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Depth Distribution of Origins of Characteristic X-Rays from Thick Targets

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If I is the intensity of any x-ray wave-length as observed in a direction making an angle ϕ with the face of the (thick) target in which the radiation originates, then

$$S_T = \frac{-1}{\mu}\left(\frac{d \log I}{d \csc \phi}\right)_{\csc \phi = 0}$$

is the depth of the *centroid of the x-ray luminosity distribution* in the target. This depth is greater than the effective mean depth of production by an amount which depends upon the angle of observation and the form of the luminosity distribution. For large angle observation of Ag $K\alpha$ at 100 kv (and probably at any voltage) the two depths are identical within the errors of measurement. The distributions of x-ray luminosity with target depth for Ag $K\alpha$ from a thick target are calculated for six tube potentials from 50 kv to 175 kv from the intensity observations of Webster, Hansen and Duveneck. There is as yet no theory of these distribution functions. The characteristics of the curves obtained are shown to be capable of fairly simple mathematical description and to be in harmony with certain physical checks.

INTRODUCTION

WHEN x-rays are produced by normal incidence of electrons upon a thick target, radiation originates at all depths within a layer extending from the target surface to some heretofore undetermined maximum depth which depends upon the characteristics of the target material and the energy of the incident electrons. It is the purpose of this paper to discuss the distribution with depth of the *x-ray luminosity* of the target, that is, the energy (of a specified wave-length) radiated per second per unit volume of target per unit target current. This luminosity is a function of electron energy as well as of depth and wave-length and for any given value of these three independent variables it is presumably different for characteristic and general radiations. While the following discussion is partly of general application it relates more particularly to the $K\alpha$ luminosity of silver targets.

MEAN DEPTH OF PRODUCTION

The work of Ham,[1] of W. P. Davey[2] and of L. G. Davey[3] established the concept of a *mean depth of production* for the unresolved x-ray output of a thick target, and showed the possibility of measuring such a depth by observation of the relative intensities of total x-radiation emerging at different angles with the target face. Webster and Hennings,[4] in a study of the penetration of cathode rays in molybdenum, measured the mean depth of production of a particular wave-length (i.e., the wave-length of the absorption limit of the target) for a number of molybdenum continuous spectra. While the next step in the problem is the determination of the depth distribution of luminosity for a particular wave-length, this

[1] W. R. Ham, Phys. Rev. **30**, 1 (1910).
[2] W. P. Davey, J. Frank. Inst. **171**, 277 (1911).
[3] L. G. Davey, Phys. Rev. **4**, 217 (1914).
[4] D. L. Webster and A. E. Hennings, Phys. Rev. **21**, 301 (1923).

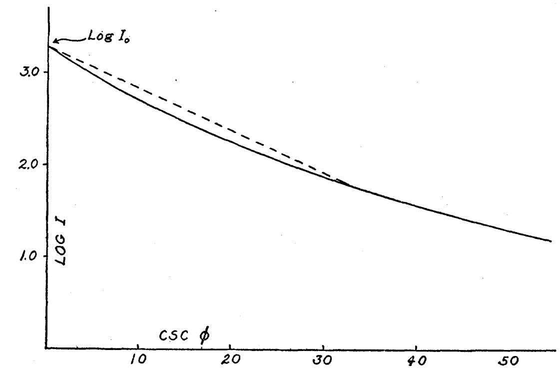

FIG. 1. Kulenkampff curve for determining mean depth of production and target absorption correction for monochromatic x-rays. (Ag $K\alpha$ at 100 kv.)

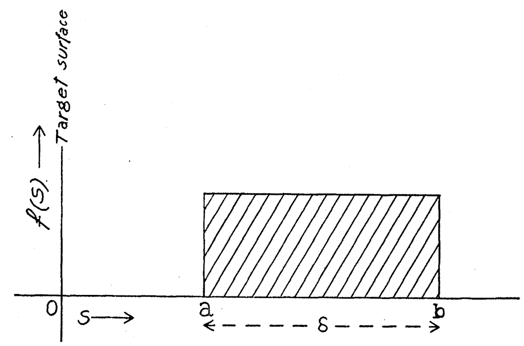

FIG 2. Assumed rectangular x-ray luminosity distribution.

matter is deferred until the next section, while we consider here an extension of the theory of the mean depth of production.

