 r_0^2 I. \qquad (E_{\text{neutron}} \text{ large, } E_\alpha \text{ large.}) \qquad (42)$$

The measured cross sections for fast neutrons are of the order $0.3 \cdot 10^{-24}$ (fluorine, Fermi). The nuclear radius of F being about $4 \cdot 10^{-13}$, we have[22]

$$\pi^3 I \approx 1/2, \quad I \approx 1/60. \qquad (43)$$

c. Light nuclei, slow neutrons

If the nucleus is light and the gain in kinetic energy sufficiently large so that the α-particle can go over the top of the barrier even with neutron energy zero, we have to leave out $e^{-2G}/4 \sin^2 \varphi_\alpha$ in (41) and find (cf. 43)

$$\Phi_{\text{dis}} = 4\pi^3 I \lambda\lambda_0 / \sin^2 \varphi_0 \approx 2\lambda\lambda_0 / \sin^2 \varphi_0.$$

$$\text{(slow neutrons, fast } \alpha\text{'s.)} \qquad (44)$$

The cross section increases in this case enormously with decreasing energy of the neutron, just as in the capture case. For gas-kinetic energy, we find (cf. (20), (23))[22]

$$\Phi_{\text{dis}} \approx 1300 \cdot 10^{-24} / \sin^2 \varphi_0. \qquad (45)$$

The experimental cross sections of Li^6 and B^{10} for slow neutrons are of the order 10^{-21} cm^2. They can, therefore, be explained without assuming a "resonance effect" due to small φ_0. The greater *a priori* probability of disintegrations with emission of particles as against such with emission of γ-rays makes the cross section for the former process larger both for fast and slow neutrons.

d. Fast neutrons, heavy nuclei: potential barrier

We now want to discuss briefly the effect of the penetrability of the potential barrier surrounding the nucleus for charged particles. If the neutron is fast and the α-particle energy smaller than the top of the barrier, the cross section (42) has to be multiplied by the penetrability

$$P = e^{-2G}/4 \sin^2 \varphi_\alpha. \text{ (fast neutrons, slow } \alpha\text{'s.)} \qquad (46)$$

Leaving out for the present the resonance denominator $4 \sin^2 \varphi_\alpha$, we have to discuss G. Assuming, as Gamow did, a large negative potential energy inside the nucleus ($r < r_0$) and the pure Coulomb potential $V = e^2 Zz/r$ outside ($r > r_0$, $z = 2$ for α-particle, 1 for proton), the integral $G(39)$ has the well-known value

$$G = \frac{2e^2 Zz}{\hbar v}\left[\text{arc cos}\left(\frac{r_0}{r_1}\right)^{\frac{1}{2}} - \left(\frac{r_0(r_1 - r_0)}{r_1^2}\right)^{\frac{1}{2}}\right], \qquad (47)$$

where
$$r_1 = e^2 Zz/E \qquad (48)$$

is the distance of closest approach in the Coulomb field and v the velocity of the particle. We assume the nuclear radius to be proportional to $Z^{\frac{1}{3}}$ which seems to be verified by experiment, then from Gamow's data on radioactive nuclei it follows that

$$r_0 = 1.9 \cdot 10^{-13} Z^{\frac{1}{3}} \text{ cm} = \tfrac{2}{3}(e^2/mc^2) Z^{\frac{1}{3}}. \qquad (49)$$

The height of the potential barrier is then

$$V_0 = e^2 Zz/r_0 = (3/2) mc^2 z Z^{\frac{2}{3}} = 0.75 z Z^{\frac{2}{3}} \text{ MV}. \qquad (50)$$

To a given energy of the particle, there corresponds a critical nuclear charge Z_0 for which $V_0 = E$, *viz.*,

$$Z_0 = (2E/3zmc^2)^{\frac{3}{2}}, \qquad (51)$$

the corresponding nuclear radius is

$$R = \left(\frac{2}{3}\right)^{\frac{3}{2}} \frac{e^2}{mc^2}\left(\frac{E}{zmc^2}\right)^{\frac{1}{2}} = 1.55\left(\frac{E}{zmc^2}\right)^{\frac{1}{2}} \cdot 10^{-13} \qquad (52)$$

