Two other M odd Z odd nuclei ${}_{27}Co^{59}$ [15] and ${}_{21}Sc^{45}$ [16] have $I=7/2$ and $g(I)$-factors 0.77 and 1.0, respectively. These $g(I)$ values which the respective writers give only as estimates are of the same order of magnitude as those of ${}_{55}Cs^{133}$, ${}_{57}La^{139}$ and ${}_{51}Sb^{123}$.

The apparent regularity in mechanical and magnetic moments pointed out above suggests that the nuclei Cs^{133}, La^{139}, Sb^{123} and possibly Co^{59} and Sc^{45} have some structural feature in common.

In conclusion the authors wish to thank Professor E. F. Burton, Director of the McLennan Laboratory, for his interest in this investigation and for the facilities placed at our disposal.

[15] K. R. More, Phys. Rev. **46**, 470 (1934). See also H. Kopfermann and E. Rasmussen, Naturwiss. **22**, 219 (1934).
[16] H. Kopfermann and E. Rasmussen, Zeits. f. Physik **92**, 82 (1934). See also H. Schüler and Th. Schmidt, Naturwiss. **22**, 758 (1934).

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Symmetry Properties and the Identity of Similar Particles

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It is demonstrated that in a system of N similar particles, the permissible wave functions are either symmetrical or antisymmetrical with respect to interchanges of the particles, if the following principles are assumed to be valid: (1) The interchange of two like particles produces no change in any measurable property of the system. In particular, if ψ is to be a permissible wave function, $\psi\bar{\psi}$ must be unaltered by such an interchange. (2) All the wave functions obtained from a given permissible wave function by permuting the similar particles are also permissible wave functions for the same eigenvalue.

It has often been remarked that the principle of antisymmetry of the wave function for electrons appears in quantum mechanics as something simply superimposed on the theory to take into account the Pauli exclusion principle. Also in the treatment of photons a corresponding principle of symmetry must be superimposed to obtain the Bose-Einstein statistics, which one must assume photons to obey in order to obtain the Planck radiation law. These and other facts have led to the empirical principle that for any given type of particle only symmetrical states occur or else only antisymmetrical states. It is the purpose of this paper[1] to show that this empirical principle follows logically from other more fundamental principles, *viz.*:

(1) *The interchange of two like particles in a dynamical system will produce no change in any measurable property of the system.*

(2) *All the mathematical quantities obtained from a given permissible mathematical quantity appearing in the theory, by permuting the similar particles, are also permissible.* We have in mind especially the quantity ψ.

These two principles together we shall call the principle of the identity of similar particles. They appear to be necessary from the physical point of view; in other words they are physical axioms. It has often been assumed that wave functions neither symmetrical nor antisymmetrical would distinguish between like particles but explicit proofs are lacking. We shall show that the symmetry or the antisymmetry follows from the principles stated above.

Let there be N particles of the type considered, besides other particles. The coordinates $(x_i, y_i, z_i, \sigma_i)$ including spin σ_i, if present, of the ith particle we shall indicate by x_i. The wave function of the system may be indicated as follows:

$$\psi=\psi(x_1, x_2, \cdots x_N; b). \quad (1)$$

* Now at the Massachusetts Institute of Technology.
[1] E. E. Witmer and J. P. Vinti, Phys. Rev. **43**, 780 (1933).

Here the b symbolizes the coordinates of all the other particles; we shall henceforth omit it. Let us now permute the particles by permuting the positions of their corresponding coordinates in the wave function. The permutation operator we shall call P, where P stands for

$$x_i' = x_j; \quad i, j = 1, 2 \cdots N, \tag{2}$$

so that

$$P\psi(x_1, x_2, \cdots x_N) = \psi[P(x_1, x_2, \cdots x_N)]$$
$$= \psi(x_1', x_2', \cdots x_N') = \psi_P(x_1, x_2, \cdots x_N). \tag{3}$$

Here x_i' stands for the coordinates of the ith particle after the permutation.

$\psi_P(x_1, x_2, \cdots x_N)$ is now a new function of the coordinates $(x_1, x_2, \cdots x_N)$. The permutation which simply interchanges the particles α and β, we shall denote by $P_{\alpha\beta}$.

