atoms, ^4S and ^3P of As and O, respectively, since there are a number of repulsive states derived from this combination. Furthermore, a linear extrapolation to convergence of the vibrational levels of the normal state gives 46,943 cm^{-1}, and such a procedure usually leads to values which are 20 to 30 percent too high. Therefore we conclude that 4.93 volts is a reliable value for the heat of dissociation.

During the present investigation, there appeared a report of work on the AsO bands by Shawhan and Morgan.[7] Their analysis of system A is different from ours, in that the electronic doubling is given as 644 cm^{-1}, and the vibration frequency in the upper state 372 cm^{-1}. In view of the good agreement we find for the doubling in the two systems, and of the fact that 372 cm^{-1} is almost impossibly low in comparison to 966 for the normal state, there seems no doubt as to the correctness of our values for this system. These authors have apparently mistaken the vibration frequency in the upper state for the electronic difference. Their analysis of system B is in essential agreement with ours, and extends it to higher vibrational states in the upper state by the equation $G'=1098(v'+\frac{1}{2})-6(v'+\frac{1}{2})^2$. Assuming the correctness of these constants, we calculate the origins of system B to be 39,862.0 and 38,838.3 cm^{-1}. In their preliminary report, Shawhan and Morgan give no statement about the specific bands with $v'>0$ which are observed, nor about their intensities. It is stated, however, that the bands were excited in the arc and the flame in emission, and also observed in absorption in the flame. If our conclusions are correct as to the predissociation, it seems probable that these additional bands were observed in absorption.

[7] E. N. Shawhan and F. Morgan, Phys. Rev. **47**, 199A (1935) (St. Louis Meeting, 1934).

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A New Type of Expansion in Radiation Problems

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A new type of expansion of $\mathbf{i}(1)e^{ikr_{12}}/r_{12}$ is developed. Here $\mathbf{i}$ is a vector function of the spherical coordinates denoted by 1 and r_{12} is the distance between two points denoted by 1 and 2. This expansion is used in the solution of Maxwell's equations and a simple general expression is found for the energy radiated from a known current distribution. A brief application to Dirac's theory of radiation is given. An expansion for $\mathbf{i}(1)/r_{12}$ is developed which can be used to find the vector potential due to a steady current distribution.

THE well-known expansion of $e^{ikr_{12}}/r_{12}$ in terms of spherical harmonics and half-order Bessel functions provides a method of evaluating $\int(e^{ikr_{12}}/r_{12})\rho(1)d\tau_1$, which is a solution of the inhomogenous wave equation. But the same method is not very satisfactory for the solution of the corresponding vector wave equation because in any coordinates except Cartesian it is rather difficult to keep the various components separate. In the present paper a new type of expansion is found which avoids this difficulty.

Guided somewhat by the expansion for the scalar case we anticipate an expansion of the form

$$\mathbf{i}(1)e^{ikr_{12}}/r_{12}=\sum a_{kl}\mathbf{A}_k'(2)\mathbf{A}_l(1)\cdot\mathbf{i}(1). \quad r_2>r_1 \quad (1)$$

Here vectors are in bold faced type, and the $\mathbf{A}$'s are solutions of the vector wave equation; $\mathbf{A}'$ is to bear the same relation to $\mathbf{A}$ as the Hankel function does to the Bessel function in the scalar case.

Now from each solution of the scalar wave equation we can construct three and only three independent solutions of the vector wave equation; for example in Cartesian coordinates we have only to multiply the scalar function by each of the three unit vectors. There are, of course, an infinite number of possible sets of functions but the ones most suited for present purposes are constructed as follows. Let ξ be a solution of the scalar wave equation, say

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$$\xi=\left(\frac{(2l+1)(l-m)!}{8l(l+1)(l+m)!}\right)^{\frac{1}{2}}\frac{1}{(kr)^{\frac{1}{2}}}J_{l+\frac{1}{2}}(kr)P_l^m(\cos\theta)e^{im\varphi}. \tag{2}$$

Then we use the three vector functions[1] defined by

$$\mathbf{A}_{1lm}=(l(l+1))^{\frac{1}{2}}\nabla\xi_{lm}\,;\qquad \mathbf{A}_{2lm}=k\nabla\times(\mathbf{r}\xi_{lm})=-(1/k)\nabla\times\mathbf{A}_{3lm}\,;\qquad \mathbf{A}_{3lm}=-(1/k)\nabla\times\mathbf{A}_{2lm}. \tag{3}$$

$\mathbf{A}'$ is obtained by replacing the $J(kr)$ in ξ by $H^{(1)}(kr)$. If k, l and m were to range through all allowed values the resulting set of functions would be complete but here k is always considered as constant.

