In Eq. (7) we have

$$\mathcal{H}'_{\nu_1\nu_2,\,11}=\beta\int\cdots\int F_{\nu_1}^{*}F_{\nu_2}^{*}\xi_1\xi_2^2\cos\vartheta_1(3\cos^2\vartheta_2-1)F_1F_1\xi_1^2\xi_2^2\sin\vartheta_1\sin\vartheta_2 d\xi_1 d\vartheta_1 d\varphi_1 d\xi_2 d\vartheta_2 d\varphi_2$$

$$=\beta\int\cdots\int F_{\nu_1}^{*}F_{\nu_2}^{*}\{48\sqrt{2}F_{310}F_{420}-96\sqrt{(3)}F_{310}F_{320}-96\sqrt{2}F_{210}F_{420}$$

$$+192\sqrt{(3)}F_{210}F_{320}\}\xi_1\xi_2\sin\vartheta_1\sin\vartheta_2 d\xi_1 d\vartheta_1 d\varphi_1 d\xi_2 d\vartheta_2 d\varphi_2,$$

and hence

$$\mathcal{H}'_{23,11}=192\sqrt{(3)}\beta,\qquad \mathcal{H}'_{33,11}=-96\sqrt{(3)}\beta,\qquad \mathcal{H}'_{24,11}=-96\sqrt{2}\beta,$$

and $\mathcal{H}'_{34,\,11}=48\sqrt{2}\beta$, all others being equal to zero.

To illustrate the evaluation of the integral Δ let us consider $\Delta_{11,\,11}$:

$$\Delta_{11,\,11}=\int\cdots\int\{F_{100}(\xi_1)F_{100}(\xi_2)\}^2\xi_1^2\xi_2^2\sin\vartheta_1\sin\vartheta_2 d\xi_1 d\vartheta_1 d\varphi_1 d\xi_2 d\vartheta_2 d\varphi_2$$

$$=\int\cdots\int F_{100}(\xi_1)F_{100}(\xi_2)\{2F_{200}(\xi_1)F_{200}(\xi_2)-2\sqrt{2}F_{200}(\xi_1)F_{100}(\xi_2)-2\sqrt{2}F_{100}(\xi_1)F_{200}(\xi_2)$$

$$+4F_{100}(\xi_1)F_{100}(\xi_2)\}\xi_1\xi_2\sin\vartheta_1\sin\vartheta_2 d\xi_1 d\vartheta_1 d\varphi_1 d\xi_2 d\vartheta_2 d\varphi,$$

or, making use of the orthogonality and normalization of the F's, $\Delta_{11,\,11}=4$.

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The Magnetic Moment of the K^{39} Nucleus

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Starting with the K^+ field given by Hartree, a 4*s* wave function has been found by numerical integration. The orthogonality correction diminishes the value of the wave function at the origin by about one-third. The experimental value of the hyperfine structure separation of the ^2S normal state is 0.015 cm^{-1}. From this is calculated a molecular magnetic moment of 1.2 nuclear magnetons, as compared with the value of 0.38 nuclear magnetons calculated by Millman, Fox and Rabi on the basis of the modified Goudsmit's formula. The disagreement seems to be due chiefly to the possibility that the single electron wave functions of Fermi and Segré are not mutually orthogonal.

THERE have recently been published two independent determinations of the h.f.s. separation of the normal state of the K^{39} atom. Jackson and Kuhn[1] have used a spectroscopic method; Millman, Fox and Rabi[2] have relied on the method of molecular beams. The numerical agreement is good, and 0.015 cm^{-1} may be taken as the value of the separation.

From this, Millman, Fox and Rabi estimate the value of the nuclear magnetic moment to be 0.38 nuclear magneton, using the semi-empirical formula due to Goudsmit[3] and Fermi and Segré.[4, 5] Since previous cases of low magnetic moments have been associated with isotopes of even mass number, the above value seemed somewhat anomalous, and we therefore thought it of interest to ascertain whether or not the "anomaly" could be removed by the use of a Hartree wave function. This we now proceed to investigate.

[1] D. A. Jackson and H. Kuhn, Nature **134**, 25 (1934); Proc. Roy. Soc. **A148**, 335 (1935).

[2] Millman, Fox and Rabi, Phys. Rev. **46**, 320 (1934). These authors were able to determine the spin as well as the h.f.s. splitting.

[3] S. Goudsmit, Phys. Rev. **43**, 636 (1933).

[4] E. Fermi and E. Segré, Zeits. f. Physik **82**, 729 (1933).

[5] E. Fermi and E. Segré, Memorie, R. Accademia d'Italia **4**, 131 (1933).

Wave Functions

Starting with the field of K^+, which has been published by Hartree,[6] we have performed a numerical integration to obtain the $4s$ wave function.[7] The results are given in Table I.

