An Interpretation of e^2/mc^2 and h/mc

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In this paper a point of view is presented according to which the quantities m and e, the mass and the charge of elementary particles, need not enter into the electrodynamics and the quantum mechanics of electrons, positrons and photons. The only constants entering into the equations, rewritten in this way, are the velocity of light and two independent lengths, namely: $a=e^2/mc^2$, and $b=h/mc$. The first of these lengths determines the scale of electrodynamic phenomena, and has no especial relation to electronic radius. The second determines the scale of quantum-mechanical phenomena. In the absence of particles electromagnetic phenomena have no definite scale. This fact, together with the possibility of creation of electron-positron pairs, leads to the belief that a theory of interactions of electrons, positrons, and photons, giving as a by-product a derivation of the ratio $a/b=\alpha=e^2/hc$, could be formulated without introducing the two other pure numbers $\beta=m/M$ and $\gamma=Gm^2/e^2$. This theory is envisaged as a limiting theory, obtainable from the future general theory by putting $\beta=\gamma=0$. It is then considered from the point of view of the necessity of giving up space-time framework for the description of physical phenomena. It is concluded that the first limiting theory should not necessitate abolition of space and time.

(1)

TO the founders of electrodynamics the problem of interaction of charged bodies was, from a certain point of view, a much more complex problem than it should appear today. Faraday and Maxwell had to deal with interactions of bodies of arbitrary mass and arbitrary charge, and this arbitrariness had to appear explicitly in their equations. Accordingly, the concepts of mass and electric charge were necessary. Today, however, we believe that if we had understood and properly formulated interactions between elementary particles, interactions between bodies of any dimensions could be calculated without any, except mathematical, difficulty. Thus, from this point of view, our problem is considerably simplified.

Let us assume that the elementary particles are the electron, the positron, and the neutron (we do not regard the photon as a particle at all, for it has none of the attributes of a classical particle). Let us further suppose that the interaction between a neutron and other particles is not strictly of electromagnetic nature—leaving open, for the present, the question as to the nature of this interaction. Then we are led to the conclusion that the first problem of electrodynamics is to account for the interaction of electrons and positrons; that is, of particles all having the same mass and numerically the same charge.

If the problem, historically, had occurred in this form, it is likely that the concepts of mass and charge would not have been introduced. For mass, as we now know, arises out of comparison of accelerations of interacting bodies, and would not have occurred at all if all the bodies acquired numerically the same accelerations under the same forces. Similarly, the concept of charge became necessary only because different bodies experienced numerically different forces in the same field, or produced numerically different fields under geometrically similar conditions.

We shall now review briefly some classical formulas, and shall try to see how such formulas may be altered, when account is taken of the simplification introduced by the fact that all particles are of the same mass and of numerically the same charge. We shall find that by so doing we achieve a new interpretation of the constants e^2/mc^2 and h/mc. We shall use h to designate Planck's constant divided by 2π.

(2)

We choose, for the sake of brevity, the representation of classical formulas in the form

$$\delta\int L\,dt=0. \tag{1}$$

We disregard, as presenting no particular interest, the non-relativistic classical treatment. For a free particle in the relativistic theory $L\,dt=dW$ must be invariant. Elementary considerations lead to

$$dW = -mc\,ds = -mc^2(1-v^2/c^2)^{\frac{1}{2}}dt, \quad (2)$$

where m is an arbitrary constant. For a number of free non-interacting particles this becomes

$$dW = -\sum_k m_k c^2(1-v_k^2/c^2)^{\frac{1}{2}}dt. \quad (3)$$

From the new point of view the constants m_k, which are put in to make Lagrangians of the individual particles additive in the presence of interactions, are unnecessary because the masses are all equal. In fact, presented with this problem without a previous acquaintance with the concept of mass, we would probably write something like this:

$$dW^* = -\sum_k (1-v_k^2/c^2)^{\frac{1}{2}}d\tau, \quad (3^*)$$

where $\tau = ct$. We shall uniformly star all quantities, definitions of which differ from the old definitions.

