An Indeterminacy Relation for Several Observables and Its Classical Interpretation

H. P. ROBERTSON, *Princeton University*
(Received September 10, 1934)

An examination of the limitations placed by the conceptual structure of quantum mechanics on the simultaneous measurement of two or more observables leads, for the case of an even number $m=2k$ of observables, to a quantitative estimate, depending only on the mean values of their commutators in the state in question, of the least value which the product of their m uncertainties may assume. This indeterminacy relation is interpreted classically to mean that, by measurement of these observables alone, a point (p, q) in phase space can be determined only to within a certain channel whose normal cross section, as measured by the value of the kth integral invariant taken over it, must be at least of order h^k.

1. INTRODUCTION

PERHAPS the most revolutionary and far-reaching consequence of quantum mechanics is the surrender which it entails of strict determinacy as a fundamental physical principle. That the limitations on determinacy thus imposed are in some measure susceptible of quantitative expression was first recognized by Heisenberg and by Bohr,[1] who showed from general considerations that the product of the uncertainties in the measurements of two canonically conjugate observables, such as p and q, must be at least of the order of Planck's constant h. The familiar precise form $\Delta p \cdot \Delta q \geqq h/4\pi$ of this indeterminacy relation for any two canonically conjugate observables, in which their uncertainties are measured by their standard deviations (root-mean-squares of deviations from the means) in the state under examination, was obtained by Kennard[2] within the framework of quantum mechanics, thus showing that it is inherent in the fundamental postulates underlying the subject. The extension of this result to any two classical observables (i.e., functions of p_i, q^i) was given by Robertson,[3] who showed that the product of their uncertainties is limited on the lower side by the mean value of their commutator in the state in question. Finally, the corresponding result for general systems was derived by Robertson and by Schrödinger,[4] and at the same time made more restrictive by the introduction of an additional term depending on the mean value of the anti-commutator of the two observables.

Now since the commutator of two observables is the quantum-mechanical analogue of their classical Poisson bracket, these formulations suggest the possibility of obtaining a classical interpretation of the uncertainty principle, at least insofar as it involves the commutator. The present writer[4] was in fact able to show that it implied that the first integral invariant $I_1=\int\sum p_i dq^i$ taken over an appropriately chosen surface in phase space, is at least of order h; since the methods employed in the (as yet unpublished) proof of this result are of importance for the present work, the derivation is given in the latter part of Section 2 below.

All of the above investigations have been concerned with the uncertainty in the simultaneous measurement of two observables. However, it is clear that the measurement of more than two observables must in general be subject to more severe limitations than those imposed by the above on each pair taken separately, for we cannot in general expect that the conditions necessary to insure minimum uncertainty in one pair will be consistent with those which insure the minimum in other pairs. This suggests the existence of an indeterminacy relation involving at once the uncertainties of all observables under consideration. In the first part of Section 2 below we prepare the way for the derivation of such a relation by presenting a proof of the known inequalities for two observables in a form susceptible of extension to the general case. This extension is given in Section 3, where in particular it is shown that the product of the uncertainties of an *even* number of observables

[1] W. Heisenberg, Zeits. f. Physik **43**, 172 (1927); N. Bohr, Nature **121**, 580 (1928).
[2] E. H. Kennard, Zeits. f. Physik **44**, 326 (1927).
[3] H. P. Robertson, Phys. Rev. **34**, 163 (1929).
[4] H. P. Robertson, Phys. Rev. **35**, 667A (1930); E. Schrödinger, Sitzungsber. preuss. Akad. Wiss. **1930**, 296.

cannot be less than a certain positive function of their commutators. On taking into account the correspondence between commutator and Poisson bracket this result leads, in Section 4, to a classical interpretation involving the higher integral invariants of phase space.

2. Indeterminacy Relation for Two Observables and Its Classical Interpretation

We begin with a derivation of the uncertainty relation for two real observables α_r $(r=1, 2)$ in a given state ψ, adopting in general the notation and conventions of Dirac.[5] For convenience we introduce in place of α_r the two new (real) observables

$$\beta_r=\alpha_r-a_r, \quad \text{where} \quad a_r=\varphi\alpha_r\psi \tag{2.1}$$

is the mean value of α_r in the state specified by ψ or its conjugate φ; the mean value of β_r in this state is then zero. Now let x_r be two arbitrary complex numbers, and consider the scalar square of the composite state $\sum(r)x_r\beta_r\psi$; this (numerical) quantity may be written

$$\Psi : \sum_{r,\, s} \psi_{rs}\bar{x}_r x_s, \tag{2.2}$$

where $\bar{x}_r$ is the conjugate complex of x_r and

$$\psi_{rs}=\varphi\beta_r\beta_s\psi=(\beta_r\beta_s)_0, \tag{2.3}$$

the subscript zero denoting, here and in all quantum-mechanical formulae in the following, mean value in the state ψ. The form Ψ is Hermitian, for $\bar{\psi}_{rs}=\psi_{sr}$, and by the fundamental postulates of quantum mechanics it cannot assume negative values. Furthermore Ψ can vanish only if $\beta_1\psi$, $\beta_2\psi$ are linearly dependent; hence for simplicity of expression we shall assume during the course of the work that these two states are linearly independent, and at the end modify the result to take account of the exceptional case.