Now $|\psi(x_1, x_2, \cdots x_N)|^2$ is in concept a measurable physical quantity, since it is a probability density. Hence the first principle above requires

$$|\psi(x_1, x_2, \cdots x_N)|^2 = |\psi(x_2, x_1, \cdots x_N)|^2$$
$$= |\psi(x_1', x_2', \cdots x_N)|^2.$$

Here we have interchanged merely the particles 1 and 2, so that

$$\psi(x_1', x_2', \cdots x_N) = P_{12}\psi(x_1, x_2, \cdots x_N).$$

Now all permutations can be expressed as a product of transpositions, i.e., interchanges of two particles. Therefore we shall restate the principle of the identity of like particles as applied to the probability density in the following manner. If $\psi_1, \psi_2, \psi_3, \cdots$ represent the *totality* of permissible eigenfunctions of states of the system of N particles, then $P_{\alpha\beta}(\psi_k\bar{\psi}_k)$ must equal $\psi_k\bar{\psi}_k$, where ψ_k is *any* of the permissible eigenfunctions. This principle will be used in the sequel to carry out proofs by the method of *reductio ad absurdum*.

We need a preliminary theorem. If

$$P_{\alpha\beta}\psi = \pm\psi \tag{4}$$

the sign must be positive for all interchanges or else negative for all interchanges.

Let us suppose that in Eq. (4) some of the interchanges give rise to plus signs and others to minus signs. Let us divide interchanges into two classes, the first class (A) containing those associated with plus signs and the second class (B) those associated with minus signs. Now there is some particle α in class A which also occurs in class B. To prove this, pick any particle n at random from class A. If it does not occur in class B, then all the interchanges $(n, 1)$, $(n, 2) \cdots (n, n-1)$, $(n, n+1)$, $\cdots (n, N)$ are in class A. But then every one of the particles 1, 2, $\cdots N$ occurs in interchanges of class A, so that if class B has any members, there must be at least one particle α common to both classes. Accordingly, let $P_{\alpha\beta}$ give a plus sign in Eq. (4) and $P_{\alpha\gamma}$ a minus sign.

Now $$P_{\beta\gamma}\psi = P_{\alpha\beta}P_{\alpha\gamma}P_{\alpha\beta}\psi, \tag{5}$$

or $$P_{\beta\gamma}\psi = P_{\alpha\gamma}P_{\alpha\beta}P_{\alpha\gamma}\psi. \tag{6}$$

Thus by Eq. (5) $P_{\beta\gamma}\psi = -\psi$, but by Eq. (6) $P_{\beta\gamma}\psi = +\psi$. We are thus led to a contradiction, so that either class A or class B must be empty. The signs must therefore all be alike.

An alternative,[2] more direct proof, is as follows. We may write $P_{\alpha\beta}\psi = C_{\alpha\beta}\psi$, where $C_{\alpha\beta} = \pm 1$.

Eq. (5) gives $C_{\beta\gamma} = C_{\alpha\beta}C_{\alpha\gamma}C_{\alpha\beta} = C_{\alpha\gamma}C_{\alpha\beta}^2 = C_{\alpha\gamma}$.

From $C_{\beta\gamma} = C_{\alpha\gamma}$ it follows by letting α, β, γ take on all values from 1 to N that the C's are all equal, so that the signs must all be plus or else all minus.

Thus if we can show that $P_{\alpha\beta}\psi = \pm\psi$, the wave function must be either symmetrical or antisymmetrical. Suppose we let

$$\psi = \rho e^{i\eta} \quad (\rho, \eta \text{ real}). \tag{7}$$

If P is any permutation, $P\psi = P\rho e^{iP\eta}$

and $\rho^2 = \psi\bar{\psi} = P(\psi\bar{\psi}) = (P\psi)(P\bar{\psi}) = (P\rho)^2$.

Thus $P\rho = \pm\rho = \rho e^{im\pi}, \quad m = 1 \text{ or } 2,$

so that $$P\psi = \rho e^{i[P\eta + m\pi]} = e^{i\epsilon}\psi, \tag{8}$$

[2] This proof was called to our attention by Dr. G. H. Shortley (private communication).

where $\epsilon = P\eta - \eta + m\pi$, a real function of the coordinates. Thus $P\psi = e^{i\epsilon(x_i)}\psi$ is a necessary and sufficient condition that $P(\psi\bar{\psi}) = \psi\bar{\psi}$.