Corresponding to this W^* the momentum of a free particle may be defined as

$$\mathbf{p}^* = \partial L^*/\partial(\mathbf{v}/c) = (\mathbf{v}/c)/(1-v^2/c^2)^{\frac{1}{2}}, \quad (4^*)$$

instead of the classical

$$\mathbf{p} = m\mathbf{v}/(1-v^2/c^2)^{\frac{1}{2}}. \quad (4)$$

For a particle moving in a given field the classical Lagrangian is:

$$L = -mc^2(1-v^2/c^2)^{\frac{1}{2}} - e\varphi + e\mathbf{A}\cdot\mathbf{v}/c, \quad (5)$$

where φ and $\mathbf{A}$ are the scalar and the vector potentials, respectively. From the new point of view there is no need of introducing a new quantity e. We can write instead

$$L^* = -(1-v^2/c^2)^{\frac{1}{2}} \pm (-\varphi^* + \mathbf{A}^*\cdot\mathbf{v}/c), \quad (5^*)$$

the sign being chosen according to the sign of the particle.[1]

Eq. (5*) leads to the equation of motion

$$\frac{d}{d\tau}\mathbf{p}^* = \frac{d}{dt}\frac{\partial L^*}{\partial \mathbf{v}} = \pm\left(\boldsymbol{E}^* + \frac{\mathbf{v}}{c}\times\boldsymbol{H}^*\right), \quad (6^*)$$

where

$$\boldsymbol{E}^* = -\nabla\varphi^* - \frac{1}{c}\frac{\partial\mathbf{A}^*}{\partial t} \quad \text{and} \quad \boldsymbol{H}^* = \nabla\times\mathbf{A}^*. \quad (7^*)$$

These are to be compared with the usual equations:

$$\frac{d}{dt}\mathbf{p} = e\boldsymbol{E} + e\frac{\mathbf{v}}{c}\times\boldsymbol{H}, \quad (6)$$

where

$$\boldsymbol{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \quad \text{and} \quad \boldsymbol{H} = \nabla\times\mathbf{A}. \quad (7)$$

Eqs. (7*) lead immediately to the first pair of Maxwell-Lorentz equations, namely:

$$\nabla\times\boldsymbol{E}^* = -\frac{1}{c}\frac{\partial\boldsymbol{H}^*}{\partial t} \quad \text{and} \quad \nabla\cdot\boldsymbol{H}^* = 0, \quad (8^*)$$

which are exactly like the corresponding classical equations. To obtain the other pair of Maxwell-Lorentz equations, it is usual to put $\delta W = 0$, with

$$W = \int\int\{-\rho\varphi + \rho\mathbf{A}\cdot\mathbf{v}/c + (1/8\pi)(E^2 - H^2)\}dVd\tau \quad (9)$$

where ρ is the charge density. This charge density is introduced in order to represent the last two terms of Eq. (5) as a volume integral. Actually, however,

$$\rho = \sum_k e_k\delta(\mathbf{r}-\mathbf{r}_k), \quad (10)$$

where $\delta(\mathbf{r}-\mathbf{r}_k) = \delta(x-x_k)\delta(y-y_k)\delta(z-z_k)$. In the new formulation we can introduce

$$\rho^* = \sum_k \pm\delta(\mathbf{r}-\mathbf{r}_k), \quad (10^*)$$

where the sign of the delta-function for each particle is chosen positive when the particle is positive, and *vice versa*. Instead of Eq. (9) we will then have:

$$W^* = \int\int\{-\rho^*\varphi^* + \rho^*\mathbf{A}^*\cdot\mathbf{v}/c + (1/8\pi a)(E^{*2} - H^{*2})\}dVd\tau. \quad (9^*)$$

As is easily seen, for example from Eq. (5*), φ^* and $\mathbf{A}^*$ are dimensionless; ρ^* is, dimensionally, the number of particles per unit volume; while $\boldsymbol{E}^*$ and $\boldsymbol{H}^*$ are of the dimension $(length)^{-1}$. Thus, the constant a in Eq. (9*) must be dimensionally

[1] We could, of course, introduce e^* which has the value $+1$ or -1, but this would still be different from introducing the constant e, the value of which must be determined experimentally.

a length. We note that it is the first and only constant that we have to introduce into our electrodynamics. This constant must, of course, be determined experimentally; but by comparison with the classical theory we know that

$$a = e^2/mc^2. \tag{11}$$

The constant c, occurring in our equations either as v/c or ct, is properly not an electrodynamic constant at all, but the limiting speed of the relativistic dynamics. Electrodynamics, of course, shows that this must also be the speed of propagation of electromagnetic waves in a vacuum.

The field equations obtained from Eq. (9*) by varying φ^* and $\mathbf{A}^*$, and putting $\delta W^* = 0$, are

$$\frac{1}{c}\frac{\partial \boldsymbol{E}^*}{\partial t} - \nabla \times \boldsymbol{H}^* = 4\pi a \rho^* (\mathbf{v}/c) \quad \text{and} \quad \nabla \cdot \boldsymbol{E}^* = 4\pi a \rho^*. \tag{12*}$$

The length a occurs here quite naturally and, obviously, without any direct connection with the electron radius. To be sure, it is here the natural unit of length with which the scale of phenomena is determined. The radius of the electron, if it could be derived on this theory, would naturally be expressed as a numerical multiple of a; but all other phenomena are also expressed in terms of a. Thus, for example, the field of a positive particle at rest would be expressed by the potential

$$\varphi^* = a/r = 1/(r/a). \tag{13*}$$

(3)

