Approximate Wave Functions for High Energy Electrons in Coulomb Fields

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In this paper we construct approximate solutions of the Dirac equations for an electron in a Coulomb field of force, which reduce at large distances to plane waves. For very high energy electrons these functions satisfy the wave equations to terms of the orders $(\alpha^2/r)\sin^2\theta/2$, $(\alpha^2/r)\sin\theta/2$. Thus they are everywhere at least as accurate as the functions obtained by first order approximation methods, and for $\theta\to 0$ they satisfy the equations exactly. They therefore give an improved method for treating problems in which high energy electrons are ejected in a narrow angular range. The functions are simple in form and quite similar to the Gordon-Temple solutions of the Schrödinger equation in parabolic coordinates.

I. INTRODUCTION

PROBLEMS in which an electron moving in a Coulomb field makes transitions to or between states of very high energy present great difficulties when one attempts their solution with the use of exact Dirac wave functions for the high energy states of the electron. These difficulties have been pointed out by Hall and Oppenheimer[1] in the case of the photoelectric effect; they occur also in the problems of production of high energy electron pairs by γ-rays or electrons, of the radiative impacts of electrons with nuclei and of the scattering of very hard radiation by bound electrons.

As remarked by Hall and Oppenheimer,[1] the difficulties result from the fact that the system of spherical coordinates, in which alone exact Dirac wave functions are available, is not at all adapted to the conditions of these problems. This is so because the high energy wave packets with which one has to deal are concentrated into very small angles and could have a simple description only in terms of wave functions having essentially the character of plane waves. Their resolution in terms of spherical waves requires the use of spherical harmonics of very high order, the order becoming infinite with the energy of the electron under consideration. This means that in calculating transition probabilities one is concerned with a sum of squares of matrix elements in which a great many terms contribute appreciably; all these terms refer to very large values of the angular momentum quantum number and the evaluation of any single term is extremely difficult.

A knowledge of the predictions of quantum electrodynamics regarding such processes is highly desirable in the present state of experiment and theory. Such information has usually been sought by the use of the plane wave functions of a high energy free electron, the Coulomb potential of the nucleus being taken into account by the Born approximation method in case the binding of the high energy electron is essential to the occurrence of the effect.[2] This procedure avoids the use of an unfortunate coordinate system and makes the calculation reasonably manageable in most cases. On the other hand, its approximate nature makes it necessary to consider carefully the limitations beyond which we can no longer have complete confidence in the correctness of the results.

The Coulomb binding of the electron is characterized by a certain length, the Bohr radius $a=h^2/(4\pi^2 mZe^2)$. This length becomes infinite for vanishing nuclear charge Z; and the ratios of other lengths appearing in the problem to the length a are the parameters characterizing the effect of the binding. It seems reasonable to suppose that when these are small the approximate procedure just described gives correct results. Now the lengths occurring are first of all the de Broglie wave-lengths of the electron in its

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[1] Harvey Hall and J. R. Oppenheimer, Phys. Rev. **38**, 57 (1931); Harvey Hall, Phys. Rev. **45**, 620 (1934).

[2] The plane wave calculation for the photoelectric effect is described in the first paper of reference 1. Correctly carried through, it leads to the result first obtained by F. Sauter (Ann. d. Physik **11**, 454 (1931)) using a different approximate method. Definitive results of the Born approximation calculations for production of pairs by γ-rays and for radiative impacts of fast electrons with nuclei were first obtained by Heitler and Sauter; for preliminary notice see Nature **132**, 892 (1933).

different states and the wave-length of the light quantum; and for high energies these are small. But one must consider also the de Broglie wave-lengths corresponding to differences of the various momenta, in particular to the momentum imparted to the nucleus. At high energies, $h\nu \gg mc^2$ for the light quantum, this gives the parameter

$$\lambda \sim \alpha(1-\alpha^2)^{-\frac{1}{2}}, \quad \alpha = 2\pi Ze^2/hc \sim Z/137$$

for the photoelectric effect and

$$\lambda \sim \alpha(h\nu/2mc^2)$$

for pair production and radiative impacts.[3] Thus even for light elements we cannot be sure that the results of the Born approximation treatment of these latter problems are right for very large $h\nu/mc^2$. At such large energies important contributions to the matrix integrals come from the region of space straight forward from the nucleus; and in this region within and near the caustic of the classical trajectories[4] the Born wave functions become completely unreliable.