We may, then, take the Hermitian form Ψ as *positive-definite*;[6] it can assume only positive values unless all x_r vanish. Expressed in terms of its real and imaginary parts ξ_{rs} and η_{rs}, the matrix element ψ_{rs} is

$$\begin{aligned}\psi_{rs}&=\xi_{rs}+i\eta_{rs}; \quad \xi_{rs}=\tfrac{1}{2}(\beta_r\beta_s+\beta_s\beta_r)_0=\xi_{sr},\\ \eta_{rs}&=-\tfrac{1}{2}i(\beta_r\beta_s-\beta_s\beta_r)_0=-\tfrac{1}{2}i[\alpha_r,\ \alpha_s]_0=-\eta_{sr},\end{aligned} \tag{2.4}$$

where $[\alpha_r,\ \alpha_s]$ is the commutator $\alpha_r\alpha_s-\alpha_s\alpha_r$ of α_r, α_s and the subscript zero again denotes mean value in the state ψ. Note that the diagonal element

$$\psi_{rr}=\xi_{rr}=\varphi(\alpha_r-a_r)^2\psi=(\Delta\alpha_r)^2 \quad (>0) \tag{2.5}$$

is the square of the standard derivation $\Delta\alpha_r$ of the observable α_r in the state ψ, which is to be taken as a measure of the indeterminacy of α_r in this state; this essentially positive quantity can vanish only if ψ is a characteristic vector of α_r, a possibility which we discard for the moment as being included under the case of linear dependence, for we should then have $\beta_r\psi=0$ (cf. also footnote 5).

The necessary and sufficient condition, in addition to the inequalities implied in (2.5), that the form Ψ be positive-definite is that

$$\det(\psi_{rs})\equiv\psi_{11}\psi_{22}-|\psi_{12}|^2>0 \tag{2.6}$$

or, on employing (2.4) and (2.5), we may write

$$(\Delta\alpha_1\cdot\Delta\alpha_2)^2\geqq(\beta_1\beta_2)_0(\beta_2\beta_1)_0, \tag{2.7}$$

where equality can hold only if the $\beta_r\psi$ are linearly dependent. This is the final form of the uncertainty principle for two observables mentioned in the preceding section[4]; on noting that the right-hand side is equal to $\xi_{12}^2+\eta_{12}^2$ and dropping the first of these essentially positive quantities, we obtain the weaker form[3]

$$\Delta\alpha_1\cdot\Delta\alpha_2\geqq\tfrac{1}{2}|[\alpha_1,\ \alpha_2]_0|. \tag{2.8}$$

This latter, more readily interpreted, form is the one with which we shall be concerned in the remainder of this section.

We turn now to the discussion of the classical analogue of (2.8), returning in the next section

---

[5] P. A. M. Dirac, *The Principles of Quantum Mechanics*, (Oxford, 1930). It is to be noted that in adopting this viewpoint care must be taken to avoid the employment of a state ψ which is a characteristic vector corresponding to a point in the continuous spectrum of any of the operators α, for the formal evaluation of the mean value in such a state of the commutator of α with any other observable leads to the value zero—contrary to the fact that, for example, the mean value of the commutator of p and q in any state whatever must be $h/2\pi i$.

[6] For the terminology and results here employed in dealing with quadratic or Hermitian forms see, for example, L. E. Dickson, *Modern Algebraic Theories*, Chap. IV (Chicago, 1930), or B. L. van der Waerden, *Moderne Algebra* II, Chap. XV (Berlin, 1931).

to the quantum-mechanical problem for more than two observables. If the mechanical system under consideration has but one degree of freedom, it seems evident that (2.8) must be interpretable in terms of area in the 2-dimensional phase space of p and q; indeed, on applying it to the canonically conjugate observables p, q themselves, the relation $\Delta p \cdot \Delta q = h/4\pi$ is, as has often been noted, to be interpreted as implying that the position of a point in phase space can be specified only to within a rectangle whose area is of order h. But instead of following through this simplest case for arbitrary functions of p, q we turn immediately to the general case of n degrees of freedom, and ask what quantity is there the analogue of area. The answer is clearly the first integral invariant

$$I_k = \int\int \sum_{i=1}^{n} dp_i dq^i, \tag{2.9}$$

and we should therefore look for an interpretation involving it, where the integration is to be taken over some 2-dimensional region invariantively associated with the observables $\alpha_r(p, q)$.

