Nuclear Energy Levels and the Model of a Potential Hole

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Part I contains the solution of the problem of a particle for which the potential energy $V(r)$ is a simple rectangular hole of finite depth, such a model being of interest in connection with the spacing of nuclear energy levels. Formulae are given by which a numerical determination of energies can be carried out for any assumed depth of the hole. In part II the results are extended to finite holes of different shape, and an argument is presented which proves the inadequacy of any such model to explain the arrangement of α-ray levels in Ra C′.

To correlate the empirical data concerning atomic nuclei with the theoretical properties of assumed nuclear models becomes a problem of increasing importance as the amount of experimental material grows. The present note is devoted to one aspect of this problem which permits a clear solution, insofar as it establishes the impossibility of accounting for the observed positions of α-particle energies by means of simple hypotheses regarding the nuclear potential, which are so fruitful in explaining α-ray decay.

Rutherford and his collaborators[1] have determined the positions of the nuclear levels in Ra C′ with a precision which permits an application of analysis. Gamow[2] was the first to attempt an explanation of their arrangement on the basis of a nuclear model consisting of an infinitely deep potential hole within which the α-particle moves. His analysis is intended to be only qualitative but shows at first sight some promising features.

There are two respects in which the model used by Gamow may be refined: (1) the hole should be taken of finite depth, (2) its walls should not be vertical. In the first part of this note we shall incorporate into the theory the first refinement and work out the arrangement of the energy levels of a particle for which the potential energy $V(r)$ is a simple rectangular hole of finite depth (a problem which has perhaps some mathematical interest independently of its connection with nuclear theory). In the second part we shall consider the other refinement and discuss the conclusions regarding nuclear levels to which this theory leads.

[1] Rutherford, Proc. Roy. Soc. **A131**, 684 (1931). Rutherford, Lewis and Bowden, Proc. Roy. Soc. **A142**, 347 (1933).
[2] Gamow, Nature **131**, 433 (1933); Cf. also Solvay Congress (1933).

I

Our present problem is simply to solve the Schrödinger equation for a potential energy $V(r)$ given by the solid line in Fig. 1. The Coulomb potential outside of this hole, while producing an appreciable width of the energy levels, is not essential in determining their positions and need not be considered in this connection.

Let us first recall the position of the levels inside a hole of *infinite* depth. If we reckon the energy W of the particle from the bottom of the hole and define $k^2 \equiv 8\pi^2 MW/h^2$, W being the reduced mass of particle and nucleus, then the solution of Schrödinger's equation, multiplied by r, is

$$P(r) = cr^{\frac{1}{2}} J_{l+\frac{1}{2}}(kr),$$

and k must be determined in such a way that

$$J_{l+\frac{1}{2}}(kr_0) = 0, \qquad (1)$$

where r_0 is the radius of the hole. In this expression, l is the usual azimuthal quantum number.

Thus is produced an infinite set of levels, the position of which depends on the values of M and r_0. In Table I we list the first few values of

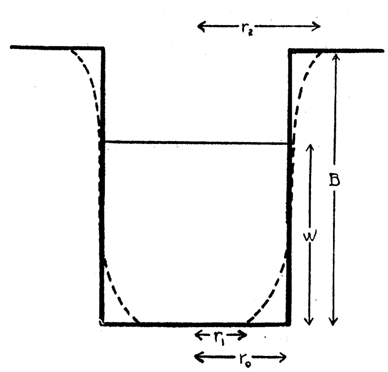

FIG. 1.

TABLE I. *Values of* Wr_0^2.

l	$Wr_0^2\cdot 10^{19}$			
0	5.081 (1)	20.32 (4)	45.73 (10)	81.29 (17)
1	10.39 (2)	30.72 (6)	61.21 (13)	101.74
2	17.10 (3)	42.58 (8)	78.18 (16)	
3	25.15 (5)	55.85 (11)	96.59	
4	34.45 (7)	70.54 (15)	116.44	
5	45.2 (9)			
6	57.0 (12)			