For simplification of the calculations, it is desirable to make the wave functions mutually orthogonal. The orthogonalization integrals are: $(1s, 2s)=0.0113$; $(1s, 3s)=0.00365$; $(1s, 4s)=0.000562$; $(2s, 3s)=0.0212$; $(2s, 4s)=0.00422$; and $(3s, 4s)=0.0506$. Let us denote by P^0 the functions after orthogonalization and normalization ($\int(P^0)^2dr=1$). Then $P^0(2s)=P(2s)-0.0113\ P(1s)$; $P^0(3s)=P(3s)-0.0212\ P(2s)-0.00341\ P(1s)$; and $P^0(4s)=1.0013[P(4s)-0.0505\,P(3s)-0.00315\,P(2s)-0.000342\,P(1s)]$. The values of the nonorthogonal wave functions $\chi=(P/r)$ at the origin are: $\chi_{1s}(0)=161.9$; $\chi_{2s}(0)=45.17$; $\chi_{3s}(0)=15.76$; and $\chi_{4s}(0)=3.124$. Before orthogonalization, $[\chi_{4s}(0)]^2=9.76$; afterwards, $[\chi_{4s}{}^0(0)]^2=4.548$. (That is, the value of the nuclear magnetic moment would be smaller by a factor of about two if the calculations were made with the nonorthogonal wave functions.)

Nuclear Moment

The h.f.s. splitting of an s-state is given[8] by $\Delta(s)=(8\pi/3)[(2I+1)/I]\mu\mu_0\psi^2(0)$, where I is the nuclear spin, μ_0 is the Bohr magneton, μ is the nuclear magnetic moment, and $\psi(0)$ is the value of the wave function (normalized so that $\int\psi^2d\tau=1$) at the nucleus. We have[9] $\psi(0)=(1/4\pi)^{\frac{1}{2}}\chi(0)$, so that $\Delta(s)=(2/3)[(2I+1)/I]\mu\times\mu_0\chi^2(0)$. For K^{39}, $I=3/2$. Substituting numerical values, we have, if $g=(\mu/\mu_0)\times1838$,

$$0.015=\frac{(2/3)(8/3)(g/1838)(9.17\times10^{-21})^2 4.548}{(0.528\times10^{-8})^3\times19.662\times10^{-17}}.$$

Solving, $g=949\Delta(s)[I/(2I+1)][1/\chi^2(0)]=1.17$. That is, we obtain a nuclear magnetic moment about three times as large as that found by Fermi. This disagreement has its origin, of course, in the fact that different eigenfunctions have been employed. We use a Hartree function; Fermi and Segré use a one-electron function based on the statistical method, which function contains parameters to be determined from the experimental term values.

Since the article of Fermi and Segré[5] is somewhat inaccessible, we shall indicate their procedure.[10] Let the Schrödinger radial function be $\rho\psi$, where ρ is the normalization factor. Then $u=r\psi$ satisfies the differential equation $(d^2u/dr^2)+(8\pi^2m/h^2)(E-eV)u=0$. The potential V is found by the statistical method, and the equation solved[11] for $E=0$ by the method of Wentzel-Brillouin. One sets $u=kR\sin\theta$, where $R^2(d\theta/dr=1$. k may be determined by matching such a solution with the power series development about the origin.

To evaluate ρ, it is supposed that at a point ξ a perturbation $\lambda\delta(r-\xi)$ is introduced, where λ is an arbitrary parameter and δ is the Dirac improper function. The first-order perturbation energy is then $\Delta E=4\pi\lambda\rho^2u^2(\xi)$. At the point ξ, the value of u will not be changed, but there will

Table I. *Normalized $P(4s)$ for potassium.*

r	P	ξ	$\epsilon=0.265$ r	P	r	P	η
0.000	0.0000	19.000	0.40	−0.0598	6.0	−0.4220	0.1199
.005	.0142	19.374	.45	− .0312	6.5	− .3937	.1573
.010	.0257	19.788	.50	− .0003	7.0	− .3609	.1881
.015	.0349	20.254	.55	+ .0307	7.5	− .3264	.2139
.020	.0421	20.779	.60	+ .0601	8.0	− .2916	.2358
					8.5	− .2578	.2547
.03	.0509	22.048	.7	.1105	9.0	− .2261	.2712
.04	.0540	23.724	.8	.1465	9.5	− .1967	.2857
.05	.0527	26.001	.9	.1679	10.0	− .1700	.2988
.06	.0480		1.0	.1761			
.07	.0409		1.1	.1734	11	− .1246	.3206
.08	.0321		1.2	.1623	12	− .0896	.3383
.09	.0220				13	− .0634	.3531
.10	.0113		1.4	.1226	14	− .0443	.3656
			1.6	.0699	15	− .0305	.3763
.12	−.0108		1.8	.0119	16	− .0209	.3856
.14	−.0321		2.0	− .0470	17	− .0138	.3937
.16	−.0511		2.2	− .1040	18	− .0095	.4008
.18	−.0673		2.4	− .1578	19	− .0063	.4071
.20	−.0801		2.6	− .2074	20	− .0042	.4128
.22	−.0896		2.8	− .2524	21	− .0028	.4179
.24	−.0958		3.0	− .2926	22	− .0018	.4225
.26	−.0989		3.2	− .3279	23	− .0012	.4267
.28	−.0993		3.4	− .3583	24	− .0008	.4305
.30	−.0972		3.6	− .3840	25	− .0005	.4340
			3.8	− .4052	26	− .0003	.4372
.35	−.0830		4.0	− .4221	27	− .0002	.4402
					28	− .0001	.4429
			4.5	− .4476	29	− .0001	.4455
			5.0	− .4530	30	− .0000	.4480
			5.5	− .4430	31	− .0000	.4502

[6] D. R. Hartree, Proc. Roy. Soc. **A143**, 506 (1934).