If the particle of energy E falls on any nucleus having higher atomic number than the "critical" Z_0 we have $r_0 = R(Z/Z_0)^{\frac{1}{3}}$ and $r_1 = RZ/Z_0$ and after slight reduction

$$G = \frac{4}{3^{\frac{2}{3}}} \frac{e^2}{\hbar c}\left(\frac{M}{mz}\right)^{\frac{1}{2}} \frac{E}{mc^2} \varphi(Z/Z_0)$$

$$= 0.241 \frac{E}{mc^2}\left(\frac{A}{z}\right)^{\frac{1}{2}} \varphi(Z/Z_0), \qquad (53)$$

where A is the atomic weight of the emitted particle and

$$\varphi(\zeta) = \zeta \text{ arc cos } (\zeta^{-\frac{1}{3}}) - \zeta^{\frac{2}{3}}(1 - \zeta^{-\frac{2}{3}})^{\frac{1}{2}}. \qquad (54)$$

For numerical calculations, it is still more convenient to use

$$G' = 2 \cdot {}^{10}\log e \cdot G = 0.209 \varphi(Z/Z_0) \frac{E}{mc^2}\left(\frac{A}{z}\right)^{\frac{1}{2}}, \qquad (55)$$

I in (41) (small energy), only the order of magnitude is the same. On the other hand, when the neutron energy changes, say, from 0 to 100,000 volts, I is very nearly constant.

[22] This estimate may, for a given case, be wrong by a factor 10 one way or the other. This uncertainty applies to the formulae (44) to (46).

so that $e^{-2G}=10^{-G'}$. Evaluation of (54) gives

for $Z/Z_0=$

1.5	2.0	2.5	3	3.5	4	5	6	8	10

$0.209\varphi=$

.026	.071	.125	.190	.259	.333	.490	.662	1.02	1.42

We are interested in the penetrability

(1) for α-particles produced by fast neutrons. The fastest neutrons emitted in appreciable number from the usually employed Be+radon source have experimentally[23] an energy of about 8 MV. Adding the 2 MV energy gained (in the average) in the α-type transformation (see above), and subtracting 1 MV for the energy of the recoil nucleus,[24] we find

$$E_\alpha=9\text{ MV},\ Z_0=14.6;$$

(2) for α-particles produced by slow neutrons; in the average,

$$E_\alpha=2\text{ MV},\ Z_0=1.53;$$

(3) for protons produced by fast neutrons. Assuming that the nucleus produced by this disintegration emits β-rays of 4 MV maximum energy, and taking $\frac{1}{2}$MV for the mass difference of the neutron and the hydrogen atom

$$E_p=4.5\text{ MV},\ Z_0=14.6.$$

The penetrabilities for these 3 cases are given for various nuclear charges in Table II. The nuclear charges refer to the nucleus *produced* by the disintegration. It is seen that the α-type disintegration by fast neutrons (9 MV α-particles) should begin to become less probable for Z larger than 20 and should be unobservable for $Z>27$, assuming that a penetrability of less than 1/10 makes the process unobservable and remembering that the initial nucleus has a charge by 2 higher than the nucleus produced. In Fermi's experiments, $Z=27$ (Co) is actually the heaviest nucleus found to emit α-particles under neutron bombardment, and the disintegrations of the α-type have only medium or weak intensity for $Z>20$, whereas most of the processes of the α-type found with lighter nuclei are "intense" in Fermi's nomenclature.

The heaviest nucleus for which *proton* emission under neutron bombardment has been established is Zn ($Z=30$) whereas we should expect weak effects of this type to persist up to nearly $Z=40$, and even higher atomic number if we take into account the possibility of resonance for the proton (factor $4\sin^2\varphi_\alpha$ in the denominator of P, (46); any value for φ_α between $-90°$ and $+90°$ being equally probable). The proton penetrates the potential barrier easier because of its small mass, the energies (9 and 4.5 MV) being in the ratio of the respective charges.

e. Slow neutrons, comparatively heavy nuclei

When the penetrability of the potential barrier for the α-particle is disregarded, gas-kinetic neutrons are about 5000 times more effective in producing α-type disintegration than fast ones. According to Table II, the penetrability for 2 MV α-particles becomes 1/5000 when the nucleus produced has charge $Z=9$; therefore, with our assumption about the energy balance slow neutrons should be more effective than fast ones in producing α-type disintegration in all nuclei up to $Z=11$ (Na). A possible resonance for either neutron or α-particle would shift this limit to about $Z=16$ (S); the same would be true if the energy gained in the disintegration process was 3 instead of 2 MV.