In the present paper we shall construct approximate Dirac wave functions for a high energy electron in a Coulomb field which, like the Born functions, have a type of symmetry adapted to the problems mentioned above and which are superior to the Born functions in just the respect which seems essential for the obtaining of more reliable answers to these problems; namely, they satisfy the Dirac equations everywhere in space to the first power of α and in the region of very small angles—within and near the caustic—they satisfy the equations to a much higher degree of approximation.

The construction of these functions follows closely the procedure which Gordon used originally to obtain the nonrelativistic wave function in parabolic coordinates.[5] This he did by forming a suitable linear combination of the spherical coordinate solutions. When one starts with normalized functions and chooses the coefficients so that a normalized function results, this procedure is essentially the application of a unitary transformation; it suffices to carry through explicitly the application of one row of the transformation matrix, which gives the linear combination used by Gordon. In the relativistic case there exists a similar unitary transformation which takes us from the spherical coordinate wave functions of a free electron to the plane wave functions. But the application of this transformation to the spherical wave functions of an electron in a Coulomb field leads to series which have no simple analytical expressions. To obtain the desired result we must resort to approximations. The approximations in question consist in the neglect of terms of the order α^2/l^2 compared to those kept and should not impair the usefulness of the results for problems in which only the contributions from large values of l need to be treated accurately.

II. Construction of the Functions

In our calculations we shall make exclusive use of the system of units which is best adapted to the problems of the high energy electron. Energies are measured in units mc^2, momenta in units mc and lengths in units $h/(2\pi mc)$, where m is the rest mass of the electron. Equations in these natural units may be obtained from those in c.g.s. units by setting $h/2\pi$, m and c equal to unity and e, the charge of the electron, equal to the square root of the fine structure constant; the return to c.g.s. units may readily be made by dimensional considerations. We use α to denote Z times the fine structure constant and write ϵ and k, respectively, for the absolute values of the energy and momentum of the electron when infinitely distant from the nucleus. Thus $k=(\epsilon^2-1)^{\frac{1}{2}}$.

We take as our starting point the spherical coordinate wave functions in the form given by Hulme,[6] writing l for the quantum number which he calls k. In our present units and notation, the values of some of Hulme's symbols are:

$$|B|/A = k/(\epsilon+1); \quad a=k; \quad b=\epsilon\alpha/k; \quad c=\alpha/k.$$

[3] By similar considerations one can determine the possible effects of screening on the results. Since the length characteristic of the screening is $Z^{2/3}a$, screening can scarcely come into account for the photoelectric effect, and for the pair production and radiative impacts it can be important only at energies much higher than that at which the validity of the Born approximation results first becomes open to question.

[4] Cf. first part of Gordon's paper (reference 5).

[5] W. Gordon, Zeits. f. Physik **48**, 180 (1928).

[6] H. R. Hulme, Proc. Roy. Soc. **A138**, 643 (1932).

