On the Born-Infeld Field Theory of the Electron

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Since the field equations of the Born-Infeld field theory impose no essential restrictions on the world lines of the singularities, a dynamical condition, equivalent to equations of motion, must be added to complete the theory. The variation problem from which Born and Infeld obtain equations of motion by varying the world lines of the singularities does not appear to be susceptible of generalization to include radiation reaction and, moreover, the equations of motion are not invariant in form under a Lorentz transformation. The definition of a magnetic charge and current vector (which vanishes for an isolated singularity in uniform rectilinear motion) makes possible the derivation from the energy-momentum conservation laws of a relation (Eq. (32)) which is the formal analogue of the dynamical assumption, in the classical theory of electromagnetic mass, that the total force on the electron, including the self-reaction, vanishes. This result is not, however, equivalent to equations of motion because the non-classical part of the field arising from the non-linear character of the field equations always adjusts itself, for arbitrary motion of the singularity, to maintain the relation. In the coordinate system in which the singularity is momentarily at rest the relation simplifies notably and points to a new dynamical condition which appears to be singled out from all other possible conditions by its compelling simplicity. The resulting equations of motion contain the usual radiation resistance term and also higher order terms in the self-interaction.

I. Introduction

In the Born-Infeld[1] theory the electron appears as a coulomb singularity with which is associated in a natural and unforced manner a definite spatial distribution of charge upon which external fields act. The way in which the field changes in going from one world point to a near by point is determined by a system of field equations which also imply the existence of an energy-momentum tensor satisfying a set of divergence equations interpreted as energy-momentum conservation laws.

The principal result of this paper is the derivation from the energy-momentum conservation laws of an equation which is the formal analogue of the dynamical assumption, in the classical theory of electromagnetic mass, that the total force on the electron, including the self-reaction, vanishes. A simple assumption then leads to dynamical equations of motion containing radiation resistance and higher order terms, whereas the method used by Born and Infeld to set up equations of motion does not appear to be susceptible of generalization to include radiation resistance.

II. The Field Equations

The building material out of which the Born-Infeld theory is constructed is a single antisymmetric field tensor of the second rank p_{kl}, satisfying the set of divergence equations[2]

$$\partial p^{kl}/\partial x^l = 0. \tag{1}$$

In space vector form (1) becomes[3]

$$\nabla \cdot D = 0, \quad \nabla x H = \partial D / c\partial t. \tag{2}$$

The electron at rest under the action of no forces is characterized by the assumptions that H vanishes identically and D is spherically symmetric. Then necessarily

$$D = -e\nabla(1/r) \tag{3}$$

with e an arbitrary constant of integration which is set equal to the experimental value of the electronic charge. Since D is the gradient of a scalar function its curl vanishes; also the divergence of H vanishes (trivial since H vanishes identically). Hence the stationary electron is described by a field tensor (3) which satisfies the set of equations

$$\begin{gathered} \nabla \cdot D = 0, \quad \nabla \cdot H = 0, \\ \nabla x H = 0, \quad \nabla x D = 0. \end{gathered} \tag{4}$$

Turning now to the case of an electron in uniform rectilinear motion we see that (2) and

[1] M. Born and L. Infeld, Proc. Roy. Soc. **A144**, 425 (1934).

[2] To avoid unessential complications only invariance under Lorentz transformations is required of the tensor equations.

[3] Except where otherwise stated the notation throughout is taken from reference 1.

(3) and the fact that D and H together form the six vector p_{kl} imply that the field tensor of the moving electron satifies the complete set of Maxwell's equations[4]

$$\partial p^{kl}/\partial x^l=0, \quad \partial p^{*kl}/\partial x^l=0. \qquad (M)$$

This result is summarized in the statement that the Eqs. (1) and the assumption that the field of the stationary electron is given by (3) imply that the field tensor p_{kl} for a system of electrons, each in uniform rectilinear motion, satisfies the complete set of Maxwell's equations (M). It is of course well known that M possesses solutions (in terms of the retarded potentials for a point charge) for a singularity or group of singularities with arbitrary world lives. Such solutions will be written $\bar{p}_{kl}$.

Interaction between the singularities is brought into the theory through the "Hamiltonian"[5] function $\mathbf{H}=(1+P)^{\frac{1}{2}}-1$ and the variation problem[6]

$$\delta\int\int\int\int \mathbf{H}dxdydzdt=0, \qquad (5)$$

which is to be solved by varying the field tensor p_{kl} subject to the restriction that the comparison tensors remain solutions of (1) and are all equal on the boundaries of the region of integration. It is understood that the world lines of the electrons (the lines in $xyzt$ space on which p_{kl} possesses a singularity of the second order) are not varied nor is any comparison tensor admitted which possesses singularities except on these lines. A necessary and sufficient condition that the variation of the integral vanish is then

$$\partial f^{*kl}/\partial x^l=0 \qquad (6)$$

with $f_{kl}=p_{kl}(1+P)^{-\frac{1}{2}}$. (1) and (6) together will be referred to as the Born-Infeld field equations and denoted by the symbol BI.

The field tensor p_{kl} defined by (3) and the corresponding tensor f_{kl} are, in fact, solutions of the Born-Infeld field equations. Thus for a single unperturbed singularity in uniform rectilinear motion both the Born-Infeld field equations and Maxwell's equations are simultaneously satisfied. This circumstance is of decisive importance in determining the form of the equations of motion.