Turning now to quantum mechanics, we first have the commutation formula

$$p_j q_k - q_k p_j = (h/i)\delta_{jk} I. \tag{14}$$

This we would now write as

$$p^*_j q_k - q_k p^*_j = (b/i)\delta_{jk} I, \tag{14*}$$

where b is a new constant, to be determined experimentally. Comparison of Eqs. (4) and (4*) shows, however, that

$$\mathbf{p}^* = \mathbf{p}/mc, \tag{15}$$

so that Eqs. (14) and (14*) will be in agreement if

$$b = h/mc. \tag{16}$$

To obtain Schroedinger's equation in the new form, we first obtain the energy of a particle. This is

$$E^* = (1 + p^{*2})^{\frac{1}{2}} \pm \varphi^* = \pm \varphi^* + 1 + \tfrac{1}{2}p^{*2} + \cdots, \tag{17*}$$

where the vector potential was dropped. Taking the first approximation, and dropping the unit, which corresponds classically to dropping mc^2 from the expression for the energy, we finally have

$$E^* = \pm \varphi^* + \tfrac{1}{2}p^{*2}.$$

Corresponding to this the Schroedinger equation is

$$\nabla^2 \psi + (2/b^2)(E^* \mp \varphi^*)\psi = 0. \tag{18*}$$

Comparison of Eq. (17*) with the usual expression for the energy shows that

$$E^* = E/mc^2. \tag{19}$$

Therefore the relation between the energy and the frequency of a photon,

$$E = 2\pi h \nu, \tag{20}$$

will become

$$E^* = 2\pi h\nu/mc^2 = 2\pi b/\lambda, \tag{20*}$$

where λ is the wave-length of light.

Eqs. (14*), (18*) and (20*) show that the scale of quantum phenomena is given by the fundamental length b, the only constant of the dimension of length occurring in the equations. Here we see a confirmation of the surmise of L. L. Whyte[2] that the primary purpose of the quantity h in quantum mechanics is the introduction of a new standard of length. In fact, Whyte writes down an equation (reference 2, p. 22) equivalent to our Eq. (20*).

(4)

The fine structure constant $\alpha = a/b$ first makes its appearance when we combine electromagnetic and quantum-mechanical equations. Thus, if we substitute the potential from Eq. (13*) into Eq. (18*), choosing the lower sign in the latter, we obtain the Schroedinger equation for a

[2] L. L. Whyte, *Critique of Physics*, Norton and Co., New York (1931).

hydrogen atom,

$$\nabla^2\psi+(2/b^2)(E^*+a/r)\psi=0. \qquad (21^*)$$

The eigen values of this equation are

$$E^*=-(1/2n^2)(a/b)^2=-\alpha^2/2n^2, \qquad (22^*)$$

where n is the quantum number. The scale of the distribution of the probability of position of the electron turns out to depend upon

$$r_0=b^2/a=b/\alpha, \qquad (23)$$

where r_0 is the radius of the first Bohr's orbit for the hydrogen atom. Finally, Rydberg's constant, expressed as a wave-length, is here expressed as

$$R_\lambda=4\pi b^3/a^2=4\pi b/\alpha^2. \qquad (24)$$

If we should now decide to use either a or b as the fundamental unit of length, expressing the space coordinates and ct in terms of this unit, there would be no dimensional constants in our theory at all; while the only dimensionless constant occurring in the theory would be α. Such a description of physical phenomena seems to be what was intended in the *item 4* of Whyte's program for research (reference 2, p. 144).

It is, of course, not suggested that the above discussion is in any sense a part of the structure of the future theory. The whole of this paper is rather in the nature of an attempt at a preliminary clearing of the ground for the future edifice.

(5)

The duality of electromagnetic phenomena expresses itself through the appearance of *two* independent standards of length. Is it too much to hope for that the next step in the development of electrodynamics, as it is here defined, should give us a theory explaining their ratio? I think not.

It may be urged that we don't know that α is *not* a function of the other two independent pure numbers appearing in our theories, namely: $\beta=m/M$ and $\gamma=Gm^2/e^2$, where M is the mass of the neutron and G the gravitational constant. My belief that these other numbers are irrelevant to the kind of a theory I have in mind is based on the following considerations: As can be seen from Eq. (9*), in the absence of particles the constant a is unnecessary, and in fact does not occur in the field equations. Thus, the only scale of length relevant to a correct description of pure radiation would seem to be the constant b occurring in Eq. (20*). Consider, now, creation of a pair electron+positron by a collision of two photons, a process now generally admitted to be possible. Such a pair immediately re-introduces the constant a. It seems to me that this is a strong evidence for supposing that this constant, and therefore α, had pre-existed in the original radiation. In other words, it seems that there must be something existing in the structure of radiation which determines the charge of the electron that may be created.