The only known method of measuring the mean depth of production of monochromatic radiation applicable to any chosen x-ray wave-length is the method of Ham[1] and the Daveys[2, 3] with the addition of an x-ray spectrometer for wave-length resolution. This is essentially the method used by Kulenkampff[5] to determine the attenuation of primary x-rays through absorption in the target, and since used by others for the same purpose and for the determination of mean depths of production. Following Kulenkampff, one sees that if all radiation (of some observed wave-length) were actually produced in a thin layer situated at a depth S below the target face, with I_0 representing the intensity *as emitted* then the intensity *as observed* would be similarly represented by $I = I_0 e^{-\mu S x}$, where μ is the x-ray absorption coefficient of the target and x the cosecant of the angle of observation measured from the target face. It follows that $\log I = \log I_0 - \mu S x$ and that both S and I_0 may be obtained graphically from a plot of $\log I$ *vs*. x. In particular

$$S = (-1/\mu) d(\log I)/dx. \tag{1}$$

Now although the equation above shows a linear relation between $\log I$ and x the real graph (Fig. 1) is not straight, for it results from a luminosity distribution which is quite different from the assumed concentrated layer, and with such a distribution the effective depth of production may be said to vary with x. When, as in real targets, luminosity is distributed over a range of depths the mean depth of production is best defined as that depth at which the entire luminosity could be concentrated without change in the externally observed intensity. We shall designate the depth so defined by $\bar{S}$ and call it hereafter the effective depth of production.

The graph of Fig. 1, which is typical of the many Kulenkampff curves which have been used in thick target studies, shows a marked upward concavity. That such curves must always possess this type of curvature unless the radiation all originates at the same depth may be shown by considering a rectangular luminosity distribution buried in a thick target as in Fig. 2, in which $f(s)$ measures the luminosity and s represents distance into the target. The emerging intensity is

$$I = \int_a^b f(s) e^{-\mu x s} ds = (-k/x\mu)[e^{-\mu b x} - e^{-\mu a x}].$$

If $y \equiv \log I$ this leads to

$$\frac{dy}{dx} = \frac{\mu \delta e^{-\mu x \delta}}{1 - e^{-\mu x \delta}} - \mu a - \frac{1}{x}, \tag{2}$$

where, as in Fig. 2, $\delta = b - a$. Upon differentiating again it is found that the second derivative is positive, which insures the upward concavity for this special distribution. But any distribution whatever can be analyzed into rectangular elements and consequently any Kulenkampff curve may be regarded as the sum of the curves belonging to its constituent rectangular distributions. Since all the constituent curves are concave up-

[5] H. Kulenkampff, Ann. d. Physik 69, 548 (1922).

ward their sum must be curved in the same sense.

We may now examine the variation of the effective depth of production with the angle of observation. By definition of $\bar{S}$ we have $I = I_0 e^{-\mu \bar{S} x}$ or

$$\bar{S} = (\log I_0 - \log I)/\mu x, \tag{3}$$

which is $(1/\mu)$ times the absolute value of the slope of the chord drawn in Fig. 1, from the intercept on the log I axis to the point of the curve having the abscissa x. It is apparent from the graph that the effective depth of production varies with the angle of observation, being zero when $\phi = 0$ and increasing with ϕ, in a manner which depends upon the form of $f(s)$, to a maximum for normal observation of the target face ($\csc \phi = 1$).

Although the angle $\csc^{-1} 0$ is fictitious, it has useful properties in this analysis. Kulenkampff[5] shows that the extrapolation of the curve to $\csc \phi = 0$ furnishes an intercept (in the case of smooth targets) which measures I_0, the intensity which the radiation would have if the target were non-absorbing. We now show that the effective depth of production for this fictitious angle is the depth of the *center of gravity* of the luminosity distribution curve. As x approaches zero, Fig. 1 shows that the chord of the curve approaches the tangent, and at the limit we have by Eq. (1)

$$\bar{S}_{x=0} = \frac{-1}{\mu}\left(\frac{d \log I}{dx}\right)_{x=0}. \tag{4}$$

To evaluate the derivative we first restrict ourselves to the conventional distribution of Fig. 2, whose derivative is given by Eq. (2). Rearranging Eq. (2) we have

$$\frac{dy}{dx} = \frac{-\dfrac{(\mu\delta)^2}{2!} - \dfrac{(\mu\delta)^3 x}{3!} - \dfrac{(\mu\delta)^4 x^2}{4!} - \cdots}{\mu\delta + \dfrac{x(\mu\delta)^2}{2!} + \dfrac{x^2(\mu\delta)^3}{3!} + \cdots} - \mu a,$$

from which it is apparent that

$$(dy/dx)_{x=0} = -\mu(a + \delta/2) = -\mu S_g, \tag{5}$$

where S_g is the distance from the target surface to the *center of gravity* of the assumed distribution. From Eqs. (4) and (5) it follows that