Taking the quantum number u equal to 0, we have, apart from a constant factor in each solution,

$$\begin{aligned} \psi_{1l}&=ik(l+1)P_{l+1}F_l, & \psi_{3l}&=-(\epsilon+1)(l+1)P_lG_l,\\ \psi_{2l}&=ikP^1{}_{l+1}e^{i\varphi}F_l, & \psi_{4l}&=(\epsilon+1)P_l{}^1e^{i\varphi}G_l \end{aligned} \tag{1}$$

and

$$\begin{aligned} \psi_{1l}'&=iklP_{l-1}F_{-l-1}, & \psi_{3l}'&=-(\epsilon+1)lP_lG_{-l-1},\\ \psi_{2l}'&=-ikP^1{}_{l-1}F_{-l-1}, & \psi_{4l}'&=-(\epsilon+1)P_l{}^1e^{i\varphi}G_{-l-1}. \end{aligned} \tag{2}$$

The $P_l{}^m$ are Legendre functions of $x=\cos\theta$; we use the ordinary unnormalized functions, not those defined by Darwin which Hulme uses. F_l, G_l, F_{-l-1}, G_{-l-1} are real and are defined by

$$\begin{gathered} \boldsymbol{G}_l=F_l+iG_l, \qquad \boldsymbol{G}_{-l-1}=F_{-l-1}+iG_{-l-1},\\ \boldsymbol{G}_l=[(l-s)-i\alpha(\epsilon+1)/k]\rho^s\int_{-1}^{1}(1-u)^{s-ib+1}(1+u)^{s+ib}e^{i\rho u}du, \qquad (3)\\ \boldsymbol{G}_{-l-1}=[\alpha(\epsilon-1)/k+i(l-1-s')]\rho^{s'}\int_{-}^{1}(1-u)^{s'-ib+1}(1+u)^{s'+ib}e^{i\rho u}du, \end{gathered}$$

where $\rho=kr$ and, as in Hulme's notation,

$$s=\{(l+1)^2-\alpha^2\}^{\frac{1}{2}}-1, \quad s'=\{l^2-\alpha^2\}^{\frac{1}{2}}-1.$$

These functions are, apart from notation and constant factors, just those of Hulme and are exact solutions of the Dirac equations in spherical coordinates. We now proceed to simplify these solutions by neglecting consistently terms of orders $(1/\epsilon^2)$ or (α^2/l^2) compared to those kept. Then apart from constant factors:

$$\begin{gathered} \boldsymbol{G}_l=\{-i+(\alpha/2)(1-\epsilon^{-1})/(l+1)\}\rho^l(1+i(d/d\rho))S_l(\rho),\\ \boldsymbol{G}_{-l-1}=\{1+i(\alpha/2)(1+\epsilon^{-1})/l\}\rho^{l-1}(1+i(d/d\rho))S_{l-1}(\rho) \qquad (3')\\ S_l(\rho)=S_l{}^*(\rho)=e^{i\rho}F(l+1-i\alpha,\ 2l+2,\ -2i\rho). \end{gathered}$$

with

Write $\rho^lS_l(\rho)=R_l(\rho)$. Then from the recurrence formulae[7]

$$\begin{aligned} &bF(a,\ b,\ x)=(b-a)F(a,\ b+1,\ x)+aF(a+1,\ b+1,\ x),\\ &xF(a,\ b,\ x)=-(b-a)F(a-1,\ b,\ x)+(b-2a)F(a,\ b,\ x)+aF(a+1,\ b,\ x) \end{aligned}$$

we find that

$$\rho^lS_l'(\rho)=-(2l+3)^{-1}\{1+\alpha^2/(l+1)^2\}R_{l+1}-\alpha(l+1)^{-1}R_l.$$

For our present purposes the term with $\alpha^2/(l+1)^2$ is to be neglected.