III. The Energy-Momentum Tensor

From the field equations Born and Infeld derive energy-momentum conservation equations

$$\partial T_k{}^l/\partial x^l=0, \qquad (7)$$

in which[7]

$$T_k{}^l=(\mathbf{H}+\tfrac{1}{2}f_{rs}p^{rs})\delta_k{}^l-f_{mk}p^{ml} \qquad (8a)$$

$$=\tfrac{1}{2}\{((1+P)^{\frac{1}{2}}+(1+P)^{-\frac{1}{2}}-2)\delta_k{}^l - (f_{mk}p^{ml}+f^{ml}p_{mk}-\tfrac{1}{2}f_{rs}p^{rs}\delta_k{}^l)\}. \qquad (8b)$$

The second form in the definition of $T_k{}^l$ is needed for later developments. It is instructive for the study of the relation between the field and particle forms of the conservation laws to consider the explicit form of the field tensors and the tensor $T_k{}^l$ in the neighborhood of a singularity. A number of definitions are needed for use at this point and later, namely: ${}^vp_{kl}(t)$ the field associated with a singularity, in uniform rectilinear motion, which at the time t is coincident with and moving with the accelerated singularity, ${}^ap_{kl}$ the acceleration field of the classical point charge, ${}^ep_{kl}$ the external field which causes the non-uniform motion, $\bar{q}_{kl}$ the difference between the actual field and the field of a classical point charge moving with the singularity, $\tilde{q}_{kl}$ the difference between the complete field and the sum of external field and classical field, and finally q_{kl} the sum $\tilde{q}_{kl}+{}^ep_{kl}+{}^ap_{kl}$. Then

$$\begin{aligned} \bar{p}_{kl}&={}^vp_{kl}+{}^ap_{kl},\\ p_{kl}&=\bar{p}_{kl}+\bar{q}_{kl},\\ &=\bar{p}_{kl}+\tilde{q}_{kl}+{}^ep_{kl},\\ &={}^vp_{kl}+q_{kl}. \end{aligned} \qquad (9)$$

The assumption that ${}^ep_{kl}$ is small in comparison with unity permits us to write it as a solution of Maxwell's equations. We wish to express the field quantities in terms of ${}^vp_{kl}$, ${}^vf_{kl}$ and q_{kl} retaining only first order terms in q_{kl}. The results are

[4] $p_{kl}{}^*$ is the dual tensor; see reference 1, p. 432.

[5] Reference 1, p. 436; P designates the invariant $\frac{1}{2}p_{rs}{}^*p^{*rs}=-\frac{1}{2}p_{rs}p^{rs}$.

[6] Reference 1, p. 436.

[7] Reference 1, p. 436; from the definition of f_{rs}, $f_{mk}p^{ml}=f^{ml}p_{mk}$.

$$
\begin{aligned}
&\tfrac{1}{2}p_{rs}p^{rs}=\tfrac{1}{2}\ {}^{v}p_{rs}\ {}^{v}p^{rs}+{}^{v}p_{rs}q^{rs},\\
&\mathbf{H}=\mathbf{H}^{v}-\tfrac{1}{2}\ {}^{v}f_{rs}q^{rs},\\
&(1+P)^{-\frac{1}{2}}=(1+P^{v})^{-\frac{1}{2}}+{}^{v}f_{rs}q^{rs}/(1+P^{v}),\\
&f_{kl}={}^{v}f_{kl}\{1+{}^{v}f_{rs}q^{rs}/2(1+P^{v})^{\frac{1}{2}}\}+q_{kl}/(1+P^{v})^{\frac{1}{2}},\\
&f_{mk}p^{ml}={}^{v}f_{mk}\ {}^{v}p^{ml}+{}^{v}f_{mk}q^{ml}+{}^{v}f^{ml}q_{mk}+\tfrac{1}{2}\ {}^{v}f_{mk}\ {}^{v}f^{ml}\ {}^{v}f_{rs}q^{rs},\\
&T_{k}{}^{l}={}^{v}T_{k}{}^{l}+{}^{q}T_{k}{}^{l},\\
&{}^{v}T_{k}{}^{l}=(\mathbf{H}^{v}+\tfrac{1}{2}\ {}^{v}f_{rs}\ {}^{v}prs)\delta_{k}{}^{l}-{}^{v}f_{mk}\ {}^{v}p^{ml},\\
&{}^{q}T_{k}{}^{l}=-{}^{v}f_{mk}q^{ml}-{}^{v}f^{ml}q_{mk}+\tfrac{1}{2}\ {}^{v}f_{rs}q^{rs}\delta_{k}{}^{l}+\tfrac{1}{2}(F^{v}\delta_{k}{}^{l}-{}^{v}f_{mk}\ {}^{v}f^{ml})\,{}^{v}f_{rs}q^{rs},
\end{aligned}
\tag{10}
$$

(with $F=-P/(1+P)=\frac{1}{2}f_{rs}f^{rs}$). In these equations ${}^{v}p_{rs}$, ${}^{v}f_{rs}$, q_{rs} may be replaced by $\bar{p}_{rs}$, $\bar{f}_{rs}$, $\bar{q}_{rs}$.