To determine the coefficients a_{kl} we first investigate the normalizing integrals and the orthogonality of our functions. Integrating between spheres of radius R_1 and R_2 we find

$$\begin{aligned}
&(2l+1)\int\mathbf{A}_1'\cdot\bar{\mathbf{A}}_1'd\tau=(lI_{l-1}+(l+1)I_{l+1}), && \int\mathbf{A}_2'\cdot\bar{\mathbf{A}}_2'd\tau=I_l,\\
&(2l+1)\int\mathbf{A}_3'\cdot\bar{\mathbf{A}}_3'd\tau=((l+1)I_{l-1}+lI_{l+1}), && \int A_1'\cdot\bar{\mathbf{A}}_2'd\tau=0,\\
&(2l+1)\int\mathbf{A}_1'\cdot\bar{\mathbf{A}}_3'd\tau=(l(l+1))^{\frac{1}{2}}(I_{l-1}-I_{l+1}), && \int\mathbf{A}_2'\cdot\bar{\mathbf{A}}_3'd\tau=0,\\
&I_l=\frac{\pi k^2}{2}\int_{R_1}^{R_2}\left|\frac{1}{(kr)^{\frac{1}{2}}}H^{(1)}_{l+\frac{1}{2}}(kr)\right|^2r^2dr\sim\frac{R_2-R_1}{2}, && k(R_2-R_1)\gg l.
\end{aligned} \tag{4}$$

We note that the functions are not quite orthogonal. As things work out it seems much simpler not to orthogonalize. Some of these results are obtained most easily by using the polar coordinate components of the $\mathbf{A}$'s, others by using Cartesian components. The polar components are quite simple expressions but the Cartesian components are a little hard to write down and since their use is quite essential, both here and in the integrals to follow, it is worth while to indicate that they may be obtained rather directly in the following way. It is readily seen that the z components of $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ may be obtained by operating on ξ with the following

$$\begin{gathered}
(l(l+1))^{\frac{1}{2}}\partial/\partial z,\\
k\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right),\quad\left(2+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\frac{\partial}{\partial z}+k^2z;
\end{gathered} \tag{5}$$

the other components are obtained similarly after cyclic permutations of x, y, z in the operators above. If now we write our solutions of the scalar wave equation ξ in Whittaker's general form,[2] the operations are readily carried out and on putting $P_l^m(\cos u)\sin ue^{imv}$ for Whittaker's $f(u,v)$ we find the various Cartesian components expressed as functions of polar coordinates. The components of $\mathbf{A}'$ may be found similarly or by substituting $H^{(1)}$ for J in the components of $\mathbf{A}$.

We next evaluate the following integrals

$$\begin{aligned}
&\int_{R_1}^{R_2}\mathbf{A}_1'(2)\frac{e^{ikr_{12}}}{r_{12}}d\tau_2=\frac{4\pi i}{k}\left[\frac{1}{2l+1}(lI_{l-1}+(l+1)I_{l+1})\mathbf{A}_1(1)+\frac{(l(l+1))^{\frac{1}{2}}}{2l+1}(I_{l-1}-I_{l+1})\mathbf{A}_3(1)\right], \qquad r_1<R_1\\
&\int_{R_1}^{R_2}\mathbf{A}_2'(2)\frac{e^{ikr_{12}}}{r_{12}}d\tau_2=(4\pi i/k)I_l\mathbf{A}_2(1),\\
&\int_{R_1}^{R_2}\mathbf{A}_3'(2)\frac{e^{ikr_{12}}}{r_{12}}d\tau_2=\frac{4\pi i}{k}\left[\frac{(l(l+1))^{\frac{1}{2}}}{2l+1}(I_{l-1}-I_{l+1})\mathbf{A}_1(1)+\frac{1}{2l+1}((l+1)I_{l-1}+lI_{l+1})\mathbf{A}_3(1)\right].
\end{aligned} \tag{6}$$

[1] The first two of these are essentially functions used for different purposes by Mie and others. See G. Mie, Ann. d. Physik **25**, 377 (1908); H. Bateman, *Electrical and Optical Wave Motion.*

[2] Whittaker and Watson, *Modern Analysis*, 4th ed., Par. 18. 61.