If now we insert the approximate values of F_l, G_l, F_{-l-1}, G_{-l-1} so found into (1) and (2) and also multiply the two approximate solutions by the respective factors

$$2^l(k/2\pi)e^{\pi\alpha/2}\Gamma(l+1-i\alpha)\{(2l+1)\,!\}^{-1}\{\epsilon(\epsilon+1)(l+1)\}^{-\frac{1}{2}}\{1-i(\alpha/2)(1-\epsilon^{-1})(l+1)^{-1}\}$$

and

$$2^l(k/2\pi)e^{\pi\alpha/2}\Gamma(l+1-i\alpha)\{(2l)\,!\}^{-1}\{\epsilon(\epsilon+1)l\}^{-\frac{1}{2}}\{1+i(\alpha/2l)(1-\epsilon^{-1})\};$$

neglecting small terms as before, we get

$$\begin{aligned} \psi_{1l}&=-NP_{l+1}\{ik(l+1)K_{l+1}+i(\alpha/2)(\epsilon+1)(K_l+iK_{l+1})\},\\ \psi_{2l}&=-N(l+1)^{-1}P^1{}_{l+1}e^{i\varphi}\{\text{same as for } \psi_{1l}\},\\ \psi_{3l}&=NP_l\{(\epsilon+1)(l+1)K_l-i(\alpha/2)k(K_l+iK_{l+1})\},\\ \psi_{4l}&=-N(l+1)^{-1}P_l{}^1e^{i\varphi}\{\text{same as for } \psi_{3l}\} \end{aligned} \tag{1'}$$

[7] Derived, in a different notation, by P. S. Epstein, Phys. Rev. **28**, 695 (1926).

and

$$\begin{aligned}
\psi_{1l}' &= N'P_{l-1}\{iklK_{l-1}+i(\alpha/2)(\epsilon+1)(K_l-iK_{l-1})\},\\
\psi_{2l}' &= -N'l^{-1}P^1{}_{l-1}e^{i\varphi}\{\text{same as for } \psi_{1l}'\},\\
\psi_{3l}' &= N'P_l\{(\epsilon+1)lK_l+i(\alpha/2)k(K_l-iK_{l-1})\},\\
\psi_{4l}' &= N'l^{-1}P_l{}^1e^{i\varphi}\ \{\text{same as for } \psi_{3l}'\},
\end{aligned} \tag{2'}$$

where $N=\{8\pi\epsilon(\epsilon+1)(l+1)\}^{-\frac{1}{2}}$, $\quad N'=\{8\pi\epsilon(\epsilon+1)l\}^{-\frac{1}{2}}$

and

$$K_l=2^l(2/\pi)^{\frac{1}{2}}ke^{\pi\alpha/2}\Gamma(l+1-i\alpha)\{(2l+1)\,!\}^{-1}R_l(\rho).$$

Here K_l is a radial function normalized in the k-scale.[5] For $\alpha=0$, (1′) and (2′) become correct wave functions normalized in the k-scale. For $\alpha\neq 0$, their normalization is inaccurate by factors of the form $1+0(\alpha^2/l^2)$; this is the sort of inaccuracy which they contain in other respects, also, and which we are neglecting.

We now proceed to form a linear combination of the approximate solutions (1′) and (2′), using coefficients which in the case $\alpha=0$ produce normalized plane wave solutions out of normalized spherical wave solutions and which therefore form a row of a unitary transformation matrix. These coefficients are, respectively:

$$C_l=i^lk^{-1}(4\pi)^{-\frac{1}{2}}(l+1)^{\frac{1}{2}} \quad \text{and} \quad C_l'=i^lk^{-1}(4\pi)^{-\frac{1}{2}}l^{\frac{1}{2}}.$$

Setting[8]

$$\varphi=\sum_{l=0}^{\infty}C_l\psi_l+\sum_{l=1}^{\infty}C_l'\psi_l',$$

we have

$$\begin{aligned}
\varphi_1 &= -N''\{kS-i(\alpha/2)(\epsilon+1)S'\}, & \varphi_3 &= N''\{(\epsilon+1)S-i(\alpha/2)kS'\},\\
\varphi_2 &= -N''i(\alpha/2)(\epsilon+1)S''e^{i\varphi}, & \varphi_4 &= N''i(\alpha/2)kS''e^{i\varphi},
\end{aligned}$$