Assuming the mass defects for all elements concerned to be correctly measured,[17a] the following α-type disintegrations should be more easily produced by slow than by fast neutrons: $N^{14}\rightarrow B^{11}$(100), $F^{19}\rightarrow N^{16}$[25](10), however, not: $C^{12}\rightarrow Be^9$, $Ne^{20}\rightarrow O^{17}$ and probably $O^{16}\rightarrow C^{13}$ which should be endothermic. The numbers in brackets give the probable increase of the cross section in going from fast to slow neutrons. For the heavier elements, the masses are not known accurately enough to predict the energy gain in the α-type disintegration; possibly the following transformations might be caused with high probability by slow neutrons: $Mg^{24}\rightarrow Ne^{21}$, $Mg^{25}\rightarrow Ne^{22}$, $Mg^{26}\rightarrow Ne^{23}$, $Al^{27}\rightarrow Na^{24}$, $Si^{28}\rightarrow Mg^{25}$, $P^{31}\rightarrow Al^{28}$, etc. The α-type disintegration caused by slow neutrons seems to provide a very good method for correlating masses of different nuclei. The kinetic energy of the projectile, the neutron, is known to be exactly zero (for purposes of atomic weight determination); consequently the α-

[23] Dunning, Phys. Rev. **45**, 586 (1934). Faster neutrons are reported as well but seem to be very rare indeed, and therefore are not likely to play an important part in causing disintegrations.

[24] True for Z of the order 20.

[25] The maximum energy of the β-rays emitted by N^{16} was assumed to be 6 MV, corresponding to an average energy of 2 MV as measured by Fermi.

TABLE II. *Reciprocal value of penetrability of potential barrier.*

$Z=$	5	7	10	15	20	25	30	40	50
α-particles									
9 MV	1	1	1	1	2.6	12	80	10^4	$4\cdot10^5$
2 MV	18	250	$2\cdot10^4$	$6\cdot10^7$	$6\cdot10^9$	$1.5\cdot10^{13}$	$4\cdot10^{16}$	10^{24}	10^{31}
protons									
4.5 MV	1	1	1	1	1.4	2.4	4.7	25	100

particles are homogeneous. The unambiguity of the energy determination is greatly increased by the fact that only very slow and very fast neutrons are effective in producing the α-type disintegration, but not neutrons of medium energy, say, a few hundred thousands of volts. When the neutron energy decreases from 5 MV, the first factor to change is the penetrability of the potential barrier for the α-particle. If this factor is small for neutron energy zero, it cannot be large for neutron energies of 1 MV. However when the neutron energy is reduced to 1000 volts or less, the cross section increases again because the neutron stays longer inside the nucleus.

The disintegration $F^{19}+n'\rightarrow N^{16}+He^4$ in particular, if it proves to be sufficiently exothermic to be possible with slow neutrons, seems to afford the most hope of determining the mass of the neutrino. For this purpose, the maximum energy of the β-rays emitted by N^{16} in the process $N^{16}\rightarrow O^{16}+e^-+n^0$ ($n^0=$neutrino) should be measured accurately, as well as the kinetic energies of the α-particles emitted in the production of N^{16}, and of those emitted in the process $F^{19}+H^1\rightarrow O^{16}+He^4$. Moreover, N^{16} seems to be the best nucleus known upon which to observe the recoil caused by β-disintegration;[26] if the maximum energy of the β-rays is 5 MV, which seems to be rather a lower limit, N^{16} should receive 800 volts recoil energy.

The disintegration group $Al^{27}+n^1\rightarrow Na^{24}+He^4$, $Na^{24}\rightarrow Mg^{24}+e^-+n^0$, $Al^{27}+H^1\rightarrow Mg^{24}+He^4$ would give an alternative way for the determination of the neutrino mass, but the first reaction of this group is less likely to occur with slow neutrons than the fluorine disintegration because of the high potential barrier of Na.

[26] Cf. H. Bethe and R. Peierls, Nature **133**, 532, 689 (1934).