Suppose, then, we attempt to locate a point $P(p, q)$ in phase space in such a way that the values at P of two functionally independent observables $\alpha_r(p, q)$ differ from some given mean values a_r by not more than an amount of absolute value δa_r. Assuming sufficient regularity of the functions in the portion of phase space under consideration, the situation can be described as follows (cf. Fig. 1, illustrating one quadrant of the region here of interest). The point P must lie within the channel formed by the four hyper-surfaces $\alpha_r(p, q) = a_r \pm \delta a_r$; in order to obtain a measure of the region over which P can vary laterally we compute I_1 over some appropriately chosen cross section of this channel. Now there exists at any point $A(p_0, q_0)$ lying on the "center" of the channel, defined as the intersection of the surfaces $\alpha_r(p, q) = a_r$, a unique direction associated with each of functions α; namely, the "normal" at A to the surface $\alpha(p, q) = a$. This normal is described as the locus of a point whose (p_i, q^i) coordinates relative to A are

$$(-\lambda\alpha_i, \lambda\alpha^i), \quad \text{where} \quad \alpha_i = [\partial\alpha/\partial q^i]_0, \quad \alpha^i = [\partial\alpha/\partial p_i]_0, \tag{2.10}$$

λ is a parameter specifying the position of the point along the normal, and the subscript zero here denotes evaluation at A.[7] We now choose as the cross section of the channel, over which I_1 is to be evaluated, that plane cross section spanned by the two normals at A to the two surfaces $\alpha_r(p, q) = a_r$; its equation in terms of parameters λ^r is therefore

$$p_i = p_{i0} - \sum_r \lambda^r \alpha_{ri}, \quad q^i = q_0{}^i + \sum_r \lambda^r \alpha_r{}^i. \tag{2.11}$$

Before proceeding to the evaluation of the integral I_1 we remark that this normal cross section of the channel is unique in the sense that of all plane cross sections it, and it alone, renders the integral an extremum; in order not to interrupt the present development we relegate the proof of this interesting result to Note B below.

In order to evaluate I_1 we must first determine the range of the parameters λ^r. Let $\lambda_1{}^r$ denote the parameters of the point A_1 in which the 2-spread (2.11) cuts the intersection of the bounding surfaces $\alpha_1 = a_1 + \delta a_1$, $\alpha_2 = a_2$; we must then have

$$\begin{aligned} \alpha_1(p_{i0} - \sum \lambda_1{}^r \alpha_{ri}, q_0{}^i + \sum \lambda_1{}^r \alpha_r{}^i) &= a_1 + \delta a_1, \\ \alpha_2(p_{i0} - \sum \lambda_1{}^r \alpha_{ri}, q_0{}^i + \sum \lambda_1{}^r \alpha_r{}^i) &= a_2. \end{aligned} \tag{2.12}$$

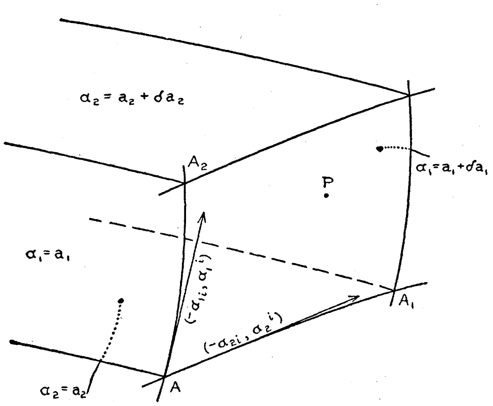

FIG. 1. Quadrant of normal cross section of channel in phase space.

[7] It may at first appear strange that the direction of this "normal" lies in the surface, as to terms of first order in λ, $\alpha(p_{i0} - \lambda\alpha_i, q_0{}^i + \lambda\alpha^i) = 0$. But it is to be remembered that the geometry of phase space is based on the bilinear concomitant $\Sigma(i)(dp_i\delta q^i - dq^i\delta p_i)$ of the two displacements d, δ; hence every displacement is "orthogonal" to itself! That the normal lies in the surface is

On expanding these functions α_r about the point A, and retaining only terms of first order in the λ_1^r, we find

$$\lambda_1^1=0, \quad \lambda_1^2=\delta a_1/(\alpha_1, \alpha_2)_0, \tag{2.13}$$

where

$$(\alpha_1, \alpha_2)\equiv\sum_{i=1}^{n}\left(\frac{\partial\alpha_1}{\partial q^i}\frac{\partial\alpha_2}{\partial p_i}-\frac{\partial\alpha_1}{\partial p_i}\frac{\partial\alpha_2}{\partial q^i}\right) \tag{2.14}$$

is the Poisson bracket formed from α_1, α_2 and the subscript zero indicates, as usual, evaluation at A. That λ_1^1 vanishes to within terms of second order in the δa_1 is of course immediately attributable to the fact that the normal to the surface $\alpha_2=a_2$ is tangent to the surface itself. Similarly the parameters λ_2^r of A_2, the point of intersection of (2.11), $\alpha_1=a_1$ and $\alpha_2=a_2+\delta a_2$, are found to be

$$\lambda_2^1=-\delta a_2/(\alpha_1, \alpha_2)_0, \quad \lambda_2^2=0. \tag{2.13$'$}$$