where

$$N''=(2\pi)^{-\frac{1}{2}}\{2\epsilon(\epsilon+1)\}^{-\frac{1}{2}},$$

$$S=(\pi/2)^{\frac{1}{2}}k^{-1}\sum_{l=0}^{\infty}i^l(2l+1)K_lP_l,$$

$$S'=(\pi/2)^{\frac{1}{2}}k^{-1}\{\sum_{l=0}^{\infty}i^{l+1}P_l(K_{l+1}+K_{l-1})+K_0-iK_{-1}\},$$

$$S''=(\pi/2)^{\frac{1}{2}}k^{-1}\{(1+x)\sum_{l=1}^{\infty}i^l(2l+1)l^{-1}(l+1)^{-1}P_l{}^1K_l+P_1{}^1K_0\}.$$

K_{-1} has not yet been defined and its value is arbitrary. In deriving the form given for S'' we have used the recurrence formula

$$lP^1{}_{l+1}(x)+(l+1)P^1{}_{l-1}(x)=(2l+1)xP_l{}^1(x),$$

which holds for $l\geq 1$ if we set $P_0{}^1=0$.

For S Gordon obtained the value[5]

$$S=e^{\pi\alpha/2}\Gamma(1-i\alpha)e^{i\rho x}F(i\alpha,\,1,\,i\rho(1-x)).$$

To evaluate S' and S'' we must use Gordon's relation[5, 9]

$$K_l=(k/2\pi i)e^{3\pi\alpha/2}\Gamma(1-i\alpha)\oint e^{i\rho s}[\rho(1-s)]^{-\frac{1}{2}}J_{l+\frac{1}{2}}[\rho(1-s)]s^{i\alpha-1}(1-s)^{-i\alpha}ds.$$

[8] This summation is extended over the ranges of l for which the solutions are valid. Terms containing $P_0{}^1$ are to be regarded as vanishing. Cf. reference 12.

[9] The contour is as specified in reference 5.

Letting this serve to define K_{-1}, we have $K_{-1}=-iK_0$. The evaluation of S' and S'' thus depends on evaluating the sums

$$\bar{S}'=(\pi/2)^{\frac{1}{2}}\sum_{l=0}^{\infty} i^{l+1}\cdot\sigma^{-\frac{1}{2}}[J_{l+\frac{1}{2}}(\sigma)+J_{l-\frac{1}{2}}(\sigma)]P_l(x),$$

$$\bar{S}''=(\pi/2)^{\frac{1}{2}}\sum_{l=1}^{\infty} i^{l}(2l+1)l^{-1}(l+1)^{-1}\sigma^{-\frac{1}{2}}J_{l+\frac{1}{2}}(\sigma)P_l^1(x),$$

where $\sigma=\rho(1-s)$ and on performing contour integrations. On using the recurrence formula

$$J_{n+1}(\sigma)+J_{n-1}(\sigma)=2n\sigma^{-1}J_n(\sigma)$$

we have

$$\bar{S}'=i\sigma^{-1}(\pi/2)^{\frac{1}{2}}\sum_{l=0}^{\infty} i^{l}(2l+1)\sigma^{-\frac{1}{2}}J_{l+\frac{1}{2}}(\sigma)P_l(x)=i\sigma^{-1}e^{i\sigma x};$$

and the contour integration gives

$$S'=e^{\pi\alpha/2}\Gamma(1-i\alpha)(1-x)e^{i\rho x}F(1+i\alpha,\ 2,\ i\rho(1-x)).$$

To evaluate $\bar{S}''$ we use the relations

$$(\pi/2)^{\frac{1}{2}}i^{l}\sigma^{-\frac{1}{2}}J_{l+\frac{1}{2}}(\sigma)=\tfrac{1}{2}\int_{-1}^{1}e^{i\sigma t}P_l(t)dt \quad\text{and}\quad P_l^1(x)=(1-x^2)^{\frac{1}{2}}dP_l(x)/dx.$$