$$\bar{S}_{x=0} = S_g.$$

Though the luminosity distribution assumed was a special one, more general conclusions may be drawn. Imagine two distinct (though permissibly contiguous) rectangular distributions in the same target, and suppose it to be possible to observe in various directions the radiation from either separately or from both together. In each of the three cases of observation suppose the Log I *vs.* x curve to be plotted and extrapolated to $x = 0$, where all the intensities and intensity derivatives used below are taken. Using subscripts 1, 2 and T to designate quantities related to the separate distributions and to their total, respectively, we have

$$I_T = I_1 + I_2 \quad \text{and} \quad (dI_T/dx) = dI_1/dx + dI_2/dx.$$

The depths to the centers of gravity of the separate distributions, as shown by Eq. (4), are

$$S_{g1} = (-1/\mu)(dy_1/dx) = (-dI_1/dx)/\mu I_1$$

and

$$S_{g2} = (-dI_2/dx)/\mu I_2.$$

From the equations of this paragraph we may obtain

$$dy_T/dx = (dI_T/dx)/I_T = -\mu\frac{S_{g1}I_1 + S_{g2}I_2}{I_1 + I_2} = -\mu S_T, \tag{6}$$

where S_T is the depth of the center of gravity of the combination. Since any real distribution may be built up by the successive addition of properly chosen rectangular blocks it follows that in any case the depth of the center of gravity of a distribution is given by Eq. (4).

The center of gravity depth is greater than the effective depth of production, even when the latter is at its maximum ($\phi = 90°$), but it is obvious in Fig. 1 that the change in slope from $x = 0$ to $x = 1$ is very small, for any reasonable extrapolation, so the difference in these two depths must also be small in this fairly typical case. We refer to these depths again in a later section.

Method of Determining Luminosity Distributions

Webster, Hansen and Duveneck[6] have measured the intensity of the $K\alpha$ lines from a thick silver target upon whose plane face cathode rays were incident normally, making observations at several voltages up to 175 kv and at various angles ϕ between the direction of observation and the target face. With the symbols used above the observed intensity (in arbitrary units) at any given electron energy is

$$I(\phi)=\int_0^\infty f(s)e^{-\mu s x}ds, \tag{7}$$

where $I(\phi)$ is known for a number of values of ϕ from experiment and $f(s)$ is the luminosity distribution function to be found. The history of this equation as recorded in the literature of mathematics does not encourage an attempt at an exact analytical solution, and it seems necessary to resort to methods of approximation. Physical considerations impose important conditions upon $f(s)$ which serve as guides in seeking solutions: the function is never negative or infinite, it is zero for all negative values of s and practically so for all positive values exceeding the foil range of cathode electrons. It is finite at $s=0$ and almost certainly goes to zero with zero slope.

Eq. (7) has appeared in physical problems before[7] and several possible methods of attack have been used or proposed. We have found most useful an as yet unpublished method suggested by Professor William V. Houston of California Institute of Technology. In this method it is assumed that the luminosity distribution is adequately representable by

$$f(s)=e^{-Hs}(a+bs+cs^2+\cdots), \tag{8}$$

an assumption which gives Eq. (7) the form

$$I(\phi)=\frac{a}{\mu x+H}+\frac{b}{(\mu x+H)^2}+\frac{2c}{(\mu x+H)^3}+\cdots.$$

Having a number of observations of I equal to the number of constants retained in $f(s)$ we can solve for these constants and thus determine the function. Since $f(s)$ must in reality be a smooth curve and since it is believed to decrease monotonically to zero at the maximum depth somewhat after the manner of an exponential, it seems certain that the assumed form will permit a satisfactory representation.

Our procedure has been to replot the data contained in Fig. 1 of reference 6 in such a way as to smooth out certain slight irregularities caused by the use of non-identical units of measurement for different angles of observation. This smoothing did nothing to correct for possible *systematic* variations of the unit of measurement with angle. If such variations existed they were probably worst at the largest grazing angles of observation, so we have not used Webster, Hansen and Duveneck's data for any angles above 8.05°, except for checking purposes. From the selected data the constants of Eq. (8) were obtained, through the method suggested by Houston, assisted by graphical devices and cut-and-try approximations which cannot be economically described.

