On the Mathematical Analysis of Cosmic-Ray Data

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In connection with cosmic-ray calculations, attention is drawn to the existence of extensive tables by Miller and Rosebrugh of the integral $\int_x^\infty e^{-u}u^n du$ for integral values of n from -2 to $+2$. Cosmic-ray data may be analyzed by a procedure which ultimately depends on Prony's method of analyzing a curve into a finite number of exponential constituents. By the use of interpolation theorems and the methods of the calculus of finite differences, observational data for exponential analysis may be derived free from the uncertainties of graphical differentiation. Methods of calculating the effect of self-scattering by the atmosphere are indicated. The exact calculation of the effect of wall thickness in a spherical ionization chamber is shown to depend on the functions tabulated by Miller and Rosebrugh.

§1. INTRODUCTION

THE principal object of this brief note is to draw attention to extensive and convenient tables by W. Lash Miller and T. R. Rosebrugh[1] of integrals of the type

$$I_n(x)=\int_x^\infty e^{-u}u^n du. \qquad (1)$$

For negative values of n, these integrals may ultimately be reduced to depend on the exponential integral. The integrals $I_{-2}(x)$, $I_{-1}(x)$, $I_0(x)$, $I_1(x)$ and $I_2(x)$ are, however, tabulated directly to 9 significant figures at intervals of 0.001 between $x=0$ and $x=1$, and at intervals of 0.01 between $x=1$ and $x=2$.

In terms of the integral

$$G(x)=\int_1^\infty u^{-2}e^{-xu}du, \qquad (2)$$

which plays an important part in atmospheric radiation problems, we obviously have,

$$G(x)=xI_{-2}(x). \qquad (3)$$

A short table of $G(x)$ is given by E. Gold[2] to 5 significant figures at intervals of 0.01 between $x=0$ and $x=1$; 0.05 between 0.1 and 1.0; 0.1 between 1 and 2; 0.2 between 2 and 3; 0.5 between 3 and 6. Enlarged tables of the integral have been computed by L. D. Weld,[3] but apparently have not been published: this is now perhaps unnecessary in view of the existence of the Miller-Rosebrugh tables referred to above.

Obviously $I_{-1}(x)=Ei(-x)$ and $I_0(x)=e^{-x}$, for which well-known tables are available.

In the following sections we deal briefly with two problems connected with the analysis of cosmic-ray data in regard to which the Miller-Rosebrugh tables are likely to prove extremely convenient, and deserve to be better known.

§2. EXPONENTIAL ANALYSIS OF COSMIC-RAY DATA

The mathematical problem of analyzing the radiation from sun and sky has many points in common with the corresponding problem relating to cosmic-rays.

If $E_0(\zeta)$ is the intensity per unit solid angle outside the earth's atmosphere, the corresponding intensity $E(\zeta, z)$ at height z above sea level is easily seen to be[4]

$$E(\zeta, z)=E_0(\zeta)e^{-\mu_0(H-Z)\sec\zeta}. \qquad (4)$$

The coefficient of attenuation is assumed to be proportional to the density, and μ_0 is its value at standard density ρ_0.

In the above formula, which is valid for any law of variation of atmospheric density with height, the curvature of the earth is neglected, and

$$Z=\int_0^z \rho/\rho_0\, dz \quad\text{and}\quad H=\int_0^\infty \rho/\rho_0\, dz. \qquad (5)$$

[1] W. Lash Miller and T. R. Rosebrugh, Trans. Roy. Soc. Canada [2] **9**, 73–107, 1903, Sect. III, University of Toronto Studies, No. 43 (Published by the Librarian, University of Toronto).

[2] E. Gold, Proc. Roy. Soc. London **A82**, 62 (1909).

[3] L. D. Weld, Phys. Rev. **40**, 713 (1932).

[4] L. V. King, Trans. Roy. Soc. London **A212**, 392 (1912).

Obviously H is the height of the "homogeneous atmosphere," and in terms of the barometric pressure p at height z, and the standard pressure p_0 at sea level we have, $(H-Z)/Z=p/p_0$, so that we may write (4) in the form

$$E(\zeta, z)=E_0(\zeta) \exp [-(\mu_0 H_0/p_0)\cdot p \sec \zeta]. \quad (6)$$

If it is assumed that the ionization produced in a spherically symmetrical ionization chamber is proportional to $E(\zeta, z)$, the total ionization at height z from all direction is given by

$$I(z)=k\int E(\zeta, z)d\omega, \quad (7)$$

where k is a constant of proportionality depending on the structure of the ionization vessel.