I_1 now becomes, in terms of the integration variables λ^r,

$$I_1=4\int_0^{\lambda_1^2}\int_0^{\lambda_2^1}\sum_{i=1}^{n}\frac{\partial(p_i, q^i)}{\partial(\lambda^1, \lambda^2)}d\lambda^1 d\lambda^2, \tag{2.15}$$

where the p_i, q^i in the Jacobian are defined as linear functions of λ^r by (2.11), and we have made use of the fact that to within terms of higher order in δa_r the entire integral is the fourfold of the integral over the parallelogram defined by the three vertices A, A_1 and A_2. But the value of the sum of the Jacobians occurring in (2.15) is found to be $-(\alpha_1, \alpha_2)_0$, whence

$$I_1=-4\delta a_1\delta a_2/(\alpha_1, \alpha_2)_0; \tag{2.16}$$

it is convenient, for comparison with the quantum-mechanical result (2.8), to write this classical one in the form

$$\delta a_1\cdot\delta a_2=\tfrac{1}{4}|(\alpha_1, \alpha_2)_0|\cdot|I_1| \tag{2.17}$$

in terms of absolute values.

We digress for the moment to indicate an illuminating alternative approach to Eq. (2.17), which in effect reduces the problem for any two observables α_r (whose Poisson bracket does not vanish at A) to that for two canonically conjugate ones. From this viewpoint we would first set up a linear canonical transformation T: $(p, q)\rightarrow(\bar{p}, \bar{q})$ which has as its $(\bar{p}_1, \bar{q}^1)$-plane the plane through A normal to the channel; the equations (2.11) of this plane may, on writing $\lambda^1=a\bar{q}^1$, $\lambda^2=b\bar{p}_1$ where $ab=1/(\alpha_1, \alpha_2)_0$, be most significantly interpreted as defining the correspondence induced in it by T. But to the degree of approximation employed in the above we have on the cross section

$$\alpha_1=a_1+b(\alpha_1, \alpha_2)_0\bar{p}_1, \quad \alpha_2=a_2-a(\alpha_1, \alpha_2)_0\bar{q}^1, \tag{2.18}$$

from which we obtain immediately the previous result (2.17) on noting that in the new coordinates the integral invariant $I_1=\int\int d\bar{p}_1 d\bar{q}^1=4\delta\bar{p}_1\delta\bar{q}^1$.

In making this comparison between (2.17) and (2.8) two points must be considered: first, the relationship between the Poisson bracket (α_1, α_2) at A and the mean value of the commutator $[\alpha_1, \alpha_2]$; secondly, the relationship between the sharp classical limits δa_r and the quantum-mechanical uncertainties $\Delta\alpha_r$. The first point is easily settled; if α_r on the left is the classical observable corresponding to the quantum-mechanical observable α_r on the right, then $(\alpha_1, \alpha_2)\sim(-2\pi i/h)[\alpha_1, \alpha_2]$,[8] and since to the approximation employed $(\alpha_1, \alpha_2)_0$ is the mean value of the Poisson bracket over the cross section, we may set

$$(\alpha_1, \alpha_2)_0=-(2\pi i/h)[\alpha_1, \alpha_2]_0. \tag{2.19}$$

The second point is somewhat more troublesome; we return for the moment to the simple case $\alpha_1=p$, $\alpha_2=q$ in order to examine it. Our procedure above seems most reasonably interpreted as being based on the assumption that all points within the rectangle $p_0\pm\delta p$, $q_0\pm\delta q$ have the same *a priori* probability, whereas those outside have zero probability; the standard deviations of p, q are then $\Delta p=\delta p/3^{\frac{1}{2}}$, $\Delta q=\delta q/3^{\frac{1}{2}}$, whence $\Delta p\Delta q=|I_1|/4\cdot 3$. Now the older quantum theory considered phase space as divided up into cells of area h, and we should therefore like to be able to conclude that the uncertainty principle $\Delta p\Delta q=h/4\pi$ could be interpreted *precisely* as stating that it is impossible to determine the location of a point (p, q) to within an area $|I_1|=h$; this would demand,

but an expression of the familiar fact that, on employing $\alpha(p, q)$ as the Hamiltonian function, $\alpha=$const. is an integral of the canonical equations.

[8] Reference 5, p. 96.

however, the relations $\Delta p = \delta p/\pi^{\frac{1}{2}}$, $\Delta q = \delta q/\pi^{\frac{1}{2}}$—and it is not generally held that $\pi = 3$![9] We dismiss the whole quibble by taking the relation between the δ and Δ for any observable α in the form

$$\delta\alpha = (\rho\pi)^{\frac{1}{2}}\Delta\alpha, \qquad (2.20)$$

where ρ is a factor of proportionality which our procedure seems to assign the value $3/\pi$, but which leads to a somewhat more satisfying interpretation if, leaning on the older form of quantum theory, it is given the value 1.