The constant a in the expression for $f(s)$ represents the luminosity of the target surface, which should vary with voltage in approximately the same manner as does the intensity of the $K\alpha$ lines from a *thin* target of silver. The two cases differ in that the thick target surface emits a greater proportion of radiation produced by fluorescence and electron rediffusion, but both of these secondary effects show voltage variations which approximately parallel the variation in intensity of the directly produced radiation, so we have ignored the difference and imposed the condition that variations of a with voltage shall agree with the *thin* target intensity variations found by Webster, Hansen and Duveneck.[8]

The function assumed in Eq. (8) takes on small negative values for large values of s, since the constant c turns out to be negative. This physically impossible result emphasizes the arbitrary nature of the functional form assumed, and in this connection we must certainly disclaim any suggestion that the terms in the series represent separate physical processes of x-ray production. However, any method of luminosity analysis based upon the angular variation of external in-

[6] D. L. Webster, W. W. Hansen and F. B. Duveneck, Phys. Rev. **44**, 258 (1933).

[7] L. Silberstein, J. O. S. A. **22**, 265 (1932); Phil. Mag. **15**, 375 (1933); C. Eckart, Phys. Rev. **45**, 851 (1934).

[8] D. L. Webster, W. W. Hansen and F. B. Duveneck, Phys. Rev. **43**, 839 (1933).

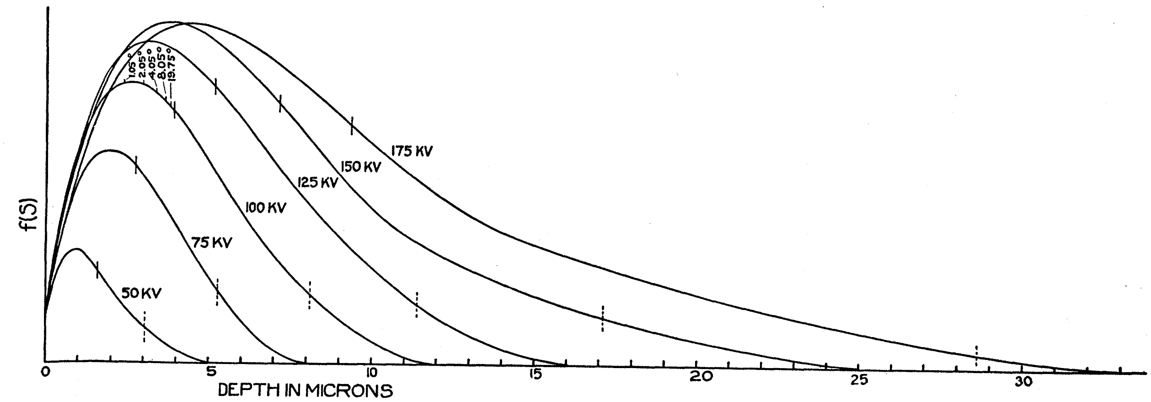

FIG. 3. Luminosity distributions of $K\alpha$ radiation from a thick silver target with cathode rays incident along normals to the plane target face. Solid vertical line segments intersecting the curves mark the positions of the centers of gravity of respective distributions. The relation of the center of gravity to the effective depths of production for various angles of observation is illustrated in the case of the 100 kv curve. For 90° (normal) observation the two depths are sensibly identical. Dotted lines crossing the curves indicate foil ranges of electrons at corresponding potentials. Ordinates are in arbitrary units which are the same for all curves.

tensity will give results which are less dependable the greater the depth, and the negative luminosities which we obtain are within the probable error of luminosity determination for the depths at which they occur, as is shown by the fact that the calculated external intensity from the integrated negative luminosity is within the error of intensity measurement. We have therefore discarded the negative part of the distribution curves and smoothed off the region of amputation in an arbitrary but, we believe, physically plausible manner.

RESULTS AND DISCUSSION

The distributions obtained are shown for a series of six cathode-ray energies in Fig. 3. The equation representing each curve contains four constants and has been derived from intensity observations at four values of the angle ϕ. Since the simultaneous solution of the four equations containing experimentally determined magnitudes was a very inexact process and since the obtained solutions have been arbitrarily adjusted in some particulars it is important to check the distributions obtained back against the original data. This is a much simpler operation than its inverse, and is in line with our view that the proof of the distribution is not in the rigor of its derivation but in its ability to recreate all the original intensity observations. The recheck was performed by plotting from each $f(s)$ in Fig. 3 a series of curves represented by $y=f(s)e^{-\mu s x}$ for five values of ϕ, including the four used in obtaining $f(s)$. The areas of the curves so obtained were found to be proportional to the corresponding original observations upon $I(\phi)$, as they should be, closely enough so that the two sets of values could be matched with an average discrepancy of between 1 and 2 percent and with a maximum under 5 percent. Uncertainties in the original data, make it useless to strive for a better fit. The distributions obtained are therefore as good as the observations permit.