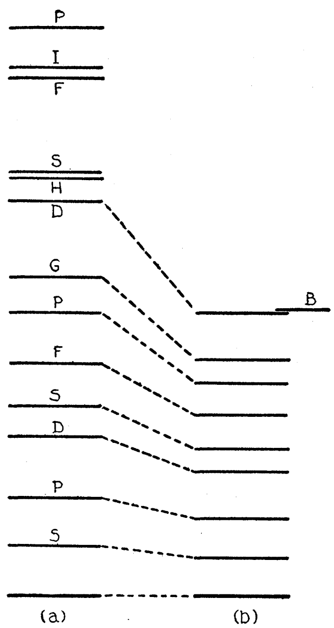

FIG. 2. (a) Arrangement of levels in an infinitely deep hole with spectroscopic designation of l-values. (b) Arrangement of levels in a hole of depth B.

$W\cdot r_0^2$, in electron-volts$\times$cm^2, taking for M the mass of an alpha-particle. The parentheses contain the ordinal numbers of the levels from the lowest upward. The corresponding levels are plotted in order in Fig. 2a. They have already been drawn by Gamow.[2]

Let us now determine what happens if the depth of the hole, B, is *finite*. If then we define $\kappa^2 \equiv 8\pi^2 M(B-W)/h^2$, we obtain for the solution inside the hole $P_i = c_1 r^{\frac{1}{2}} J_{l+\frac{1}{2}}(kr)$, and outside $P_0 = c_2 r^{\frac{1}{2}} H_{l+\frac{1}{2}}(i\kappa r)$. Here H is the Hankel function which vanishes for large positive imaginary values of the argument. These solutions, as well as their first derivatives, must be continuous at r_0, a circumstance which leads to the relation

$$\frac{J_{l+\frac{3}{2}}(kr_0)}{J_{l-\frac{1}{2}}(kr_0)} = \frac{H_{l+\frac{3}{2}}(i\kappa r_0)}{H_{l-\frac{1}{2}}(i\kappa r_0)}. \tag{2}$$

This equation is to be used for determining k.

The right hand side of (2), for which we shall write $R_l(\kappa r_0)$, can be expanded as a quotient of finite sums. Thus one finds

$$\begin{aligned}
-R_0(x) &= 1+x^{-1},\\
-R_1(x) &= 1+3x^{-1}+3x^{-2},\\
-R_2(x) &= \frac{1+6x^{-1}+15x^{-2}+15x^{-3}}{1+x^{-1}},\\
-R_3(x) &= \frac{1+10x^{-1}+45x^{-2}+105x^{-3}+105x^{-4}}{1+3x^{-1}+3x^{-2}},\\
-R_4(x) &= \frac{1+15x^{-1}+105x^{-2}+420x^{-3}+945x^{-4}+945x^{-5}}{1+6x^{-1}+15x^{-2}+15x^{-3}}.
\end{aligned} \tag{3}$$

If the barrier is infinitely high, then according to (3) $\kappa\to\infty$, and every $R(\kappa r_0)$ takes on the value -1. In that case, Eq. (2) states: $J_{l+\frac{3}{2}}+J_{l-\frac{1}{2}}=0$. But this is identical with $J_{l+\frac{1}{2}}(kr_0)=0$. Thus we have arrived at condition (1).

Let us now put $kr_0 = (8\pi^2 MW/h^2)^{\frac{1}{2}} r_0 \equiv x$, $(8\pi^2 BM/h^2)^{\frac{1}{2}} r_0 \equiv b$, so that (2) finally takes the form

$$J_{l+\frac{3}{2}}(x)/J_{l-\frac{1}{2}}(x) = R_l(b^2-x^2)^{\frac{1}{2}}. \tag{4}$$

The solutions x of this equation determine the permissible W's. They can be obtained with little labor by plotting the R's (cf. (3)) as well as the ratios of the Bessel functions. The energies, in electron-volts, are then given by

$$5.148\times 10^{-20} x^2/r_0^2.$$

It is easily seen that the finite height of the barrier depresses all the levels of Table I, as might be expected. One can find the maximum depression of any given level by the following consideration.