The above are obtained by using Cartesian components and the expansion of $e^{ikr_{12}}/r_{12}$ in spherical coordinates. We may remark that equations almost identical with (4) and (6) are obtained if $\mathbf{A}'$ and $\mathbf{A}$ are interchanged.

Considering the left-hand side of Eq. (1) as a function of the variables 2 (with $r_2 > r_1$) it must be possible to expand it in a series of A's; doing this we find the expansion coefficients to be $(4\pi i/k)\bar{\mathbf{A}}(1)\cdot\mathbf{i}(1)$. Thus we have

$$\mathbf{i}(1)e^{ikr_{12}}/r_{12} = (4\pi i/k)\sum \mathbf{A}_s'(2)\bar{\mathbf{A}}_s(1)\cdot\mathbf{i}(1). \qquad r_2 > r_1 \quad (7)$$

Here s includes all the numbers needed to characterize an $\mathbf{A}$.

Solutions of Maxwell's Equations and an Expression for the Radiation from a Given Current Distribution

The principal use of this expansion seems to be in connection with Maxwell's equations. Suppose that the charge and current are confined to a finite region, say a sphere of radius R, and that they vary sinusoidally in time so that they can be written $\mathbf{i}(xyz)e^{-i\omega t}$, $\rho(xyz)e^{-i\omega t}$. Then all the other quantities will also have a common factor $e^{-i\omega t}$; this we will remove. Then we easily find

$$\begin{aligned} \mathbf{A} &= (i/\omega)\sum a_s \mathbf{A}_s', \\ \mathbf{E} &= (-1/c)\sum (a_{2lm}\mathbf{A}_{2lm}' + a_{3lm}\mathbf{A}_{3lm}'), \\ \mathbf{B} &= (-i/c)\sum (a_{2lm}\mathbf{A}_{3lm}' + a_{3lm}\mathbf{A}_{2lm}'), \\ a_s &= \int \bar{\mathbf{A}}_s\cdot\mathbf{i}\,d\tau, \quad r > R. \end{aligned} \qquad (8)$$

(We use Heaviside-Lorentz units.) The scalar potential is not written since if $\mathbf{i}$ and hence $\mathbf{A}$ are known, everything else may be found without knowledge of ρ or the scalar potential—the converse is not true, of course.

It should be noted that in any practical case, such for example as the computation of the field due to a radio antenna, the series all converge very rapidly because near the origin the $\mathbf{A}$'s vanish like higher and higher powers of r as l increases. Thus $\mathbf{A}_{31m}$ is finite at the origin, $\mathbf{A}_{32m}$ is proportional to r near $r=0$, etc.

From these results it is possible to find a particularly simple expression for the total energy radiated per unit time by a given current distribution. The result is

$$\int \mathbf{S}\cdot d\sigma = (1/2c)\sum (|a_{2lm}|^2 + |a_{3lm}|^2). \quad (9)$$

The form of this last expression suggests that its scope can be extended to include coordinate systems other than spherical. Thus if we consider a complete orthogonal set of vector functions which includes among others $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ a measure of the radiation from a current distribution is the square of the "norm on the subspace" spanned by $\mathbf{A}_2$ and $\mathbf{A}_3$. To go further we must decide what it is that distinguishes $\mathbf{A}_2$ and $\mathbf{A}_3$ from other functions, for instance $\mathbf{A}_1$. One essential distinction is that $\mathbf{A}_2$ and $\mathbf{A}_3$ have zero divergence, while $\mathbf{A}_1$ has zero curl; the reason this is important is that a current distribution with zero curl cannot radiate and a function with zero curl will have no component in the subspace spanned by the functions with zero divergence.[3] The other important characteristic is of course that $\mathbf{A}_2$ and $\mathbf{A}_3$ are solutions of a wave equation with the proper value of k. Suppose now we know a complete set of solutions of the scalar wave equation in some coordinates other than spherical; from each of these we can construct three solutions of the vector wave equation, one lamellar and two solenoidal. All the latter functions could of course be obtained (assuming them to be orthogonal inside a sphere) by a rotation of the subspace spanned by $\mathbf{A}_2$, $\mathbf{A}_3$; but the norm on the subspace is invariant under this rotation so that the total radiation can be computed exactly as before by summing the squares of the coefficients of the expansion of $\mathbf{i}$.