Now

$$\sum_{l=1}^{\infty}(l+\tfrac{1}{2})l^{-1}(l+1)^{-1}P_l(t)P_l(x)=\begin{cases}-\tfrac{1}{2}\ln[(1-t)(1+x)]+\ln 2-\tfrac{1}{2}, & t\leqslant x;\\ -\tfrac{1}{2}\ln[(1+t)(1-x)]+\ln 2-\tfrac{1}{2}, & t\geqslant x;\end{cases}$$

being just the Green's function in the extended sense for Legendre's equation.[10] For $-1+\delta<x<1-\delta$, $\delta>0$ the convergence is uniform and we can interchange differentiation, integration and summation. Then

$$\bar{S}''=(1-x^2)^{\frac{1}{2}}[i(2\sigma)^{-1}(1+x)^{-1}(e^{i\sigma x}-e^{-i\sigma})+i(2\sigma)^{-1}(1-x)^{-1}(e^{i\sigma x}-e^{i\sigma})].$$

On carrying out the contour integration one finds that the term $e^{i\sigma}$ contributes nothing, that in $e^{-i\sigma}$ just cancels the term $P_1^1K_0$ and we have

$$S''=e^{\pi\alpha/2}\Gamma(1-i\alpha)(1-x^2)^{\frac{1}{2}}e^{i\rho x}F(1+i\alpha,\ 2,\ i\rho(1-x)).$$

Being continuous in x, this result holds without the restriction $-1+\delta<x<1-\delta$.

Thus the resulting approximate wave function is:

$$\begin{aligned}\varphi_1&=-\bar{N}\{kf-i(\alpha/2)(\epsilon+1)(1-\cos\theta)f'\},\\ \varphi_2&=-\bar{N}i(\alpha/2)(\epsilon+1)\sin\theta e^{i\varphi}f',\\ \varphi_3&=\bar{N}\{(\epsilon+1)f-i(\alpha/2)k(1-\cos\theta)f'\},\\ \varphi_4&=\bar{N}i(\alpha/2)k\sin\theta e^{i\varphi}f',\end{aligned}\tag{4}$$

where

$$\bar{N}=(2\pi)^{-\frac{1}{2}}\{2\epsilon(\epsilon+1)\}^{-\frac{1}{2}}e^{\pi\alpha/2}\Gamma(1-i\alpha),$$

$$f=e^{ikr\cos\theta}F(i\alpha,\ 1,\ ikr(1-\cos\theta)),$$

$$f'=e^{ikr\cos\theta}F(1+i\alpha,\ 2,\ ikr(1-\cos\theta)).$$

[10] Courant-Hilbert, *Methoden der Mathematischen Physik*, pp. 322–323 (2nd edition).

An identical repetition of the above arguments, starting with Hulme's functions for $u=-1$, leads to the second function:

$$\begin{aligned}\varphi_1'&=-\bar{N}i(\alpha/2)(\epsilon+1)\sin\theta e^{-i\varphi}f',\\ \varphi_2'&=\bar{N}\{kf-i(\alpha/2)(\epsilon+1)(1-\cos\theta)f'\},\\ \varphi_3'&=-\bar{N}i(\alpha/2)k\sin\theta e^{-i\varphi}f',\\ \varphi_4'&=\bar{N}\{(\epsilon+1)f-i(\alpha/2)k(1-\cos\theta)f'\}.\end{aligned}\tag{5}$$

The two functions (4) and (5) become, either for $\alpha\to 0$ or for $r\to\infty$, the ordinary plane wave functions for a free electron of positive kinetic energy and positive or negative spin component. Functions for negative kinetic energy

may be obtained from them by the transformation

$$\varphi_1\to\varphi_3,\quad \varphi_2\to\varphi_4,\quad \varphi_3\to-\varphi_1,\quad \varphi_4\to-\varphi_2,\quad \alpha\to-\alpha,$$

as is readily seen from the form of the wave equations.[11]