Now let P be $P_{\alpha\beta}$, a simple transposition. Then $\eta = \eta_{s,\,\alpha\beta} + \eta_{a,\,\alpha\beta}$, where $\eta_{s,\alpha\beta} = \frac{1}{2}(\eta + P_{\alpha\beta}\eta)$ and $\eta_{a,\alpha\beta} = \frac{1}{2}(\eta - P_{\alpha\beta}\eta)$, so that $P_{\alpha\beta}\eta_{s,\,\alpha\beta} = \eta_{s,\alpha\beta}$, and $P_{\alpha\beta}\eta_{a,\alpha\beta} = -\eta_{a,\alpha\beta}$. Then, easily, $\epsilon = -2\eta_{a,\alpha\beta} + m\pi$. Thus if $P_{\alpha\beta}\eta$ differs from η only by a constant, ϵ is a constant. Application of $P_{\alpha\beta}$ to the equation,

$$P_{\alpha\beta}\psi = e^{i\epsilon}\psi$$

then shows at once that $e^{i\epsilon} = \pm 1$, and we should have proved our theorem. However, there is no reason for assuming $\eta_{a,\alpha\beta}$ to be a constant, so that we are forced to attack the problem in another way.

Wigner, Hund and Heisenberg[3] showed that the eigenfunctions of a set of N similar particles fall into a number of non-combining sets with different symmetry properties. One of these sets S is symmetrical with respect to interchanges, another A is antisymmetrical. When $N \geqq 3$ there are also sets C neither symmetrical nor antisymmetrical. The sets C have equivalence degeneracy which cannot be removed by any perturbation symmetrical in the like particles. Now if ψ is a permissible eigenfunction, then $P_{\alpha\beta}\psi$ must also be an eigenfunction corresponding to the same value of energy (since $P_{\alpha\beta}H = H$, where H is the energy operator). From the linearity of the wave equation any linear combination $c_1\psi + c_2P_{\alpha\beta}\psi$ must also be an eigenfunction corresponding to the same energy as ψ. In case $P_{\alpha\beta}\psi$ is simply a constant times ψ, this linear combination reduces essentially to ψ. Otherwise $c_1\psi + c_2P_{\alpha\beta}\psi$ is an independent eigenfunction, and we have degeneracy. This degeneracy is the above-mentioned equivalence degeneracy non-removable by any perturbation symmetrical in the like particles. The function $c_1\psi + c_2P_{\alpha\beta}\psi$ must thus be considered equally as permissible as ψ.

For in the case of removable degeneracy one can always apply some perturbation which will pick out certain of the eigenfunctions as preferred; which set is singled out by this procedure depends, of course, on the type of perturbation (e.g., an electric field applied to the hydrogen atom selects as preferred eigenfunctions different ones than those selected by a magnetic field applied to a hydrogen atom). In the case of non-removable degeneracy on the other hand there is no perturbation symmetrical in the like particles which will select any of the eigenfunctions as preferred. We must therefore treat them all impartially and consider $c_1\psi + c_2P_{\alpha\beta}\psi$ as permissible as ψ.

One might, of course, simply lay down the postulate that if ψ is a permissible eigenfunction, then the eigenfunction $P_{\alpha\beta}\psi$ is also permissible, and hence, from the linearity of the wave equation, any linear combination $c_1\psi + c_2P_{\alpha\beta}\psi$. To permit ψ and reject $P_{\alpha\beta}\psi$ would be to treat the similar particles α and β differently, so that this postulate (No. 2 above) is physically acceptable even without the degeneracy argument.

We have then as permissible eigenfunctions

$$\psi_{s,\alpha\beta} = \psi + P_{\alpha\beta}\psi, \quad \text{and} \quad \psi_{a,\alpha\beta} = \psi - P_{\alpha\beta}\psi,$$

$\psi_{s,\,\alpha\beta}$ being symmetrical with respect to $P_{\alpha\beta}$ and $\psi_{a,\,\alpha\beta}$ antisymmetrical. If δ is an arbitrary constant, the eigenfunctions

$$\Phi_{1,2} \equiv \psi_{s,\,\alpha\beta} \pm e^{i\delta}\psi_{a,\,\alpha\beta}$$

must likewise be permissible. In general, however,

$$P_{\alpha\beta}(\Phi_1\bar{\Phi}_1) \neq \Phi_1\bar{\Phi}_1. \qquad (9)$$

This is demonstrated in the following manner.