The only restriction on the functions used that has not been mentioned is that it must be possible to normalize the functions in the same way the $\mathbf{A}$'s are normalized, i.e., so that the average value of $|\mathbf{A}|^2$ over the surface of a sphere becomes asymptotic to $(\sin(kr+\delta)/r)^2$ as

[3] Provided the function vanishes properly at infinity. The lamellar function obtained by dividing a current which is zero outside a finite region into lamellar and solenoidal parts satisfies this requirement.

$r\to\infty$. Unless some rational way can be found to normalize the functions resulting from other choices of coordinates this seems to limit the method outlined above to spherical polar, prolate spheroidal, oblate spheroidal, confocal ellipsoidal coordinates, and those designated by Eisenhart[4] as Type VI.

Treatment of the Quantum-Mechanical Radiation Problem

The functions $\mathbf{A}_2$, $\mathbf{A}_3$ may be made the basis of a rather neat treatment of Dirac's[5] theory of the radiation from an atom. Thus consider a large spherical hohlraum of radius R and let this be filled with radiation describable by a vector potential $\mathbf{A}(\nabla\cdot\mathbf{A}=0)$. This $\mathbf{A}$ can be expanded in a series in which only functions of the type $\mathbf{A}_2$ and $\mathbf{A}_3$ will appear.

$$\mathbf{A}=\sum u_s\mathbf{A}_s. \tag{10}$$

The u_s are functions of the time only and we may treat them as coordinates, since if we know them we know the radiation field. Introducing momenta conjugate to these coordinates, and changing units for convenience, we can write the energy of the field in Hamiltonian form

$$\begin{aligned} &W=\sum(\tfrac{1}{2}p_s^2+2\pi^2\nu_s^2q_s^2)\,; \quad q_s=\alpha_s^{\frac{1}{2}}u_s\,; \\ &p_s=(1/\alpha_s^{\frac{1}{2}})(\partial W/\partial u_s)\,; \quad \alpha_s=(1/c^2)(R/2). \end{aligned} \tag{11}$$

If now there is an atom in the hohlraum and we know the energy of interaction between it and the field, we can treat this interaction as a perturbation and so find the probability that the atom will change (say) from state two to state one and a radiation field oscillator of type s will simultaneously change from the nth to the $(n+1)$st state. Taking the interaction energy to be $(1/c)\mathbf{A}\cdot\mathbf{i}e^{2\pi i\nu t}$ the result is

$$\frac{(n+1)R}{2h\nu c\alpha_s}\left|\frac{1}{c}\int\mathbf{A}_s\cdot\mathbf{i}d\tau\right|^2. \tag{12}$$

For the reverse transition $(n+1)$ is replaced by n. The radius of the hohlraum R enters because there are $2Rd\nu/c$ radiation components in $d\nu$ of type s and one must sum over these. Remembering that $n=\rho_\nu c^3/8\pi h\nu^3$ we see that we obtain both the A and B coefficients of Einstein. Thus

$$\mathbf{A}_{s,\,21}=(c/h\nu)\left|(1/c)\int\mathbf{A}_s\cdot\mathbf{i}d\tau\right|^2. \tag{13}$$

If for convenience we put the atom at the origin of coordinates the point of the present method of calculation is plain; the dipole, quadrupole, etc., types of radiation are completely and naturally separated. Thus transitions involving radiation of type $\mathbf{A}_{31m}$ are generally much stronger than those associated with $\mathbf{A}_{32m}$ or $\mathbf{A}_{21m}$ because the former is finite at the origin whereas the latter two vanish like r. We notice that the designations dipole, quadrupole, etc., are not sufficient to describe the various types of radiation, for example, radiation of type $\mathbf{A}_{21m}$ is in general weak compared with the dipole type $\mathbf{A}_{31m}$ but has almost the same symmetry properties. To get a complete description then radiation of the third type with $l=1, 2\cdots$ will be called electric dipole, quadrupole, etc., and radiation of the second general type will be designated as magnetic dipole, etc.

Without more definite knowledge of the current distribution it cannot be said whether the electric quadrupole or the magnetic dipole will be stronger; one easily can think of cases where either one but not the other is zero.[6]

The relative intensities of the various Zeeman components can be computed by the present method but the results will not be given as they have already been obtained as far as the quadrupole terms.[7, 8] These formulas can be derived from quite general principles by group theory or by the methods first used for the dipole case by Honl[9] and Kronig and Goudsmit.[10] In connection with the latter we may mention that for given l a sum of $\mathbf{A}_s$ can be built up which gives radiation that is unpolarized and with intensity independent of angle.