For the highest level, i.e., the one most strongly depressed, x will be but slightly smaller than b. The depression is a maximum when $b=x$. In this case the right hand side of (4) is $R_l(0)$, which tends toward $-\infty$ for every l. Hence this relation is satisfied when $J_{l-\frac{1}{2}}(x)=0$. Now the root of $J_{l-\frac{1}{2}}$ determines the energy in an infinitely deep hole, but of a level with an l-value smaller by 1 than that under consideration. Therefore the maximum displacement of a level is from its value in Table I to the value immediately above it in the same column of the table. By virtue of this fact one can circumvent actual calculations when a mere estimate of the effect of finite depth upon the "normal" position of the energy levels is desired, since it permits a determination of the quantum number of the *highest* energy state as well as its approximate position. To show the effect of finite depth of the hole upon the "normal" arrangement of the levels (Fig. 2a) we have calculated graphically the roots of 4, taking for B a value which will produce a maximum depression of the levels assumed by Gamow[3] for Ra C′. The results are plotted in Fig. 2b. All levels above B merge into a continuum. Comparison with Gamow's postulated level scheme shows that, to be sure, the finite depth produces the desired compression, but there is no strong crowding of the upper levels as in Gamow's diagram. This example is now merely hypothetical, however, since Gamow's assignment has been superseded by the recent determination of the levels due to Rutherford, Lewis and Bowden.

II

The considerations of part I will be helpful as we turn to the next question: what precise form of the potential walls will give the arrangement of levels required experimentally? This can of course only be found by trying various potential energy curves, such as the dotted one in Fig. 1, with the known levels in a finite rectangular hole serving as a guide. Given the dotted curve, one can simply start with the eigenvalue and eigenfunction calculated in part I, integrate the latter numerically from r_1 to r_0 and from r_2 to r_0, and see what change in the eigenvalue is necessary to make the solutions join. In doing this one is readily convinced that *very implausible* potential energy curves are required to produce the spacing of the levels proposed either by Gamow or by Rutherford, Lewis and Bowden. The major discrepancy which one encounters appears plainly from the following argument.

Suppose the energy difference between the lowest and, say, the nth level to be known. We can then, by choosing r_0 properly, construct a finite rectangular hole which will produce this energy difference. To obtain the minimum value of r_0 compatible with the given energy difference we will agree to place the barrier height B just above the position of the nth level. It is then at once evident (and may be shown rigorously) that, if the true potential hole differs at all from the assumed rectangular hole but still produces the same energy difference between the lowest and the nth level, its radius must be smaller than r_0 near the bottom and greater than r_0 near the top. Hence, if r is the radius of the actual hole near the top, $r > r_0$.

Let us now consider Gamow's level scheme. It assigns to the energy difference between the lowest and the 8th level the value 17.8×10^5 e. v. According to Table I this difference is $37.5 \times 10^{-19} \times r_0^{-2}$ e. v. if the hole is infinitely deep, $25.64 \times 10^{-19} r_0^{-2}$ e. v. if the hole has its minimum possible depth. In the first case, $r_0 = 1.45 \times 10^{-12}$ cm, in the second, $r_0 = 1.2 \times 10^{-12}$. Hence we are led to suppose that, even with the most favorable choice of B,

$$r > 1.2 \times 10^{-12} \text{ cm}.$$

This is hopelessly irreconcilable with the value[4] 8.2×10^{-13} cm $< r < 8.6 \times 10^{-13}$ cm which is required to obtain the correct decay constant for Ra C′.

The situation is no less unsatisfactory if we assume the assignment of levels given by Rutherford, Lewis and Bowden. Here the difference between the lowest and the 16th level is 27×10^5 e. v. This, according to Table I, is at least $61.2 \times 10^{-19} r_0^{-2}$ e.v. Thus $r_0 \sim 1.5 \times 10^{-12}$ cm, and

$$r > 1.5 \times 10^{-12} \text{ cm}.$$

In view of these considerations we must regard the model of a potential hole as inadequate to explain the distribution of α-particle energies within nuclei. The α-particle can not be said to move essentially in a potential field due to the other nuclear constituents; its potential appears to depend strongly on its own state of motion, in such a manner that, if the α-particle changes its energy, the entire nuclear configuration is altered.

[3] Gamow, Nature **131**, 433 (1933).

[4] Cf. Mott, Handb. d. Physik **XXIV**, 1, p. 807.