If it is further assumed that outside the earth's atmosphere $E_0(\zeta)$ is independent of zenith distance and azimuth, we have from (6) and (7), writing $d\omega=2\pi \sin \zeta\, d\zeta$, measuring height in terms of barometric pressure p, and denoting $\lambda_0=\mu_0 H_0/p_0$,

$$I(p)=2\pi k E_0\int_0^{\frac{1}{2}\pi} e^{-\lambda_0 p \sec \zeta} \sin \zeta\, d\zeta. \quad (8)$$

We readily find, in the notation of §1 that

$$I(p)=2\pi k E_0 \lambda_0 p I_{-2}(\lambda_0 p), \quad (9)$$

and hence that

$$\psi=-p^2\frac{d}{dp}\left(\frac{I}{p}\right)=2\pi k E_0 e^{-\lambda_0 p}. \quad (10)$$

If the incident radiation consists of several constituents having different coefficients of attenuation μ_{0n}, and ionization-chamber constants k_n, the ionization is additive, and we replace (9) by[5]

$$\psi=-p^2\frac{d}{dp}\left(\frac{I}{p}\right)=2\pi\sum(n)k_n E_{0n}e^{-\lambda_{0n}p}. \quad (11)$$

It will be noticed that the left-hand side contains only observed quantities I and p, while the exponential form of the right-hand side makes it possible to utilize Prony's method of exponential analysis[6] to determine whether curves obtained from balloon or aeroplane observations can be analyzed into a finite number of exponential constituents. Records of the variation of ionization with height (or recorded barometric pressure) give I as a function of p, from which a graph of I/p at *equal* intervals of p may be determined. If a series of observations for unequal intervals of p is to be analyzed, interpolations for equal intervals may be calculated by the use of Newton's formula.[7] Utilizing well-known theorems in the calculus of finite differences, the slopes $d(I/p)/dp$ may be readily computed, so that the left-hand side of (9) may be tabulated at equal intervals independently of the uncertainties of graphical differentiation. The observational material is now ready for the application of Prony's method which enables $2\pi k_n E_{0n}$ and λ_{0n} to be evaluated. Computation should be carried out successively for two, three, or more exponential terms, the constants $2\pi k_n E_{0n}$ and λ_{0n} so determined being utilized, in each case, with the use of the Miller-Rosebrugh tables, to re-synthesize the original ionization curve,

$$I(p)=2\pi\sum_n k_n E_{0n}(\lambda_{0n}p)I_{-2}(\lambda_{0n}p) \quad (12)$$

until a satisfactory fit has been obtained. Should this not prove to be possible for a reasonably small number of exponential terms, it must be concluded that the extra-atmospheric spectrum of the incident radiation is, at least in part, continuous, the problem then resolving into the more difficult one of solving from observational data an equation of the type

$$\psi(p)=-p^2\frac{d}{dp}\left(\frac{I}{p}\right)=2\pi\sum_n k_n E_{0n}e^{-\lambda_{0n}p}+\int_0^\infty 2\pi k E(\lambda)e^{-\lambda p}d\lambda. \quad (13)$$

The procedure outlined in this section supplies

[5] The left-hand side of (11) is identical with $\psi(p)=I-pdI/dp$ introduced by B. Gross (Zeits. f. Physik **83**, 217 (1933)), and now utilized in plotting experimental data.

[6] Whittaker and Robinson, *Calculus of Observations*, §180, p. 369 (Blackie and Son, London, 1924).

[7] Reference 6, p. 35.

in part the need expressed by K. K. Darrow[8] for a convenient method of analyzing cosmic-ray data analogous to the use of Fourier's theorem in the analysis of periodic effects. It is suggested as a procedure supplementary to that utilized by L. D. Weld, and may show some advantage as regards ease of computation. Although either method is likely to be somewhat laborious, records of variation of ionization with barometric pressure are now becoming more numerous and accurate, and better adapted to analysis.[9]

The reader will find no difficulty in adapting the method of computation just outlined to the analysis of under-water ionization curves.