[7] As usual, it has been assumed that it is unnecessary to make the field self-consistent.

[8] E. Fermi, Zeits. f. Physik **60**, 322 (1930).

[9] The value of the nuclear moment reported by us at the Pittsburgh meeting of the Am. Phys. Soc. was too small by this factor 4π, which had been omitted.

[10] Certain details have been omitted by Fermi and Segré. These we have tried to reconstruct as accurately as possible.

[11] In the article by F. & S., the term "∞s eigenfunctions" is used. We interpret this to mean s eigenfunctions with $E=0$. Such eigenfunctions, which were used earlier by Fermi in his calculations of the Rydberg corrections, do not differ much from the optical ones in the interior of the atom.

be a phase change $\Delta\theta$. If this were π, the number of nodes would be different by unity. Therefore, put $\Delta E=-(\Delta\theta/\pi)(dE/dn)$. The discontinuity at ξ introduced by the perturbation into the first derivative of the wave function is $\Delta u'=(8\pi^2m/h^2)\lambda u(\xi)$. Now $u'=kR'\sin\theta+kR\cos\theta\times(d\theta/dr)$. Therefore

$$\begin{aligned}\Delta u'&=kR\sin\theta\Delta(R'/R)+(k/R\sin\theta)\Delta(\sin\theta\cos\theta)\\&=kR\sin\theta\Delta(R'/R)+(k/2R\sin\theta)\Delta(\sin 2\theta)\\&=kRR'\sin\theta\Delta(1/R)+(k/R\sin\theta)\cos 2\theta\Delta\theta\end{aligned}$$

provided that R' is not appreciably altered.[12] Since $\Delta u=0$, we have $(R+\Delta R)\sin(\theta+\Delta\theta)=R\sin\theta$, or $\Delta R=-R\cot\theta\Delta\theta$. Accordingly, $\Delta(1/R)=(1/R)\cot\theta\Delta\theta$, and $\Delta u'=kR'\cos\theta\Delta\theta+(k/R\sin\theta)\cos 2\theta\Delta\theta$. Now let us choose[13] the point ξ so that $\theta=\pi/2$. Then[14] $(\Delta\theta)_{\pi/2}=-(8\pi^2m/h^2)\lambda R^2$. Substituting in the expression for ΔE, we finally obtain $\rho^2=(2m/h^2k^2)(dE/dn)$. The value of k was determined for 15 atoms by Amaldi and Fermi,[15] and correction factors with respect to the formula $\psi^2(0)=(1/\pi a^3)(Z/2Rh)\times(dE/dn)$ were calculated for various values of Z. For small Z, the factor is approximately unity. If, then, we have a Rydberg series, we can readily compute $\psi^2(0)$ from the empirical value of dE/dn, so that this formula is quite useful.

The calculations of Fermi and Segré seem to us somewhat incomplete, however, since they have not found a set of mutually orthogonal single-electron wave functions. This is, apparently, the principal reason for the disagreement between our results and theirs, as may be seen by a comparison of the nonorthogonal $4s$ functions calculated by the two methods. The statistical method gives $\chi_{4s}^2(0)=13.7$; while the Hartree method results in $\chi_{4s}^2(0)=9.76$. The agreement is, in our opinion, as good as is to be expected.

We do not at present feel able to estimate how accurate the results from the Hartree method are. Experience to date seems to indicate that the Hartree functions are fairly reliable, especially when applied to scattering problems. We conclude that the value of the magnetic moment found from the Hartree method is probably the most accurate so far, but may be subject to modification when better wave functions are known. Because of this uncertainty, we have not calculated a value of the magnetic moment from the splitting of the $^2P_{3/2}$ state, for even if agreement with the value from the $^2S_{1/2}$ state were obtained, it might not be overly significant.[16]

Finally, our value of 1.2 nuclear magnetons is not as abnormally low as that found by Millman, Fox and Rabi, so that we do not consider that there is any basis for regarding the structure of the K^{39} nucleus as exceptional.

[12] This will probably be the case, since R is the amplitude of the oscillating function.

[13] This brings up the question as to whether the procedure is actually independent of the choice of ξ. Fermi and Segré do not discuss the point.

[14] This is in agreement with formula (30), p. 152, reference 5, for $\theta=\pi/2$, but we are unable to check the general validity of this formula.

[15] Reference 30 (reference 5). Apparently, this work is still unpublished.

[16] The ^2P state is probably perturbed, according to Fermi and Segré (reference 5, p. 141).