$$P_{\alpha\beta}\Phi_1 = \Phi_2, \quad P_{\alpha\beta}\Phi_2 = \Phi_1 \qquad (10)$$

and $\quad P_{\alpha\beta}(\Phi_1\bar{\Phi}_1) = (P_{\alpha\beta}\Phi_1)(P_{\alpha\beta}\bar{\Phi}_1) = \Phi_2\bar{\Phi}_2,$

so that

$$P_{\alpha\beta}(\Phi_1\bar{\Phi}_1) - \Phi_1\bar{\Phi}_1 = \Phi_2\bar{\Phi}_2 - \Phi_1\bar{\Phi}_1$$
$$= -2(e^{-i\delta}\psi_{s,\,\alpha\beta}\bar{\psi}_{a,\,\alpha\beta} + e^{i\delta}\bar{\psi}_{s,\,\alpha\beta}\psi_{a,\,\alpha\beta}), \qquad (11)$$
$$= -4\rho_s\rho_a \cos(\gamma_s - \gamma_a - \delta), \qquad (12)$$

where $\psi_{s,\,\alpha\beta} = \rho_s e^{i\gamma_s}$ and $\psi_{a,\,\alpha\beta} = \rho_a e^{i\gamma_a}$, the ρ's and γ's being real. Now $\cos(\gamma_s - \gamma_a - \delta)$ does not in general vanish, so that (9) follows.

We have thus shown that if the result of $P_{\alpha\beta}$ operating on a permissible eigenfunction ψ is not a constant times ψ, that there must be *other* permissible eigenfunctions for which the proba-

[3] E. Wigner, Zeits. f. Physik **40**, 492 (1927); **40**, 883 (1927); **43**, 624 (1927); F. Hund, Zeits. f. Physik **43**, 778 (1927); W. Heisenberg, Zeits. f. Physik **41**, 239 (1927).

bility density is not symmetric. This result violates our hypothesis that the probability density must be symmetric for any permissible eigenfunction. We must therefore have

$$P_{\alpha\beta}\psi = c\psi, \tag{13}$$

where c is a constant. Application of $P_{\alpha\beta}$ to this equation shows that $c=\pm 1$; the preliminary theorem then shows that the signs are the same for all interchanges.

We have therefore demonstrated that the wave functions must be either symmetrical or antisymmetrical if the two principles stated above are assumed to be valid.

This idea can be generalized. Using Dirac's notation, the ψ we have been using is

$$(q_1' \cdots q_{3N}' | H') \equiv (q' | H'),$$

where H is the energy. Let

$$(p_1' \cdots p_{3N}' | H') \equiv (p' | H').$$

Then[4]

$$(p'|H') = h^{-3N/2} \int_{-\infty}^{\infty} e^{-(2\pi i/h)(p_1' q_1' + p_2' q_2' + \cdots)} dq' (q'|H').$$

If we apply the permutation $P_{\alpha\beta}$ to this equation, we see that $(p'|H')$ will be symmetrical or antisymmetrical according as $(q'|H')$ is symmetrical or antisymmetrical. It is assumed that the application of $P_{\alpha\beta}$ to

$$e^{-(2\pi i/h)(p_1' q_1' + p_2' q_2' + \cdots p'_{3N} q'_{3N})}$$

is to be made in such a way that p_α and p_β are interchanged at the same time that q_α and q_β are interchanged. Thus the transformation function is symmetrical with respect to an interchange of particles and leads to the result that $(p'|H')$ is antisymmetrical for electrons and protons.

This can be generalized still further. We can require on the basis of the principles stated at the beginning of this article that the probability associated with two sets of observables ξ and η shall be invariant with respect to interchanges of like particles. In Dirac's symbolism $|(\xi'|\eta')|^2$ shall be invariant with respect to interchanges of similar particles. This will lead to the result that $(\xi'|\eta')$ must be either symmetrical or antisymmetrical, provided that the idea of permuting particles applies to the quantity in question.

One of us (J. P. V.) wishes to acknowledge interesting discussions with Professor P. M. Morse and Dr. G. H. Shortley in regard to some of the points in this paper.

[4] P. A. M. Dirac, *Principles of Quantum Mechanics*, p. 106.

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Ion Distribution During the Initial Stages of Spark Discharge in Nonuniform Fields

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A method is given whereby a direct study of ion distribution during the initial stages of spark discharges may be made by means of a Wilson cloud chamber and a potential impulse of less than 10^{-7} sec. duration. Photographs are shown for ion distribution between a point and plane with the point both negative and positive. The mechanisms responsible for the initiation in these cases has not been discovered. It was found that a maximum field of 10^5 to 10^6 volts/cm was a necessary condition for ionization. With the point negative this could cause sufficient auto-electronic emission to start the discharge. With the point positive it is shown that it is improbable that free electrons in the gas are necessary for initiation of ionization.

The methods ordinarily used in the investigation of the initial stages of spark discharge give little direct information concerning the ion distribution in the gap space before sufficient luminosity for visual or photographic observations has developed. A knowledge of this ion distribution is essential for an understanding of the process involved in the starting of a spark discharge. This paper is the report of an attempt which has been made to determine it in a direct manner.

The method with preliminary results was