It is difficult to evaluate the probable errors of the distributions in Fig. 3, or, indeed, to define a suitable measure of probable error. We find, however, that a distribution will fail to check back upon the data from which it came, causing discrepancies of 5 or 10 percent, if the maximum of the distribution be raised or lowered by as much as 10 percent relative to the ordinate at $s=0$, or if the extreme depth of the distribution be changed by 20 percent.

Present knowledge of electron scattering and stopping processes is probably not complete enough to permit a theoretical calculation of the luminosity distribution functions, and even with such knowledge the computation would be formidably complex. A partial check on the experimental distributions may be obtained, however, by comparing the maximum depths in Fig. 3 with measured foil ranges of electrons of corresponding energies. A foil range means the thickness of the thickest foil which normally incident electrons will pierce. Since the path in such a foil is far from straight, the foil range is, in silver,[9] probably less than half the real range. Foil range for different elements is a function of the density of the foil, so it is possible to use measurements by Schonland[10] and others on aluminum to deduce by interpolation a series of *silver* foil ranges for electrons having potentials pertinent to the curves of Fig. 3. The ranges so obtained are denoted by vertical dotted lines crossing the distribution curves.

The relation of these ranges to the luminosity distributions was an afterthought and has not been injected in any way into the solutions themselves, so the agreement shown in the figure may be accepted as a useful confirmation. The existence of luminosity below the depth of the foil range is presumably caused by fluorescence.

The centers of gravity of these distributions all agree in depth with the values obtained by the graphical method (Eq. 4) within the limits of error of our determinations of the latter. Since both methods start from the same data, this is not exactly an independent check on the correctness of the distribution curves but is a demonstration that the same result can be had by two methods for computing the same quantity when one method depends heavily upon the correctness of the distributions.

Webster, Hansen and Duveneck,[6] after correcting their thick target silver $K\alpha$ intensities for target absorption by extrapolating the curve of log I *vs.* csc ϕ to csc $\phi=0$, found the corrected intensity $I_0(U)$ to vary with U, the ratio of tube voltage to K-ionizing voltage, according to the equation $I_0(U)=K(U-1)^{1.65}$ up to about $U=4$. Using for $I_0(U)$ the areas under the curves of Fig. 3 we find a similar relation to hold, over the same range of voltage, except that the exponent is 1.58. This difference does not seem too large to be accounted for by the combined non-systematic errors of the two methods so it would be unwarranted to conclude from this one case that either method is inherently prone to yield either high or low results.

Along the 100 kv curve of Fig. 3 are marks indicating the effective depths of production of this distribution for several values of ϕ. For ninety degree observation this depth differs from the center of gravity depth by less than the width of the vertical line indicating the latter.

Conclusion

Summarizing, the luminosity, $f(s)$, of silver $K\alpha$ radiation produced in a thick target by normal cathode rays when plotted against depth, s, in the target results in a distribution curve which starts with a finite $f(s)$ at $s=0$, rises to a maximum and falls away gradually to zero at a depth several times the depth of the maximum. This general character is possessed by all distributions investigated, from 50 kv to 175 kv. The depth of the maximum in microns is given approximately by $S_m=0.8\ (U-1)$. The depth to the center of gravity of the distribution, which is almost identically the effective depth of production for normally emerging rays, is given approximately by $S_g=1.45\ (U-1)$. The maximum depth limit is difficult to state since the curve has no sharp intersection with the s axis, but the depth at which the distribution has decreased to 1/10 of its value at the maximum may be stated approximately as $S_l=3.3\ (U-1)$, at least up to $U=5$, after which S_l seems to increase slightly faster with U. The height of the maximum of $f(s)$ varies from 2.5 times its value at the target surface in the case of the 50 kv curve to 6.5 times for the 175 kv curve. The height of the maximum increases with U less rapidly as U increases and attains a maximum between $U=6$ and $U=7$. The area under the curves, giving the total emitted energy, varies with U according to the formula found by Webster, Hansen and Duveneck in a different treatment of the same data.

It is a pleasure to acknowledge a number of helpful conversations with Professor Carl Eckart of the University of Chicago upon the solution of Eq. (7).

[9] E. J. Williams, Proc. Roy. Soc. **A130**, 310 (1931).

[10] B. F. J. Schonland, Proc. Roy. Soc. **A104**, 235 (1923); **108**, 187 (1925); C. E. Eddy, Proc. Camb. Phil. Soc. **25**, 50 (1929); R. W. Varder, Phil. Mag. **29**, 726 (1915).