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[4] L. P. Eisenhart, Annals of Math. **35**, 284 (1934); Phys. Rev. **45**, 427 (1934).

[5] P. A. M. Dirac, Proc. Roy. Soc. **A114**, 243 (1927).

[6] For examples of actual cases where the magnetic dipole term is important see E. U. Condon, Astrophys. J. **79**, 217 (1934).

[7] A. Rubinowicz, Zeits. f. Physik **53**, 267 (1929).

[8] H. C. Brinkman, Dissertation, Amsterdam.

[9] H. Honl, Zeits. f. Physik **31**, 340 (1925).

[10] R. de L. Kronig and S. Goudsmit, Naturwiss. **13**, 90 (1925).

It is of interest to note that exactly the same results for the Einstein A coefficients can be obtained by computing in a classical way the radiation from an atom with a given current distribution. By Eq. (9) it is seen that the Einstein A is that given by Eq. (13) except for a factor of two that appears in such cases as this.[11]

The Vector Potential in the Steady Current Case

It is of some interest to investigate the expansion of $\mathbf{i}(1)/r_{12}$ even though use of the expansion does not seem to lead to any results not previously known, with the possible exception of a formula similar to Eq. (9) for the inductance of a coil. We might start by letting k approach zero in the expressions above but this is likely to be rather awkward as the $\mathbf{A}$'s all approach zero while the $\mathbf{A}'$'s become infinite. All told it seems best to start anew in the same way as before by inventing three vector functions satisfying a vector Laplace equations and developing an expansion of $\mathbf{i}(1)/r_{12}$ in terms of these. Such functions are

$$\begin{aligned} A_1&=\mathbf{r}\xi-[1/(2l+3)]\nabla(r^2\xi),\\ A_2&=\nabla\times A_1=\nabla\times(\mathbf{r}\xi),\\ A_3&=\nabla\times A_2=\cdot\nabla\nabla A_1, \end{aligned} \tag{14}$$

$$\xi=\left[\frac{(l-m)!}{l(l+1)(l+m)!}\right]^{\frac{1}{2}} r^l P_l^m e^{im\varphi}.$$

The functions A' are the same except that the r^l in ξ is replaced by r^{-l-1} and the $-1/(2l+3)$ in the definition of A_1 is changed to $+1/(2l-1)$. In the limit $k\to 0$ and taking no account of constant factors

$$\mathbf{A}_2\to A_2,\quad \mathbf{A}_1\to A_3,\quad \mathbf{A}_3\to A_3 \text{ and } \mathbf{A}_3-(l+1)\mathbf{A}_1\to A_1.$$

[11] J. Frenkel, *Elementary Theory of Wave Mechanics*, Oxford, p. 134.

There is an interesting crossing over of the functions in the integrals that follow

$$\int_{R_1}^{R_2}\frac{A_1'(2)}{r_{12}}d\tau_2=U_1A_3(1),$$

$$\int_{R_1}^{R_2}\frac{A_2'(2)}{r_{12}}d\tau_2=U_2A_2(1),$$

$$\int_{R_1}^{R_2}\frac{A_3'(2)}{r_{12}}d\tau_2=U_3A_1(1), \qquad r_1<R_1$$

with

$$U_1=[4\pi/(2l-1)^2(2l-3)(l+1)](1/R_1^{2l-3}-1/R_2^{2l-3}),$$

$$U_2=[4\pi/(2l-1)(2l+1)](1/R_1^{2l-1}-1/R_2^{2l-1}),$$

$$U_3=(4\pi l/2l+1)(1/R_1^{2l+1}-1/R_2^{2l+1}).$$

Also

$$\int_{R_1}^{R_2}A_i'\cdot\bar{A}_i'd\tau=0 \quad (i\neq j)$$

$$\int_{R_1}^{R_2}A_i'\cdot\bar{A}_i'd\tau=U_i,$$

and

$$\mathbf{i}(1)/r_{12}=\sum A_s'(2)\bar{A}_s(1)\cdot\mathbf{i}(1). \qquad r_1<r_2$$

If the vector potential and the magnetic field due to a given steady current are determined by means of this expansion, it will be found that B is apparently a linear combination of A_2 and A_3. Actually, however, the terms in A_2 vanish because their coefficients depend on integrals of the form $\int A_3\cdot\mathbf{i}d\tau$ and these are zero because $\nabla\cdot\mathbf{i}=-(1/c)\dot{\rho}=0$. We note that the electric field due to a stationary charge distribution can also be expressed by using A_3 only.

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