As a direct consequence of (7) and (10)

$$
\begin{aligned}
\frac{\partial}{\partial x^{l}}\,{}^{q}T_{k}{}^{l}&=-\frac{\partial}{\partial x^{l}}\,{}^{v}T_{k}{}^{l}\\
&=-\frac{\dot{v}_{i}}{c}\frac{\partial}{\partial v_{i}}\,{}^{v}T_{k}{}^{4},
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
-\int\!\!\int\!\!\int\frac{\partial}{\partial x^{l}}\,{}^{q}T_{k}{}^{l}d\tau&=\frac{\dot{v}_{i}}{c}\frac{\partial}{\partial v_{i}}\int\!\!\int\!\!\int{}^{v}T_{k}{}^{4}d\tau\\
&=\frac{d}{cdt}\int\!\!\int\!\!\int{}^{v}T_{k}{}^{4}d\tau,
\end{aligned}
\tag{12}
$$

integrating about the singularity over a region on the surface of which P is small in comparison with unity. A simple calculation yields[8] (letting $\beta=(1-v^{2}/c^{2})^{-\frac{1}{2}}$)

$$
\begin{aligned}
\int\!\!\int\!\!\int{}^{v}T_{k}{}^{4}d\tau&=(4\pi c/b^{2})m_{0}v_{k}\beta,\quad k=1,2,3,\\
&=(4\pi c/b^{2})m_{0}c\beta,\quad k=4.
\end{aligned}
\tag{13}
$$

The left-hand member of (12) must therefore be interpreted as external force plus radiation resistance. In this way the field conservation laws are transformed into a form appropriate to the particle standpoint. However, the Eqs. (12) are not equations of motion for, as will be shown, the field equations do not in any way restrict the motion.

[8] m_0 is the electronic mass; reference 1, p. 446. b is the "absolute" field; reference 1, p. 437.

IV. The Charge and Current Densities

The equation[9]

$$\partial f^{kl}/\partial x^{l}=4\pi\rho^{k} \tag{14}$$

defines the electric charge and current densities ρ^{k}. Let us add the equation

$$\frac{\partial}{\partial x^{l}}p^{*kl}=\frac{\partial}{\partial x^{l}}\tilde{q}^{*kl}=4\pi\sigma^{k}, \tag{15}$$

defining the four vector σ^{k}, and interpret σ^{k} as the magnetic charge and current density vector. The preceding discussion enables us to state at once that $\tilde{q}_{kl}$ and σ^{k} vanish identically in the case of an isolated singularity in uniform rectilinear motion.

V. The Field Equations for Arbitrary Motions of the Singularities

In this section we show, by sketching a method for the calculation of $\bar{q}_{kl}$, that the field equations do not impose any essential restrictions on the motion of the singularities. It is supposed that the world lines of the singularities are arbitrarily prescribed, subject to the broad restriction that the curvature of any given world line should never exceed in order of magnitude the curvature in the world line of a classical particle moving in the field of all the other singularities. This restriction insures that in the neighborhood of a singularity

$$\partial\bar{f}^{*kl}/\partial x^{l}=O({}^{e}p_{rs}). \tag{16}$$

($O(x)$ is a function which vanishes with the first

[9] Reference 1, p. 441.

power of x.) With prescribed world lines $\bar{p}_{kl}$ is a known function of the space-time coordinates, a solution of (1) and hence $\bar{q}_{kl}$ must satisfy the Eq. (1):

$$\partial \bar{q}^{kl}/\partial x^l = 0. \tag{17}$$

Because of (17) there exists a potential vector χ_k in terms of which

$$\bar{q}_{rs}^* = \frac{\partial}{\partial x^r}\chi_s - \frac{\partial}{\partial x^s}\chi_r. \tag{18}$$

Then in the neighborhood of a singularity the field Eq. (6) becomes, with the help of the remark following (10),

$$\frac{\partial}{\partial x^l}\left[(1+\bar{P})^{-\frac{1}{2}}\left(g^{kr}\frac{\partial}{\partial x^r}\chi^l - g^{ls}\frac{\partial}{\partial x^s}\chi^k\right)\right] - \bar{f}^{*kl}\frac{\partial}{\partial x^l}\left[(1+\bar{P})^{-\frac{1}{2}}\bar{f}^{*rs}\frac{\partial}{\partial x^r}\chi_s\right] = -\frac{\partial}{\partial x^l}\bar{f}^{*kl} + O(\bar{q}_{rs}^2), \tag{19}$$

a set of four non-homogeneous linear partial differential equations of the second order in the unknown functions χ_k if the remainder term $O(\bar{q}_{rs}^2)$ is neglected. The appropriate solution of (19) determines a field tensor $\bar{q}_{kl}$ which is a solution of (17) and also determines the field tensor $\tilde{q}_{kl}$ by means of the equations $\tilde{q}_{kl} = \bar{q}_{kl} - {}^e p_{kl}$. The Eq. (19) is also obtained if $\bar{p}_{kl}$ is interpreted as the *sum* of the fields of the classical point charges on prescribed world lines. $\bar{q}_{rs}$, the difference between p_{rs} and $\bar{p}_{rs}$, is then the non-classical part of the field arising from the nonlinear character of the field equations and, with neglect of the remainder term, is given throughout space by the appropriate solution of (19).

VI. The Born-Infeld Equations of Motion

The preceding discussion brings up the question how is it possible for Born and Infeld to derive equations of motion from the field equations? The answer is simply that their equations of motion are not consequences of the field equations, but are obtained from the variation problem (5) by putting in place of p_{rs} the tensor ${}^v p_{rs} + {}^e p_{rs}$ and varying the world line in the resulting definite function of the position and velocity of the singularity. It is clear that two distinct variation principles are thus associated with Eq. (5); one yielding the field equations, the other the equations of motion. In this manner they obtain a Lagrangian function[10]

$$\Lambda(r(t), v(t), t) = m_0 c^2/\beta - \int\int\int \varphi_l^{(e)}\, {}^v\rho^l d\tau, \tag{20}$$

in which $\varphi_l^{(e)}$ is the potential four vector of the external field and ${}^v\rho^l$ is the charge four vector for a singularity in uniform rectilinear motion with the velocity v. More generally we might return to the variation problem and, using an approximation to the field of the classical point charge more accurate than ${}^v p_{rs}$, obtain a correction term to be added to (20) of the form (for small values of the velocity)

$$\sum_{i=0}^{N}\sum_{j=1}^{N} a_{ij} v^{(i)} \cdot v^{(j)}, \tag{21}$$

which would contribute a sum of odd derivatives of the velocity to the equations of motion.[11] It does not seem possible to obtain a radiation resistance term (an even derivative of the velocity) from any modification of this form of the variation problem.