APPENDIX 1. THE EFFECT OF SLOW NEUTRONS WITH NONVANISHING ANGULAR MOMENTUM $l\neq0$

For $l=1$, the wave function outside the nucleus is

$$\psi_1=\sin(kr+\delta_1)/k^2r-\cos(kr+\delta_1)/k. \quad (56)$$

Inside the nucleus ψ_1 has the same form as ψ_0, cf. (7), only with y having a different meaning. For convenience, we expand ψ in a power series in kr and δ:

$$\psi_1=\delta_1/k^2r+\tfrac{1}{3}kr^2+0(k^3r^4)+0(r\delta). \quad (57)$$

Neglecting all powers of δ and kr higher than the first, we find by joining the outside and inside wave functions together

$$\begin{aligned} C_1y(r_0)^{-\frac{1}{2}}\cos\varphi(r_0)&=\delta_1/k^2r+\tfrac{1}{3}kr^2,\\ -C_1y(r_0)^{+\frac{1}{2}}\sin\varphi(r_0)&=-\delta_1/k^2r^2+\tfrac{2}{3}kr, \end{aligned} \quad (58)$$

wherefrom

$$\begin{aligned} C_1&=\frac{kr_0(\lambda_1)^{\frac{1}{2}}}{(\lambda_1/r_0)\cos\varphi_1-\sin\varphi_1},\\ \delta_1&=\tfrac{1}{3}k^3r_0^3\frac{2(\lambda_1/r_0)\cos\varphi_1+\sin\varphi_1}{(\lambda_1/r_0)\cos\varphi_1-\sin\varphi_1}. \end{aligned} \quad (59)$$

Apart from the resonance denominator, δ_1 is smaller than the phase δ for $l=0$, given in (11), by a factor $(kr_0)^2$, the elastic scattering cross section therefore by a factor $(kr_0)^4=10^{-15}$ for gas-kinetic neutrons. C_1^2 is likewise smaller by a factor $(kr_0)^2=5\cdot10^{-7}$ than the corresponding quantity for $l=0$ (cf. (14)). To obtain, e.g., a capture cross section of only 10^{-25} cm^2 for neutrons of $E=kT$ and $l=1$, the phase φ_1 would have to lie in a region of $1/10°$ breadth near the value $\mathrm{tg}^{-1}(\lambda_1/r_0)$ which is very unlikely. It is slightly better for higher energy, because all effects due to "p-neutrons" decrease with decreasing energy; the elastic scattering as E^2, the capture and α-type disintegration as $E^{\frac{1}{2}}$. There is just a chance that there might be an element which has no bound p-level for the neutron and therefore cannot capture s-neutrons, but for which incoming p-neutrons have a phase φ_1 within 1° of the "right" value $\mathrm{tg}^{-1}(\lambda_1/r_0)$: Such an element would capture p-neutrons of energies round 100,000 volt with a cross section of the order $3\cdot 10^{-25}$ cm^2.

Appendix 2. Attempt to Determine the Potential Energy for Neutrons in the Nucleus[26a]

It is not likely that the approximation made in this paper, i.e., taking the nucleus as a rigid body and representing it by a potential field acting on the neutron, is really adequate. Anyhow, if it is made—and it is the only practicable approximation in many cases—it should be made consistently, and therefore the potential to be assumed is of interest.

We assume the potential to be constant, equal to $-V_0$, inside the nucleus, and treat the neutrons simply as a Fermi gas enclosed in a volume

$$\Omega=(4\pi/3)r_0^3.$$

Since $N=A-Z$ (A = atomic weight) neutrons are in the nucleus, their average kinetic energy is according to the Fermi statistics formula

$$E_0=\frac{3}{5}\frac{(2\pi\hbar)^2}{2M}\left(\frac{3N}{8\pi\Omega}\right)^{2/3}=\frac{3}{5}\frac{3^{4/3}\pi^{2/3}}{2^{7/3}}\frac{\hbar^2}{Mr_0^2}N^{2/3}. \quad (60)$$

Inserting for r_0 the value (49) and taking for N/Z the average value 1.25 (true for medium atomic weight), we find

$$E_0=2.39\left(\frac{3}{2}\right)^2\left(\frac{\hbar c}{e^2}\right)^2\frac{m}{M}mc^2(1.25)^{\frac{2}{3}}=24 \text{ MV}. \quad (61)$$