We are now in a position to give a precise classical interpretation of the indeterminacy relation (2.8), for on eliminating $\Delta\alpha_1$, $\Delta\alpha_2$ and $[\alpha_1, \alpha_2]_0$ with the aid of (2.20), (2.19) and (2.17) we find

$$|I_1| \geq \rho h. \qquad (2.21)$$

This highly significant inequality may now be interpreted as follows: Whereas purely classically the simultaneous measurement of two observables α_r may in theory be so performed as to constrain the representative point in phase space of lie on the intersection of $\alpha_r = a_r$, the quantum-mechanical uncertainty principle asserts that such observations can at most constrain the representative point to lie within a channel about $\alpha_r = a_r$, whose normal cross section, as measured by the value of the first integral invariant I_1 taken over it, is of order h.

3. Indeterminacy Relation for Several Observables

We consider now the extension of the relations obtained in the previous section for two observables to the case of m observables α_r $(r = 1, 2, \cdots, m)$. For a given state ψ we may define $\beta_r = \alpha_r - a_r$ as in (2.1) and construct the m-ary Hermitian form Ψ, Eq. (2.2), in which the indices r, s now run through the extended range. This form, whose coefficients are given by (2.3), is positive-definite unless there exists a linear dependence between the states $\beta_r\psi$—a possibility which we again exclude for the moment.

Now the most restrictive inequalities which we can hope to obtain on the present line of attack are clearly those which state that Ψ is positive-definite—i.e., that its determinant, together with the $m-1$ determinants obtained from it by dropping out successively the last $m-1$ rows and columns, be positive. These conditions do yield immediately inequalities of the form

$$(\Delta\alpha_1 \cdot \Delta\alpha_2 \cdots \Delta\alpha_{m'})^2 > F_{m'-2}[(\Delta\alpha_r)^2, \psi_{rs}], \qquad (3.1)$$

where $r, s = 1, 2, \cdots m'$, $m' = 1, 2, \cdots m$, $F_{m'-2}$ is a (positive) multinomial of degree $m'-2$ in the $(\Delta\alpha_r)^2$. However, these results as they stand seem to hold little interest because of the rather unmanageable multinomials, and so although some useful information concerning the product of the m uncertainties can be obtained from the inequality for $m' = m$ by a judicious use of the remaining ones, we prefer to seek the (assuredly weaker) classically interpretable extension of (2.8) by a less direct attack.

Our modified objective is to obtain from the form Ψ an inequality which asserts that the product of all m uncertainties is greater than some positive function of the mean values $[\alpha_r, \alpha_s]_0$ of the commutators alone, i.e., of the imaginary parts η_{rs} of the elements ψ_{rs}. As the first step toward its attainment we take note of an inequality which will eventually enable us to eliminate the real parts ξ_{rs} of the off-diagonal elements: by applying to the positive definite form

$$\sum_{r,s} \tfrac{1}{2}(\psi_{rs}\bar{x}_r x_s + \bar{\psi}_{rs}\bar{\bar{x}}_r\bar{x}_s) = \sum_{r,s} \xi_{rs}\bar{x}_r x_s, \qquad (3.2)$$

a theorem of Hadamard[10] which states that the product of the diagonal elements of a positive-definite quadratic or Hermitian form is not less than its determinant, we obtain the inequality

$$(\Delta\alpha_1 \cdot \Delta\alpha_2 \cdots \Delta\alpha_m)^2 \geq \det(\xi_{rs}); \qquad (3.3)$$

equality holds here only if all off-diagonal elements vanish. This reduces our problem to that of showing that the determinant of the real parts ξ_{rs} of the coefficients of the positive-definite Hermitian form Ψ is greater than some positive function of the imaginary parts η_{rs}. Now in case $m = 2k$ is even the determinant of the antisymmetric elements is the square of the

[9] See, however, W. W. R. Ball, *Mathematical Recreations and Essays*, p. 297 (London, 1914), for references to weighty empirical evidence to the contrary!

[10] For an elementary proof of this theorem see O. Szász, Math. u. Naturwiss. Ber. aus Ungarn **27**, 172 (1909).

Pfaffian function[11]

$$\mathit{ff}_k(\eta_{rs}) = \sum(-1)^p \eta_{r_1r_2}\eta_{r_3r_4}\cdots\eta_{r_{m-1}r_m}, \quad (3.4)$$

where the indices $(r_1, r_2, \cdots, r_m)$ represent a permutation of $(1, 2, \cdots, m)$ with p derangements and the sum is extended over the $1\cdot 3\cdot 5 \cdots (m-1)$ distinct products. And it can in fact be shown that under our conditions

$$\det(\xi_{rs}) > \det(\eta_{rs}) \quad (=\mathit{ff}_k{}^2), \quad (3.5)$$

equality setting in only if the matrix of Ψ is of rank k; if m is odd this relation is of course trivial, as then $\det(\eta_{rs}) = 0$, but as we shall see in the sequel we may expect a classical interpretation along the lines initiated in Section 2 above only for even m. I have not been able to find this lemma (3.5) in the literature, and have therefore presented a simple proof of it in Note A below.