In addition to the difficulty in connection with the radiation resistance there is another which seems more fundamental. The equations of motion will be invariant in form only if $\Lambda\beta$ is a scalar invariant. But since the time cross section for the space integration depends on the coordinate system in which the integral is defined, the quantity $\beta\int\int\int\varphi_l^{(e)}\,{}^v\rho^l d\tau$ is not in general a scalar invariant. This non-invariant result from an apparently invariant procedure appears to arise from the fact that the equation $\delta\int_{t_1}^{t_2}\Lambda dt = 0$ follows from $\delta\int\int\int\int H dx dy dz dt = 0$ only if the space and time integrations separate (i.e., the limits for the time integration are independent of the space coordinates). However the

[10] Reference 1, p. 449.

[11] See Courant-Hilbert *Methoden der Mathematischen Physik*, p. 171 for statement of the variation problem and derivation of the Euler equations for the general case in which Λ is a function of derivatives of arbitrary order.

statement that the space and time integrations separate is not invariant and different equations of motion are obtained for different choices of the coordinate system in which the separation occurs.

The interaction of an electron with a plane monochromatic wave supplies a simple illustration of the preceding remarks.[12] In a coordinate system in which the electron is moving momentarily in the direction of propagation of the wave with velocity v the quantity $\beta\int\int\int\varphi_l{}^{(e)v}\rho^l d\tau$ has the value

$$\frac{2e}{cy}\left(\int_0^\infty \frac{\sin yz}{(1+z^4)^{\frac{3}{2}}}dz\right)\beta\frac{dx^l}{dt}\varphi_l{}^{(e)}(x, y, z, t)$$

with $y=2\pi r_0/\lambda\beta=(2\pi r_0/\lambda_0)(1+v/c)$ (λ_0 the wavelength measured by an observer moving with the electron). Now y is obviously not invariant, consequently neither is $\Lambda\beta$.

VII. An Alternative Form of the Conservation Equations

The analysis is facilitated by writing the Eqs. (6) and (15) in the equivalent forms

$$\frac{\partial}{\partial x^k}f_{lm}+\frac{\partial}{\partial x^l}f_{mk}+\frac{\partial}{\partial x^m}f_{kl}=0, \tag{22}$$

$$\frac{\partial}{\partial x^k}p_{lm}+\frac{\partial}{\partial x^l}p_{mk}+\frac{\partial}{\partial x^m}p_{kl}=8\pi\sigma_{klm}, \tag{23}$$

in which σ_{klm} is a complete antisymmetric tensor of the third order related to σ^k through the equation[13]

$$\sigma_{klm}=j_{klmn}\sigma^n. \tag{24}$$

To begin, Eq. (14) is multiplied by p_{km}:

$$p_{km}\frac{\partial}{\partial x^l}f^{kl}=4\pi p_{km}\rho^k.$$

Then

$$\frac{\partial}{\partial x^l}(p_{km}f^{kl})-f^{kl}\frac{\partial}{\partial x^l}p_{km}=4\pi p_{km}\rho^k,$$

$$-f^{kl}\frac{\partial}{\partial x^l}p_{km}=-\tfrac{1}{2}\frac{\partial}{\partial x^m}(f^{kl}p_{kl})+\tfrac{1}{2}p^{kl}\frac{\partial}{\partial x^m}f_{kl}+4\pi f^{kl}\sigma_{klm},$$

$$\tfrac{1}{2}p^{kl}\frac{\partial}{\partial x^m}f_{kl}=\frac{\partial}{\partial x^l}(p^{kl}f_{km}). \tag{25}$$

From these results and the identity $f^{kl}\sigma_{klm}=-f_{km}{}^*\sigma^k$ follows[14]

$$\frac{\partial}{\partial x^l}\{p_{km}f^{kl}+p^{kl}f_{km}-\tfrac{1}{2}f_{rs}p^{rs}\delta_m{}^l\}=4\pi(p_{km}\rho^k+f_{km}{}^*\sigma^k). \tag{26}$$

Now if the conservation Eq. (7) is multiplied by two and added to (26) there is obtained the symmetrical and strikingly simple equation

$$\frac{\partial}{\partial x^m}((1+P)^{\frac{1}{2}}+(1+P)^{-\frac{1}{2}})=4\pi(p_{km}\rho^k+f_{km}{}^*\sigma^k). \tag{27}$$

Let us fix attention on a particular singularity e_i and integrate (27) over the space between two surfaces S and S' enclosing e_i. S is taken such that in a coordinate system in which the singularity is instantaneously at rest it is spherical and centered at e_i; moreover it is assumed possible to take the radius of S so great that on S the scalar invariant P is small in comparison with unity. (This assumption excludes the interesting case of two singularities separated momentarily by a distance of less than 10^{-12} cm.) That value of the radius is taken for which P is roughly as small as possible. S' is any surface having arbitrarily small linear dimensions and symmetric with respect to reflection through the singularity. Then, letting $\mathbf{M}=(1+P)^{\frac{1}{2}}+(1+P)^{-\frac{1}{2}}-2$,

$$\iiint_{S-S'}\frac{\partial}{\partial x^m}\mathbf{M}d\tau=\iint_S \mathbf{M}\cos(x^m, s)ds+\iint_{S'}\mathbf{M}\cos(x^m, s)ds,\quad m=1, 2, 3. \tag{28}$$

On S

$$\mathbf{M}\cong\tfrac{1}{4}P^2=(D^2-H^2)^2/4b^4. \tag{29}$$

[12] Compare reference 1, p. 450.