The effect of scattered atmospheric radiation, if appreciable, complicates the analysis. If a law of scattering by air is known, the methods developed by the writer for dealing with the somewhat similar problem of scattering of light in the atmosphere may prove serviceable. In this problem self-illumination of the atmosphere contributes a term $\Phi(Z, \zeta)$ to the right-hand side of Eq. (4). This function is determined from the approximate solution of an integral equation. It is shown[10] to have a maximum value at $Z=\frac{1}{2}H$ or $p=\frac{1}{2}p_0$. When the complete expression for $E(Z, \zeta)$ is integrated for radiation from all directions, the part due to self-illumination or self-scattering will have its maximum shifted to a higher altitude depending on the coefficient of attenuation. The detailed examination of the problem as applied to cosmic radiation must, however, be deferred to a future investigation. It would appear, however, that the criterion $d^2(\log \psi)/dp^2>0$, easily seen from (13) to be necessary if primary constituents of an exponential type alone are considered, may require modification when the additional terms due to atmospheric self-scattering (assumed to have the same coefficient of attenuation as the primary radiation) are taken into account. This latter factor may, in fact, give an adequate explanation of the points of inflection observed in the more recent graphs of log ψ plotted against p.[11]

[8] K. K. Darrow, *Data and Nature of Cosmic Rays*, Bell Syst. Tech. J. **9**, 165 (1932).

[9] A. H. Compton and R. J. Stephenson, *Cosmic-Ray Ionization at High Altitudes*, Phys. Rev. **45**, 441 (1934).

[10] L. V. King, Trans. Roy. Soc. London **A212** (1912), Eq. (36).

[11] A. H. Compton and R. J. Stephenson, *Cosmic-Ray Ionization at High Altitudes*, Phys. Rev. **45**, 446 (1934).

§3. Effect of Wall Thickness in a Spherical Ionization Chamber

In a recent paper, C. Eckart[12] has examined the absorption of a homogeneous beam of radiation by the walls of a spherical ionization chamber of internal radius r, external radius R, the material having a coefficient of attenuation κ. The average ionization per unit volume is shown to be proportional to

$$H=\xi\int_0^1 e^{-\kappa\xi}z^2dz, \tag{14}$$

where

$$\xi=(t^2+r^2z^2)^{\frac{1}{2}}-rz, \quad \text{and} \quad t^2=R^2-r^2. \tag{15}$$

The integral in (14) may be evaluated without approximation in terms of the functions tabulated by Miller and Rosebrugh. From the first equation in (15) we have $2rz=(t^2/\xi)-\xi$, so that in terms of ξ,

$$H=-\left(\frac{3}{8r^3}\right)\int_{\xi_0}^{\xi_1} e^{-\kappa\xi}\left(\frac{t^6}{\xi^4}-\frac{t^4}{\xi^2}-t^2+\xi^2\right)d\xi. \tag{16}$$

Introducing the dimensionless variables, $x=\kappa\xi$, $\tau=\kappa t$, $\rho=\kappa r$, $\tau'=(\tau^2+\rho^2)^{\frac{1}{2}}-\rho$, Eq. (16) may be written

$$H=\tfrac{3}{8}\rho^{-3}[\tau^6I_{-4}(x)-\tau^4I_{-2}(x) -\tau^2I_0(x)+I_2(x)]_\tau^{\tau'}. \tag{17}$$

The last three integrals are directly tabulated, while

$$I_{-4}(x)=\tfrac{1}{3}e^{-x}/x^3-\tfrac{1}{6}e^{-x}/x^2+\tfrac{1}{6}I_{-2}(x). \tag{18}$$

The use of the exact formula (17) makes the computation of H a comparatively easy matter, so much so that experiments for analyzing cosmic radiation at any one locality by varying t is suggested. It should be possible, as in Section 2, to transform Eq. (17) in such a way that we have in the left-hand side a function of H, $\partial H/\partial t$, etc., and on the right a number of exponential terms, so that Prony's method of analysis may be utilized. The correction to be applied for self-scattering by the spherical walls is likely to prove difficult and its calculation only to be attempted if experiments of high precision of the type considered were seriously considered.[13]

[12] C. Eckart, Phys. Rev. **45**, 451 (1934).

[13] The problem of scattering of solar radiation by a spherically curved atmosphere has recently been considered by C. L. Pekeris, Skrifter utgitt av Det Norske Videnskaps, Akademi i Oslo, 1934, p. 12.