On combining the two inequalities (3.3) and (3.5) thus won, we obtain the desired extension of (2.8) to the case of any *even* number $m = 2k$ of real observables α_r:

$$\Delta\alpha_1\cdot\Delta\alpha_2\cdots\Delta\alpha_{2k} \geq (1/2^k)\,|\mathit{ff}_k([\alpha_r, \alpha_s]_0)|, \quad (3.6)$$

where equality holds non-trivially only if all ξ_{rs} $(r \neq s)$ vanish and there exist exactly k linear dependences between the states $\beta_r\psi$. This result is of little interest unless the rank of the matrix $([\alpha_r, \alpha_s]_0)$ is m, as otherwise $\mathit{ff}_k = 0$; note that in particular this trivial case arises in any state ψ whatever if there exists between the α_r a relation of the form

$$\sum_r x_r \alpha_r = \alpha, \quad (3.7)$$

where α is an observable which commutes with all α_r. However, for $k > 1$, a functional dependence between the operators α_r does not suffice to make ff_k vanish; we shall have occasion to return to this point in the sequel.

It is of some interest to compare this result with the best inequality which can be obtained by the repeated application of (2.8) above. Consider the product $\Delta\alpha_1\cdot\Delta\alpha_2\cdots\Delta\alpha_{2k}$ as broken up in any way into the product of k pairs of uncertainties; corresponding to this resolution there is exactly one term in ff_k, in the sense that it is the product of the k commutators of the observables in each pair. Hence by applying (2.8) to each of these pairs we may conclude that the entire product $\Delta\alpha_1\cdots\Delta\alpha_{2k}$ is not less than $1/2^k$ times the absolute value of any term in ff_k—and so, in particular, (2.8) alone implies the inequality obtained from (3.6) on dividing the right-hand side by the number $1\cdot 3\cdot 5 \cdots (m-1)$ of distinct terms in ff_k.

[11] See, for example, G. Kowalewski, *Einführung in die Determinantentheorie*, p. 149 (Leipsig, 1909).

4. Classical Interpretation of Indeterminacy Relation for Several Observables

Just as in the above we surmised that the uncertainty principle for two observables should find its classical interpretation in terms of the first integral invariant taken over an appropriately chosen 2-dimensional surface in phase space, so in the general case of $m = 2k$ observables we may expect the classical analogue to involve the kth integral invariant

$$I_k = \frac{1}{k!}\int\int\cdots\int\int \sum_{i_1\cdots i_k} dp_{i_1}dq^{i_1}\cdots dp_{i_k}dq^{i_k} \quad (4.1)$$

taken over some m-dimensional surface invariantively associated with the m classical observables $\alpha_r(p, q)$. We have here divided by $k!$ in order to avoid the duplication of integrands in the sum corresponding to permutations of the indices $(i_1, i_2, \cdots, i_k)$; k can of course not exceed n, the number of degrees of freedom of the classical system, and the integral of maximum order

$$I_n = \int\int\cdots\int\int dp_1\,dq^1\cdots dp_n\,dq^n \quad (4.2)$$

is the Liouville measure of volume in phase space.

The development for $m = 2$ given in Section 2 clearly points the way in which the present extension is to be effected. On assuming functional independence of the m observables α_r and sufficient regularity to insure the construction, we choose any point $A(p_0, q_0)$ on the intersection of the m hyper-surfaces $\alpha_r(p, q) = a_r$ and evaluate the integral I_k over the normal plane cross section through A of the channel formed by the $2m$ surfaces $\alpha_r(p, q) = a_r \pm \delta a_r$. This normal plane is again defined by (2.11), in which the summation index r now runs from 1 to m; here we remark that the I_k so defined is again an extremum (cf. Note B). In order to determine the range of the m parameters λ^r, denote by $\lambda_t{}^r$ the parameters of the point A_t in which the m-spread (2.11) cuts the intersection of the surfaces $\alpha_1 = a_1 \cdots, \alpha_t = a_t + \delta a_t, \cdots, \alpha_m = a_m$. These parameter values are defined implicitly in terms of the δa_r by

$$\alpha_s(p_{i0} - \sum \lambda_t{}^r \alpha_{ri}, q_0{}^i + \sum \lambda_t{}^r \alpha_r{}^i) = a_s + \delta_{st}\delta a_s, \quad (4.3)$$

where $\delta_{st} = 1$ if $s = t$, 0 otherwise; retaining only terms of first order these conditions yield

$$\sum_r (\alpha_s, \alpha_r)_0 \lambda_t{}^r = \delta_{st}\delta a_s. \quad (4.4)$$

We shall in the following need only the value of the determinant of the $\lambda_t{}^r$; on evaluating the determinant of both sides of this equation we find

$$\text{ff}_k{}^2[(\alpha_r, \alpha_s)_0] \cdot \det(\lambda_t{}^r) = \delta a_1 \cdots \delta a_m, \quad (4.5)$$

where we have made use of the fact that the determinant of the antisymmetric $(\alpha_r, \alpha_s)_0$ is the square of their Pfaffian function, defined by equation (3.4).