[13] j_{klmn} is the complete antisymmetric tensor of the fourth order; see reference 1, p. 431

[14] With the identity $p^{*kl}f_{km}{}^*=p_{km}f^{kl}-\tfrac{1}{2}p_{rs}f^{rs}\delta_m{}^l$ the left-hand member of (26) can be written in symmetrical form.

D has the same order of magnitude on S as the perturbing field ${}^e p_{rs}$ there or at e_i arising from the other singularities in the field (merely another way of saying that S is so chosen that on it P is roughly as small as possible). Now, even for a radiation field, $r^2(D^2-H^2)^2 < O({}^e p_{rs}{}^2)$ and consequently

$$\left|\iint_S \mathbf{M} \cos(x^m, s) ds\right| < O({}^e p_{rs}{}^2), m=1, 2, 3. \quad (30)$$

The other surface integral is uniformly bounded, independent of the linear dimensions of S', because the singularity of the argument at e_i is only of the second degree. Moreover a reflection through the singularity merely changes the sign of the argument without changing its value (in the limit as the linear dimensions of S' approach zero). Hence

$$\lim_{S'\to 0} \iint_{S'} \mathbf{M} \cos(x^m, s) ds = 0, \quad m=1, 2, 3,$$

and[15]

$$\iiint \nabla \mathbf{M} d\tau = 0 \quad (31)$$

with neglect of terms having the order of magnitude $O(ep_{rs}{}^2)$. Thus (27) implies

$$\iiint (p_{km}\rho^k + f_{km}{}^*\sigma^k) d\tau = 0, \quad m=1, 2, 3, \quad (32)$$

neglecting again quadratic terms in the perturbing field at e_i. Classical analogy suggests interpreting $(p_{km}\rho^k + f_{km}{}^*\sigma^k)d\tau$ as the force on the charge occupying the element of volume $d\tau$. With this interpretation (32) is equivalent to the statement that the total force on the electron vanishes. But since the BI field equations do not determine the motion neither can (32) which is a consequence of the field equations. The electrons may move as they please, but always the non-classical part of the field, $\bar{q}_{rs}$, which is determined from the motion by the field equations has just the value required to insure the truth of (32). This remark applies with equal force to Eq. (12) which is indeed equivalent to (32). It is better then not to attach too much significance to the above interpretation although it cannot be said to be false and does point unmistakably to a reasonable dynamical assumption.

Hitherto only three of the Eqs. (27) have been considered. But, from (10),

$$\mathbf{M} = \mathbf{M}^v + \tfrac{1}{2} F^v \, {}^v f_{rs} q^{rs} + O(q_{rs}{}^2) \quad (33)$$

and hence both the Eqs. (32) and the equation obtained by applying to (27) for $m=4$ an integration over the interior of S are contained in the set of equations

$$4\pi \iiint (p_{km}\rho^k + f_{km}{}^*\sigma^k) d\tau = \delta_m{}^4 \left\{ \frac{\partial}{2c\partial t} \iiint F^v \, {}^v f_{rs} q^{rs} d\tau - \iiint F^v \frac{\dot{v}_i}{c} \frac{\partial}{\partial v_i} (1+P^v)^{\frac{1}{2}} d\tau \right\}. \quad (34)$$

15 Wherever a volume integration appears it is understood that the integration is over the region inclosed by S.

VIII. The Conservation Equations in the Coordinate System in Which the Singularity is Instantaneously at Rest

The equation

$$\frac{\partial}{\partial x^l} {}^v f^{kl} = 4\pi \, {}^v \rho^k + \frac{\dot{v}_i}{c} \frac{\partial}{\partial v_i} {}^v f^{k4} \quad (35)$$

defines the unperturbed charge four vector ${}^v\rho^k$ which is associated with a singularity moving with the instantaneous velocity v. Let

$${}^q\rho^k = \rho^k - {}^v\rho^k \quad (36)$$

represent the distortion in the charge and current densities produced by the acceleration and the external field. Then

$$p_{km}\rho^k = {}^v p_{km} \, {}^v\rho^k + q_{km} \, {}^v\rho^k + p_{km} \, {}^q\rho^k. \quad (37)$$

Symmetry considerations lead at once to the conclusion

$$\int\int\int {}^{v}p_{km}\ {}^{v}\rho^k d\tau = 0. \tag{38}$$

Furthermore the integral over $p_{km}{}^{q}\rho^k$ can be expressed in a simple form. For

$$\frac{\partial}{\partial x^l} f^{kl} = 4\pi({}^{v}\rho^k + {}^{q}\rho^k) \tag{39}$$

and hence, using (10),

$$4\pi\ {}^{q}\rho^k = \frac{\dot{v}_i}{c}\frac{\partial}{\partial v_i}{}^{v}f^{k4} + \frac{\partial}{\partial x_l}(f^{kl}f_{rs}q^{rs}/2(1+P)^{\frac{1}{2}}) + \frac{\partial}{\partial x^l}(q_{kl}/(1+P)^{\frac{1}{2}}) + O(q_{rs}{}^2). \tag{40}$$