On substituting (4) or (5) in the wave equations,[11] we obtain as the terms which fail to cancel out in the left member quantities of orders

$$(\alpha/2\epsilon)\sin^2\theta/2,\quad (\alpha/2\epsilon)\sin\theta/2,\quad (\alpha^2/r)\sin^2\theta/2,$$

and

$$(\alpha^2/r)\sin\theta/2$$

compared to the wave functions themselves. Thus, for very large ϵ we are left, as in the case of the Born functions, with a discrepancy of order α^2; but for $\theta\to 0$, in the region where the Born functions fail, the discrepancy in the present case becomes vanishingly small.

The functions as given are normalized in the scale of k_x, k_y and k_z; those we have obtained here are for $k_x=k_y=0$, $k_z=k$. The functions for any other direction of k may be obtained by a rotation of coordinates of the type required by Dirac wave functions.[12] Apart from this the use of the functions is similar to that described for the nonrelativistic case by Sommerfeld.[13]

The set of functions thus obtained is not a complete orthogonal set when $\alpha\neq 0$, but in the solution of problems it is to be used as if it were such a set. This is explained as follows: In the exact solution of the sort of problem for which these functions are intended one is concerned with the evaluation of a sum of the form

$$\sum_{lu} V'_{n';\ lu}V_{lu;\ n},$$

where V and V' are interaction energies, and l, u are angular momentum quantum numbers designating the spherical coordinate wave functions; n and n' denote states of the electron. In problems such as the photoelectric effect, pair production and radiative impacts one has $V'=V$, $n'=n$; in other problems such as that of the scattering of very hard radiation this is not so. The use of the set of functions here derived is to be justified by the relation

$$\sum_{lu} V'_{n';\ lu}V_{lu;\ n}\approx\sum_{lu}(V'_{n';\ lu})'(V_{lu;\ n})'$$
$$=\sum_{\sigma}\int d\omega\cdot V'_{n';\ \theta\varphi\sigma}V_{\theta\varphi\sigma;\ n}.$$

Here $(V_{lu;\ n})'$ and $(V'_{n';\ lu})'$ are calculated with the modified wave functions typified by (1′) and (2′); and $V_{\theta\varphi\sigma;\ n}$ and $V'_{n';\ \theta\varphi\sigma}$ are calculated with the approximate wave functions typified by (4) and (5), suitably renormalized, $d\omega$ being an element of solid angle containing the direction $(\theta,\ \varphi)$ and σ denoting the two spins. The approximate equality of the first two members is based on the fact that only very large values of l contribute appreciably to the sums and that for large l the matrix elements calculated from (1′) and (2′) are presumably very nearly equal to those calculated with the exact functions; the equality of the second and third members follows from the unitary character of the transformation from the set of functions typified by (1′) and (2′) to the set typified by (4) and (5).

The approximate equality of two sums which is involved in the above argument cannot be regarded as established with certainty by the physical arguments given in the introduction. On the other hand it could be proved rigorously only by making special considerations for each individual problem, and in doing this it would be necessary to overcome a large part at least of the difficulties which inhere in the exact solution of these problems using spherical coordinates. Thus the type of function which we have here constructed is best regarded as the basis of a method which is admittedly only approximate and subject to possible failures, but physically reasonable and probably reliable for the cases for which it was devised.

The writer wishes to thank Professor J. R. Oppenheimer, who suggested the construction of these functions, for much helpful advice.

[11] For the Dirac wave equations completely written out in the form which is the basis of the present work, see for example H. R. Hulme, Proc. Roy. Soc. **A133**, 382 (1931). The equations are to be put into natural units as we have previously explained.

[12] C. G. Darwin, Proc. Roy. Soc. **A118**, 654 (1928).

[13] A. Sommerfeld, Ann. d. Physik **11**, 256 (1931).