In terms of the integration parameters λ^r

$$I_k = \frac{1}{k!}\int \cdots \int \sum_i \frac{\partial(p_{i_1}, q^{i_1}, \cdots, p_{i_k}, q^{i_k})}{\partial(\lambda^1 \cdots \lambda^{2k})} \times d\lambda^1 \cdots d\lambda^{2k}, \quad (4.6)$$

where the p_i, q^i are the linear functions of λ^r defined by (2.11). The integrand is the sum

$$\sum_{i_1 \cdots i_{2k}} \begin{vmatrix} -\alpha_{1\,i_1} & \alpha_1{}^{i_1} & \cdots & -\alpha_{1\,i_k} & \alpha_1{}^{i_k} \\ -\alpha_{2 i_1} & \alpha_2{}^{i_1} & & & \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ -\alpha_{2k\,i_1} & \alpha_{2k}{}^{i_1} & \cdots & -\alpha_{2k\,i_k} & \alpha_{2k}{}^{i_k} \end{vmatrix}, \quad (4.7)$$

and on expanding these Jacobian determinants by columns of 2-rowed minors[12] we find as the value of the sum $(-1)^k k! \text{ff}_k[(\alpha_r, \alpha_s)_0]$; hence

$$I_k = (-1)^k \text{ff}_k[(\alpha_r, \alpha_s)_0] \int \cdots \int d\lambda' \cdots d\lambda^{2k}. \quad (4.8)$$

Now in the approximation here employed the volume of the entire λ-region is 2^m times the volume of the parallelopidon whose m defining vertices A_t adjacent to the vertex at the origin have the λ-coordinates $(\lambda_t{}^1, \lambda_t{}^2, \cdots, \lambda_t{}^m)$; but this latter is measured by the determinant of the $\lambda_t{}^r$, whose value is given by (4.5). Hence

$$I_k = (-1)^k \cdot 2^m \cdot \delta a_1 \cdots \delta a_m / \text{ff}_k[(\alpha_r, \alpha_s)_0], \quad (4.9)$$

or, on taking absolute values and writing in a form suggesting comparison with (3.6),

$$\delta a_1 \cdot \delta a_2 \cdots \delta a_{2k} = (1/2^{2k}) \cdot |\text{ff}_k[(\alpha_r, \alpha_s)_0]| \cdot |I_k|. \quad (4.10)$$

Before proceeding to the interpretation of this result in the light of (3.6), we note the at first apparently discordant fact that whereas this classical construction fails if the observables α_r are even functionally dependent, the quantum-mechanical result is trivial only if there exists a linear dependence among their commutators. But our procedure in this section has been to retain only the zero and first order terms in the expansion of the functions α_r, and hence a functional dependence is for our purposes equivalent to a linear dependence of the form (3.7) which suffices to render the quantum result trivial.

On expressing the δa_r, $(\alpha_r, \alpha_s)_0$ in terms of the $\Delta\alpha_r$, $[\alpha_r, \alpha_s]_0$ by means of (2.20), (2.19) and comparing this classical result with the quantum indeterminacy relation, we find

$$|I_k| \geq (\rho h)^k. \quad (4.11)$$

This leads for $k = n$ to the familiar notion that the least physically significant volume in phase space is of order h^n. For $k < n$ it is to be interpreted as asserting that the simultaneous observation of $2k$ dynamical variables α_r can at most constrain the representative point in phase space to lie within a channel, about the intersection of the surfaces $\alpha_r = \text{const.}$, whose normal cross section is of order h^k.

[12] Cf. reference 11, p. 38.

NOTES

A. PROOF OF AN INEQUALITY

We append here a proof of the lemma employed in Section 3: *If* $\Psi : \Sigma\psi_{rs}\bar{x}_r x_s$ *is a positive-definite Hermitian form, then*

$$\det(\xi_{rs}) > \det(\eta_{rs}), \tag{A.1}$$

where $\xi_{rs}(=\xi_{sr})$ and $\eta_{rs}(=-\eta_{sr})$ are the real and imaginary parts of ψ_{rs}, the indices r, s assuming the values 1, 2, $\cdots$, $m=2k$.

Under these conditions

$$\Xi : \Sigma\xi_{rs}\bar{x}_r x_s, \qquad \mathrm{H} : \Sigma i\eta_{rs}\bar{x}_r x_s$$

are both Hermitian forms, the former of which is itself positive-definite, as shown by Eq. (3.2) above. Hence there exists an affine transformation T: $x_r \rightarrow y_p$ which simultaneously transforms Ξ into the unit Hermitian form and H into diagonal form:[13]

$$\Xi : \sum_{p=1}^{m} |y_p|^2, \qquad \mathrm{H} : \sum_{p=1}^{m} \lambda_p |y_p|^2. \tag{A.2}$$