Accordingly, after some calculation (for which see appendix),

$$4\pi p_{km}\ {}^{q}\rho^k = \frac{1}{2}\frac{\partial}{\partial x^l}({}^{v}f_{km}\ {}^{v}f^{kl}\ {}^{v}f_{rs}q^{rs}) - \frac{\dot{v}_i}{c}{}^{v}f_{km}\ {}^{v}f^{k4}\frac{\partial}{\partial v_i}(1+P^v)^{\frac{1}{2}} + O(q_{rs}{}^2), \tag{41}$$

and, combining (37), (38) and (41) with (34)

$$4\pi\int\int\int(q_{km}\ {}^{v}\rho^k + {}^{v}f_{km}{}^{*}\sigma^k)d\tau = \frac{1}{2c}\frac{\partial}{\partial t}\int\int\int(F\ {}^{v}\delta_m{}^4 - {}^{v}f_{km}\ {}^{v}f^{k4}){}^{v}f_{rs}q^{rs}d\tau$$

$$-\int\int\int(F\ {}^{v}\delta_m{}^4 - {}^{v}f_{km}\ {}^{v}f^{k4})\frac{\dot{v}_i}{c}\frac{\partial}{\partial v_i}(1+P^v)^{\frac{1}{2}}d\tau. \tag{42}$$

The surface integral discarded to obtain (42) is bounded by the quantity[16] $2\pi|q_{rs}|(e^3/b^3r^4) = 2\pi(e/b)|q_{rs}|(r_0/r)^4$ which falls off with decreasing interaction at least as rapidly as the cube of the external field (r is roughly the smallest linear dimension of S). Now $F^v = ({}^{v}B^2 - {}^{v}E^2)/b^2$, ${}^{v}f_{k4}{}^{v}f^{k4} = -{}^{v}E^2/b^2$ and $b^2\ {}^{v}f_{km}{}^{v}f^{k4} (m \neq 4)$ is just the vector product of ${}^{v}E$ and ${}^{v}B$. Consequently $(F^v\delta_m{}^4 - {}^{v}f_{km}{}^{v}f^{k4})$ vanishes in the coordinate system in which the electron is momentarily at rest and also

$$\frac{\partial}{\partial t}\{(F^v\delta_m{}^4 - {}^{v}f_{km}\ {}^{v}f^{k4}){}^{v}f_{rs}\} = 0(\dot{v}).$$

Letting the superscript O designate this coordinate system, (42) reduces to[17]

$$\int\int\int({}^{o}q_{4m}\ {}^{o}\rho^4 + {}^{o}f_{km}{}^{*}\ {}^{o}\sigma^k)d\tau^o = 0. \tag{43}$$

It must be considered as significant that the unpleasant distortion charge vector ${}^{q}\rho^k$ drops out of (32) to leave just (43) when the problem is set up in the one coordinate system (the instantaneous rest system) which is physically singled out from all others. Finally to bring out explicitly the manner in which classical and non-classical terms enter into (43) it is written in the suggestive form

$$\int\int\int({}^{a}p_{4m}{}^{o} + {}^{e}p_{4m}{}^{o}){}^{o}\rho^4 d\tau^o = -\int\int\int({}^{o}\tilde{q}_{4m}\ {}^{o}\rho^4 + {}^{o}f_{km}{}^{*}\ {}^{o}\sigma^k)d\tau^o. \tag{44}$$

[16] $r_0 = (e/b)^{\frac{1}{2}} = 2.28 \times 10^{-13}$ cm; see reference 1, p. 446.
[17] Note that ${}^{o}\rho^4$ is $({}^{v}\rho^4)_{v=0}$ and not the quite distinct quantity $(\rho^4)_{v=0}$.

It is clear that neither side of (42) constitutes a four vector or possesses transformation properties simply related to those of a four vector. This circumstance is not surprising for only under exceptional conditions is it true that the space integral of a four vector possesses any simple transformation properties.

IX. The Equations of Motion

The noteworthy feature of (44) which makes it important and justifies the laborious analysis by means of which it is derived is the occurrence in it of the integral $\int\int\int({}^ap_{4m}{}^o+{}^ep_{4m}{}^o){}^o\rho^4 d\tau^o$, which differs only slightly from the quantity which in the classical electromagnetic theory is interpreted as the total force on the electron and required to vanish in order to supply the theory with dynamical equations of motion. The difference is only that in the classical expression the acceleration field of a volume distribution of charge appears in the place of ${}^ap_{4m}{}^o$. Here, just as in the classical theory, a dynamical condition is needed to determine the motion. But (44) leaves no doubt as to the simplest physically satisfactory condition. We must require that the motion shall be such that the right-hand member of (44) which represents a completely unclassical "force" on the electron shall vanish. Then the equation

$$\int\int\int(ap_{4m}{}^o+{}^ep_{4m}{}^o){}^o\rho^4 d\tau^o=0 \qquad (45)$$

determines the required motion. The vanishing of $\tilde{q}_{kl}$ and σ_k when the motion of the singularity is uniform and rectilinear suggests that (45) is a condition which makes the departure from the classical field small, or even, in a rough sense, as small as possible. In this respect the condition (45) is related to the Born-Infeld equations of motion. In terms of the field tensors measured in an arbitrary coordinate system (45) can be written

$$\frac{1}{\beta}\frac{dx^{om}}{dx^k}\int\int\int({}^ap_{4m}{}^o+{}^ep_{4m}{}^o){}^o\rho^4 d\tau^o = -\frac{dx^j}{cdt}\int\int\int({}^ap_{kj}+{}^ep_{kj}){}^o\rho^4 d\tau^o=0. \qquad (46)$$

The integration over $d\tau^o$ in (46) implies a cut perpendicular to the t^o axis, the time cross section for an observer moving with the electron. For this reason the substitution of ${}^v\rho^4 d\tau$ for ${}^o\rho^4 d\tau^o$ in (46) is not permitted although ${}^v\rho^4 d\tau$ is a scalar invariant.