It is, of course, not categorically asserted that gravitation, for instance, can have nothing to do with the phenomenon of a pair creation. But it seems to me that, if it has any effect at all, it will be to introduce a small correction to the value of α; possibly a correction proportional to some power of the small constant γ. I believe, therefore, that a theoretical derivation of at least a good first approximation to α should be one of the by-products of the next important step in the development of quantum electrodynamics.

The theory, a possibility of which is here discussed, must naturally be something vastly different from numerous attempts, recently in vogue, at so manipulating known material as to make a combination approximately equal to the experimental value of a physical constant. The three fundamental aspects of this theory should be: first, an explanation of the way in which electrons and positrons interact, one that would make it clear that all the requirements of the theory can be satisfied only if the interaction corresponds to $e^2=hc\alpha$; second, an explanation of the phenomenon of quantization, making clear that the theory demands that $h=e^2/\alpha c$; and, finally, an explanation of the nature of radiation, explaining why the limiting velocity of relativistic dynamics should be $c=e^2/\alpha h$. As was pointed out by Bohr, at the theoretical conference in Copenhagen last year, it is obviously impossible, starting with a theory valid for all possible values of the constants occurring in it, to derive any relation between such constants. It is for this reason that the relation $a/b=\alpha$, with a particular numerical value of α, must be a *necessary condition* for the validity of the theory.

(6)

We may hope that, ultimately, a theory will be formulated in which all the three independent pure numbers, α, β and γ will find their proper explanation. I hold it unlikely, however, that such a theory could be conceived at once. It seems to me much more likely that the progress will be through two intermediate limiting theories. The first such theory, the theory discussed above, would correspond to the limiting case $\beta=\gamma=0$; or $M=\infty$ and $G=0$. The second limiting theory, it seems likely, will be the theory with β finite, but $\gamma=0$. And finally would come the general theory.

Now, it has been often pointed out that without a possibility of constructing a measuring means, with which a coordinate system can be mapped out with any desired degree of accuracy, space and time, and therefore geometry, lose their significance; that, becoming vague, they had best be given up.[3] While entirely in agreement with such general conclusions, I wish to call attention to the fact that, on the basis of the point of view here presented, the necessity of giving up a space-time framework may not be as imminent as it may seem.

Thus, in constructing our first limiting theory, we put $M=\infty$. But this means that all the quantities, $h/M\Delta v$, h/Mc, and e^2/Mc^2, which control the accuracy of locating nuclei within crystal lattices, become zeros in this theory. This means that within the first limiting theory there is no inherent need for giving up space-time description. So far, at least, the various uncertainties connected with the electron may be ascribed to the electron, and not to the space-time framework.

[3] As the latest, and one of the best, examples of this kind see E. Schroedinger, Naturwiss. **22**, 518 (1934).

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Approximate Wave Functions and Atomic Field for Mercury

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Approximate wave functions and atomic field for mercury have been calculated by the method of the self-consistent field, and tables of preliminary results are given. A considerable amount of the calculations was carried out on the differential analyzer, and a short account of the application of the differential analyzer to this work is given.

§1. Introduction

The heaviest atom for which calculations of the approximate atomic field and wave functions by the method of the "self-consistent field"[1] has so far been completed is Cs. During the past year, calculations have been in progress for Hg, and though the approximation to the self-consistent field has not yet been carried far enough for the results to be considered as final, it seems desirable to publish the results so far obtained, as they may provide a good enough approximation for some purposes, and it may be some time before better results are available.

The amount of work involved in the determination of the self-consistent field of Hg is very considerable, not only on account of the number of one-electron wave functions to be determined, but also because of the sensitiveness of the $(5d)^{10}$ group, and to a less extent of the $(4f)^{14}$ group, to a change in the estimate of the atomic field. The $(5d)^{10}$ group shows in rather a pronounced way the phenomenon of "overstability" already noted[2] for the $(3d)^{10}$ group of Cu^+, and the $(4f)^{14}$ group, though not overstable, is inconveniently sensitive.

[1] D. R. Hartree, Proc. Camb. Phil. Soc. **24**, 89, 111 (1928); Proc. Roy. Soc. **A141**, 282 (1933); **143**, 506 (1934). The Proc. Camb. Phil. Soc. papers will be referred to as I, and the Proc. Roy. Soc. papers as II and III, respectively.

[2] See II, pp. 287, 295.