The (real) numbers λ_p are the $m=2k$ roots of the secular equation $|\lambda\xi_{rs}-i\eta_{rs}|=0$. On interchanging rows and columns, and taking into account the symmetry properties of the ξ_{rs} and η_{rs}, this determinant becomes $|\lambda\xi_{rs}+i\eta_{rs}|=0$; hence only even powers of λ occur in its expansion

$$|\xi_{rs}|\lambda^{2k}+\cdots+(-1)^k|\eta_{rs}|=0. \tag{A.3}$$

The $2k$ real roots λ_p are therefore equal and opposite in pairs, and their product is $(-1)^k|\eta_{rs}|/|\xi_{rs}|$. On renumbering the y_p we may take $\lambda_\alpha>0$, $\lambda_{\alpha+k}=-\lambda_\alpha$, $(\alpha=1, 2, \cdots, k)$, and the original form $\Psi=\Xi+\mathrm{H}$ becomes

$$\Psi : \sum_{\alpha=1}^{k} [(1+\lambda_\alpha)|y_\alpha|^2+(1-\lambda_\alpha)|y_{k+\alpha}|^2]. \tag{A.4}$$

Since Ψ is assumed positive-definite each $\lambda_\alpha<1$, and from the above the product of all λ_p, multiplied by $(-1)^k$, is

$$|\eta_{rs}|/|\xi_{rs}|=(\lambda_1\cdot\lambda_2\cdots\lambda_k)^2<1;$$

hence $|\xi_{rs}|>|\eta_{rs}|$, q.e.d. Note that if we weaken the original assumption that Ψ is positive-definite to $\Psi\geqq 0$ for $x_r\neq 0$, then $|\xi_{rs}|\geqq|\eta_{rs}|$, where equality can set in only if the matrix (ψ_{rs}) is of rank k.

B. EXTREMAL PROPERTY OF I_k

We here show that the integral I_k defined as in Section 4, i.e., over the normal cross section (2.11) of the channel $\alpha_r(p, q)=a_r\pm\delta a_r$, is an extremum as compared with any other plane cross section through A. This we accomplish by evaluating the integral over the cross section spanned by m arbitrary vectors $(-\gamma_{ri}, \gamma_r{}^i)$ through A and showing that the conditions

$$\partial I_k/\partial\gamma_r{}^i=0, \qquad \partial I_k/\partial\gamma_{ri}=0 \tag{B.1}$$

for an extremum lead to the previous definition.

Take, then, as the new m-spread

$$p_i=p_{i0}-\sum_{r=1}^{m}\lambda^r\gamma_{ri}, \qquad q^i=q_0{}^i+\sum_{r=1}^{m}\lambda^r\gamma_r{}^i. \tag{B.2}$$

On evaluating the integral I_k over this cross section—omitting here the steps involved, as they are completely analogous to those involved in the derivation of Eqs. (4.3)–(4.9) above—we find it to be

$$I_k=(-1)^k\cdot 2^m\cdot\delta a_1\cdots\delta a_m(C)^{\frac{1}{2}}/A, \tag{B.3}$$

where C, A are the determinants whose elements are

$$c_{st}=\sum_{j=1}^{n}(\gamma_{sj}\gamma_t{}^j-\gamma_s{}^j\gamma_{tj}), \qquad a_{st}=\sum_{j=1}^{n}(\alpha_{sj}\gamma_t{}^j-\alpha_s{}^j\gamma_{tj}), \tag{B.4}$$

respectively, and we have made use of the fact that because of the skew-symmetry of c_{rs} we may write $\mathit{ff}_k(c_{rs})=(C)^{\frac{1}{2}}$. The condition for the vanishing of the derivative of I_k with respect to any of its arguments γ may now be written

$$(\partial/\partial\gamma)\log C-2(\partial/\partial\gamma)\log A=0. \tag{B.5}$$

On defining c^{st}, a^{st} as the normalized cofactor (cofactor divided by determinant) of c_{st}, a_{st} in C, A, respectively, and applying the usual rule for the differentiation of a determinant, the conditions (B.1) for an extremum become

$$\sum_{s,t}(c^{st}\partial c_{st}/\partial\gamma-2a^{st}\partial a_{st}/\partial\gamma)=0, \tag{B.6}$$

where γ represents the $2mn$ arguments $\gamma_r{}^i$, γ_{ri}. On substituting the values of c_{st}, a_{st} from (B.4) these conditions become

$$\sum_s(c^{sr}\gamma_{si}-a^{sr}\alpha_{si})=0, \qquad \sum_s(c^{sr}\gamma_s{}^i-a^{sr}\alpha_s{}^i)=0. \tag{B.7}$$

But these imply, on multiplication by c_{tr} and summation with respect to r, that the vectors $(-\gamma_{ri}, \gamma_r{}^i)$ which render I_k an extremum are obtained from the m vectors $(-\alpha_{si}, \alpha_s{}^i)$ by the non-singular linear transformation whose matrix is $(\Sigma_t c_{rt}a^{st})$, and therefore also define the normal cross section employed in Sections 2 and 4, q.e.d.

It is to be observed that we have only shown that our definition leads to an extremum; this is in fact about all of interest here that can be shown. We cannot, for example, conclude that this extremum is a minimum, although we can make the invariantive statement that it is less in absolute value than the integral over the "Euclidean" normal cross section spanned by the "vectors" $(\partial\alpha/\partial p_i, \partial\alpha/\partial q^i)$, even though this latter is not invariant under canonical transformations.

---

[13] See, for example, B. L. van der Waerden, reference 6, p. 148.