The binding energy is V_0-E_0, it has experimentally the value ~ 8 MV, which makes

$$V_0=32 \text{ MV}. \quad (62)$$

With this value for V_0, the phase φ_0 of the wave function of a neutron with energy E at the boundary r_0 of the nucleus becomes[26b]

$$\varphi_0=\frac{(2M)^{\frac{1}{2}}}{\hbar}(V_0+E)^{\frac{1}{2}}r_0-\frac{\pi}{2}$$

$$=\tfrac{2}{3}\sqrt{2}\frac{e^2}{\hbar c}\left(\frac{M}{m}\right)^{\frac{1}{2}}\left(\frac{V_0+E}{mc^2}\right)^{\frac{1}{2}}Z^{\frac{1}{3}}-\frac{\pi}{2}$$

$$=2.36Z^{\frac{1}{3}}(1+0.015E)-\tfrac{1}{2}\pi \quad (63)$$

if E is measured in million volts. For Ag($Z=47$)

$$\varphi_0=2.2\pi+0.04\pi E, \quad (64)$$

which means that the s-wave function makes $2\frac{1}{2}$ complete oscillations in the nucleus, and φ_0 changes by $1°=\pi/180$, if E changes by

$$1/(180\cdot 0.04)=0.14 \text{ MV}.$$

Appendix 3. Effect of Gradual Decrease of Potential at the Boundary of the Nucleus

We represent the potential energy of the neutron at the boundary of the nucleus by the Eckart potential[27]

$$V=-V_0+V_0e^{2x/l}/(1+e^{2x/l})$$

$$=-V_0/(1+e^{2x/l}), \quad (65)$$

where $x=r-r_0$ and l is essentially the range of the forces between neutron and nucleus (V decreases, e.g., from $\frac{3}{4}V_0$ to $\frac{1}{4}V_0$ in a distance $1.1l$). Eckart has given the analytical form of the wave functions in the potential (65): If the wave function Ψ behaves like an outgoing wave e^{ikx} for large positive x (great distances from the nucleus), its value for large negative x (inside the nucleus) is

$$\Psi=a_1e^{ik_0x}+a_2e^{-ik_0x} \qquad (k_0=(2MV_0)^{\frac{1}{2}}/\hbar) \quad (66)$$

with

$$a_1{}^*=\frac{\Gamma(1+2i\beta)\Gamma(2i\alpha)}{\Gamma(1+i(\alpha+\beta))\Gamma(i(\alpha+\beta))},$$

$$a_2=\frac{\Gamma(1-2i\beta)\Gamma(2i\alpha)}{\Gamma(1+i(\alpha-\beta))\Gamma(i(\alpha-\beta))}, \quad (67)$$

[26a] Similar calculations with similar results have been made by Van Vleck (Phys. Rev. in press) and Goudsmit (unpublished).

[26b] From the definition (7), (8) it follows that $\varphi(r=0)=-\frac{1}{2}\pi$ if the potential is constant inside the nucleus.

[27] C. Eckart, Phys. Rev. **35**, 1303 (1930). We put Eckart's $B=0$, our l is Eckart's l divided by π, our $k=2\pi/\lambda'$, our $k_0=2\pi/\lambda$, our V_0 = Eckart's A, our $E=W-A$.

where $\alpha = \frac{1}{2}k_0 l; \quad \beta = \frac{1}{2}kl.$ (68)

and the asterisk denotes the conjugate complex value.

From (66) it is easy to derive the asymptotic form for *large positive* x of a wave function whose behavior *inside* the nucleus is known to be (cf. (7))

$$\psi = C\cos(k_0 x + \chi)/(k_0)^{\frac{1}{2}}. \qquad (69)$$

The asymptotic form for large positive x is:

$$\psi = Re\frac{(a_1^* e^{i\chi} - a_2^* e^{-i\chi})C}{(|a_1|^2 - |a_2|^2)(k_0)^{\frac{1}{2}}} e^{ikx}.$$

$$(Re = \text{real part.}) \qquad (70)$$

So far, the formulae are rigorous. We now make use of the fact that the energy of the neutron, and therefore β, is small. Neglecting higher powers of β than the first, we may write

$$a_1^* = a_0(1 + (\kappa + i\lambda)\beta); \quad a_2 = a_0(1 - (\kappa + i\lambda)\beta) \qquad (71)$$