There remains the problem of exhibiting explicitly the equations of motion implicit in (45). For small values of the velocity the electric field in the neighborhood of a classical point charge, computed from the retarded potentials, is given by

$$D(x,y,z,t)=e\mathbf{1}\{1/r^2-\mathbf{1}\cdot\dot{v}/2c^2r+(\mathbf{1}\cdot\dddot{v}/8c^4)r-(\mathbf{1}\cdot v^{\text{iv}}/15c^5)r^2+(\mathbf{1}\cdot v^{\text{v}}/48c^6)r^3\cdots\}$$
$$+e\{-\dot{v}/2c^2r+2\ddot{v}/3c^3-(3\dddot{v}/8c^4)r+(2v^{\text{iv}}/15c^5)r^2-(5v^{\text{v}}/144c^6)r^3+\cdots\}. \qquad (47)$$

r is the scalar distance from the point x, y, z at which the field is wanted to the instantaneous position of the electron. $\mathbf{1}$ is the unit vector from the electron to the point x, y, z. Also[18]

$$b\,{}^o\rho^4=\frac{e}{2\pi r_0^3(r/r_0)(1+(r/r_0)^4)^{\frac{3}{2}}}. \qquad (48)$$

Letting ${}^oD^e$ represent the external electric field (45) becomes

$$b\int\int\int{}^oD^e\,{}^o\rho^4 d\tau^o=\int\int\int\left\{\frac{2\dot{v}}{3c^2r}-\frac{2\ddot{v}}{3c^3}+\frac{\dddot{v}}{3c^4}r-\frac{v^{\text{iv}}}{9c^5}r^2+\frac{v^{\text{v}}}{36c^6}r^3\right\}\frac{e^2 d\tau^o}{2\pi r_0^3(r/r_0)(1+(r/r_0)^2)^{\frac{3}{2}}} \qquad (49)$$

or, introducing the symbol $G(m, n)$ for the integral

[18] Reference 1, p. 444.

$$\int_0^\infty \frac{x^m dx}{(1\mp x^4)^{n/2}},$$

$$b\int\int\int {}^oD^e\,{}^o\rho^4 d\tau^o = (4e^2/3c^2r_0)G(0,3)\dot{v} - (4e^2/3c^3)G(1,3)\ddot{v} + (2e^2r_0/3c^4)G(2,3)\dddot{v}$$
$$-(2e^2r_0^2/9c^5)G(3,3)v^{\text{iv}} + (e^2r_0^3/18c^6)G(4,3)v^{\text{v}}. \quad (50)$$

Now[19]

$$G(4,3) = G(0,3) = \tfrac{1}{2}G(0,1) = 0.9270 = (3c^2r_0/4e^2)m_0, \qquad G(1,3) = G(3,3) = 0.5. \quad (51)$$

Using the substitution[20] $x = \tan \frac{1}{2}w$

$$G(2,3) = \tfrac{1}{2}\left(\frac{\partial}{\partial\theta}\int_0^{\pi/2} \frac{d\omega}{(1-\sin^2\theta\sin^2\omega)^{\frac{1}{2}}}\right)_{\theta=\pi/4} = 0.422. \quad (52)$$

(By "Peirce's Tables" p. 121.)
Eq. (50) now takes the form

$$b\int\int\int {}^oD^e\,{}^o\rho^4 d\tau^o = m_0\dot{v} - (2e^2/3c^3)\ddot{v} + (2e^2/3c^3)(r_0/c)\{0.422\dddot{v} - 0.167(r_0/c)v^{\text{iv}} + 0.077(r_0/c)^2v^{\text{v}}\}. \quad (53)$$

We see that the equations of motion contain the usual radiation resistance term $(2e^2/3c^3)\ddot{v}$ and additional terms which are important when the electron interacts with very high frequency radiation. But clearly the derivation of the right-hand member of (53) from (54) involves approximations which limit its validity, for the problem of interaction with radiation, to wavelengths somewhat greater than the characteristic length $2\pi r_0$ (perhaps, $h\nu \leqq (137/3)m_0c^2$).[21]

The generalization of (53) to an arbitrary coordinate system is accomplished by going back to (46) or more simply by introducing two four vectors βF_k, βG_k taking the values

$$(F_1, F_2, F_3) = b\int\int\int {}^oD^e\,{}^o\rho^4 d\tau^o, \qquad F_4 = 0,$$
$$(G_1, G_2, G_3) = 0, \qquad G_4 = \frac{2e^2}{3c^4}\left\{\dot{v}^2 - 0.422\frac{r_0}{c}3\dot{v}\cdot\ddot{v} + \cdots\right\} \quad (54)$$

when the velocity of the singularity vanishes. Then

$$F_k = \frac{b^2}{\beta}\frac{dx^{ol}}{dx^k}\int\int\int {}^ep_{l4}{}^o\,{}^o\rho^4 d\tau^o$$
$$= b^2\frac{dx^j}{dx^4}\int\int\int {}^ep_{kj}\,{}^o\rho^4 d\tau^o, \quad (55)$$

or

$$(F_1, F_2, F_3) = b\int\int\int \left\{{}^eD + \frac{1}{c}(vx\,{}^eH)\right\}{}^o\rho^4 d\tau^o, \qquad F_4 = -\frac{b}{c}\int\int\int v\cdot{}^eD\,{}^o\rho^4 d\tau^o. \quad (56)$$

The equations of motion have the form

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[19] Reference 1, p. 440, p. 446.
[20] Reference 1, p. 439.