with $a_0 = \Gamma(1)\Gamma(2i\alpha)/\Gamma(1+i\alpha)\Gamma(i\alpha) = ae^{i\gamma}$, (72)

where κ, λ, a and γ are supposed to be real. Then

$$|a_1|^2 - |a_2|^2 = 4\kappa\beta a^2,$$

$$a_1^* e^{i\chi} - a_2^* e^{-i\chi} = 2a[i\sin(\gamma + \chi + \lambda\beta) + \kappa\beta\cos(\gamma + \chi)], \qquad (73)$$

and (70) goes over into

$$\psi = -\frac{2aC\sin(\gamma + \chi - \lambda\beta)}{4\kappa\beta a^2 (k_0)^{\frac{1}{2}}} \times \sin(kx - \kappa\beta\,\mathrm{ctg}\,(\gamma + \chi)). \qquad (74)$$

Evidently, we have to identify (cf. (5), (11), $x = r - r_0$!)

$$\gamma + \chi = \varphi_0 \qquad (75)$$

$\delta = kr_0 + \kappa\beta\,\mathrm{ctg}\,\varphi_0$, therefore (cf. (11)) $\kappa\beta = k\lambda_0$ and (cf. (68))

$$\lambda_0 = \tfrac{1}{2}l\kappa, \qquad (76)$$

and furthermore we have to put (cf. (5))

$$C\sin\varphi_0/2\kappa\beta a(k_0)^{\frac{1}{2}} = 1/k,$$

$$C = l\kappa a(k_0)^{\frac{1}{2}}/\sin\varphi_0. \qquad (77)$$

From the definition of κ, (71), we find, using the well-known formula[28] for $|\Gamma(1 - ix)|$

$$2\kappa = \lim_{\beta=0}\frac{d}{d\beta}\log|a_1|^2$$

$$= \lim_{\beta=0}\frac{d}{d\beta}\log\left(\frac{\beta\sinh^2\pi(\alpha+\beta)}{\alpha\sinh 2\pi\beta\sinh 2\pi\alpha}\right) \qquad (78)$$

$$= \lim_{\beta=0}\left(\frac{1}{\beta} - \frac{2\pi}{\mathrm{tgh}\,2\pi\beta} + \frac{2\pi}{\mathrm{tgh}\,\pi(\alpha+\beta)}\right) = \frac{2\pi}{\mathrm{tgh}\,\pi\alpha}.$$

Inserting this into (76), we have

$$\lambda_0 = (\pi/2)(l/\mathrm{tgh}\,\pi\alpha). \qquad (79)$$

For rapid changes of potential, or more accurately, if l is small compared with the wavelength $\hbar/(2MV_0)^{\frac{1}{2}}$ of the particles inside the potential field, α is small (cf. 68) and (79) goes over into (12), *viz.*

$$\lambda_0 = 1/k_0 = \hbar/(2MV_0)^{\frac{1}{2}}.$$

In our case, however, l is assumed to be 1.5×10^{-13} cm (cf. paragraph 3), whereas $\hbar/(2MV_0)^{\frac{1}{2}} = 1.0 \cdot 10^{-13}$ with $V_0 = 32$ MV. Therefore, $\pi\alpha = 2.36$ and tgh $\pi\alpha = 0.982$, so that practically

$$\lambda_0 = \tfrac{1}{2}\pi l. \qquad (80)$$

To evaluate (77), we calculate

$$a^2 = \lim_{\beta=0}|a_1|^2 = \frac{\sinh^2\pi\alpha}{2\pi\alpha\sinh 2\pi\alpha} = \frac{\mathrm{tgh}\,\pi\alpha}{4\pi\alpha}$$

$$C = l\left(\frac{\pi}{4\alpha\,\mathrm{tgh}\,\pi\alpha}\right)^{\frac{1}{2}}\frac{(k_0)^{\frac{1}{2}}}{\sin\varphi_0} = \frac{(\pi l/2\,\mathrm{tgh}\,\pi\alpha)^{\frac{1}{2}}}{\sin\varphi_0} \qquad (81)$$

which is exactly formula (14) with λ_0 given by (79).

[28] Jahnke-Emde, *Table of Functions*, p. 89.