[21] A more fundamental restriction implied by the consistent neglect of quadratic terms in the external field is $|\dot{v}| \ll c^2/r_0$.

$$-(F^k+G^k)=\frac{d}{dt}m_0\beta\frac{dx^k}{dt}-\frac{2e^2}{3c^3}\frac{d}{dt}\beta\frac{d}{dt}\beta\frac{dx^k}{dt}+\cdots. \tag{57}$$

The plane wave fields,

$$({}^eD,\ {}^eH)=(A,\ B)e^{2\pi i(\nu t-(lx+my+nz)/\lambda)}$$

lead to the force vector

$$(F_1,\ F_2,\ F_3)=eg\left(\frac{2\pi r_0}{\lambda_0}\right)\left\{{}^eD+\frac{1}{c}(vx{}^eH)\right\}, \tag{58}$$

in which ${}^eD$ and ${}^eH$ are evaluated at the singularity and $g(2\pi r_0/\lambda_0)$[22] is a factor less than unity representing the fact that the grip which the field obtains on the electron decreases in strength as the frequency goes up. It is important to note that the λ_0 occurring in the argument of g is the wavelength for an observer moving with the singularity. Thus $g(2\pi r_0/\lambda_0)$ is a scalar invariant, but not a constant.

Appendix: derivation of Eq. (41).

From (40)

$$4\pi p_{km}\ {}^q\rho^k=p_{km}\frac{\dot v_i}{c}\frac{\partial}{\partial v_i}{}^vf^{k4}+\frac{\partial}{\partial x^l}\{\tfrac{1}{2}(f_{rs}q^{rs})f_{km}f^{kl}+f_{km}q^{kl}\}-\left(\frac{\partial}{\partial x^l}p_{km}\right)(f^{kl}-{}^vf^{kl}), \tag{59}$$

$$\begin{aligned}-\left(\frac{\partial}{\partial x^l}p_{km}\right)(f^{kl}-{}^vf^{kl})&=(f^{kl}-{}^vf^{kl})\left(\frac{\partial}{\partial x^k}p_{ml}+\frac{\partial}{\partial x^m}p_{lk}\right)\\&=-\tfrac{1}{2}(f^{kl}-{}^vf^{kl})\frac{\partial}{\partial x^m}p_{kl}\\&=-\tfrac{1}{2}\frac{\partial}{\partial x^m}\{\tfrac{1}{2}(f_{rs}q^{rs})f_{kl}f^{kl}+f_{kl}q^{kl}\}+\tfrac{1}{2}p^{kl}\frac{\partial}{\partial x^m}(f_{kl}-{}^vf_{kl}),\end{aligned} \tag{60}$$

$$\begin{aligned}\tfrac{1}{2}p^{kl}\frac{\partial}{\partial x^m}(f_{kl}-{}^vf_{kl})&=-\tfrac{1}{2}p^{kl}\left\{\frac{\partial}{\partial x^k}(f_{lm}-{}^vf_{lm})+\frac{\partial}{\partial x^l}(f_{mk}-{}^vf_{mk})\right\}\\&\qquad-\tfrac{1}{2}\frac{\dot v_i}{c}\frac{\partial}{\partial v_i}\{\delta_m{}^4\ {}^vf_{kl}+\delta_k{}^4\ {}^vf_{lm}+\delta_l{}^4\ {}^vf_{mk}\}p^{kl}\\&=\frac{\partial}{\partial x^l}\{\tfrac{1}{2}(f_{rs}q^{rs})f^{kl}f_{km}+f^{kl}q_{km}\}-\tfrac{1}{2}\left(\frac{\dot v_i}{c}\frac{\partial}{\partial v_i}{}^vf_{kl}\right)p^{kl}\delta_m{}^4+\left(\frac{\dot v_i}{c}\frac{\partial}{\partial v_i}{}^vf_{km}\right)p^{k4}.\end{aligned} \tag{61}$$

Combined, these equations yield

$$4\pi p_{km}\ {}^q\rho^k=-\frac{\partial}{\partial x^l}{}^qT_m{}^l-\frac{\dot v_i}{c}\frac{\partial}{\partial v_i}{}^vT_m{}^4+\tfrac{1}{2}\frac{\partial}{\partial x^l}({}^vf_{km}\ {}^vf^{kl}\ {}^vf_{rs}q^{rs})-\frac{\dot v_i}{c}{}^vf_{km}\ {}^vf^{k4}\frac{\partial}{\partial v_i}(1+P^v)^{\frac{1}{2}}+O(q_{rs}{}^2) \tag{62}$$

$$=\tfrac{1}{2}\frac{\partial}{\partial x^l}({}^vf_{km}\ {}^vf^{kl}\ {}^vf_{rs}q^{rs})-\frac{\dot v_i}{c}{}^vf_{km}\ {}^vf^{k4}\frac{\partial}{\partial v_i}(1+P^v)^{\frac{1}{2}}+O(q_{rs}{}^2). \tag{41}$$

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[22] Reference 1, p. 450 for explicit form and tabulated values.