Collision of Two Light Quanta

G. BREIT* AND JOHN A. WHEELER,** *Department of Physics, New York University*
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The recombination of free electrons and free positrons and its connection with the Compton effect have been treated by Dirac before the experimental discovery of the positron. In the present note are given analogous calculations for the production of positron electron pairs as a result of the collision of two light quanta. The angular distribution of the ejected pairs is calculated for different polarizations, and formulas are given for the angular distribution of photons due to recombination. The results are applied to the collision of high energy photons of cosmic radiation with the temperature radiation of interstellar space. The effect on the absorption of such quanta is found to be negligibly small.

TWO simultaneously acting light waves with vector potentials

$$\mathbf{A}_j=\mathbf{a}_j^* \exp\{-(\omega_j t-\mathbf{k}_j\mathbf{r})\}+\mathbf{a}_j \exp\{i(\omega_j t-\mathbf{k}_j\mathbf{r})\} \quad (1)$$

are considered as acting on an electron. Under the influence of the waves a single electron wave function $\psi^{(0)}$ is changed, and the perturbed function may be expanded according to powers of a, a^*. The phenomena of pair production and of recombination have to do with the terms in $a_1^*a_2^*$ and a_1a_2, respectively, as is obvious from the theory of quantization of light waves. We consider first the pair production. We let the function $\psi^{(0)}$ represent an electron in a negative energy state. It is convenient for practical applications to normalize $\psi^{(0)}$ so as to have the electron density equal to unity. It is also unnecessary to use quantized light waves in the pair production problem, since the results with quantized waves are known to be identical with those obtained by means of ordinary waves. As a result of the calculation one finds that at a time t after the application of the waves the wave function contains a term which may be interpreted as referring to an electron in a positive energy state with a momentum and a spin coordinate which are functions of the original momentum and spin and of the momenta and polarizations of the light quanta. The density of electrons corresponding to this wave function may be put into the form

$$(e^2/mc^2)^2|a_1|^2|a_2|^2B|1-\exp(-it\delta W/\hbar)|^2/(\delta W)^2. \quad (2)$$

Here B is a dimensionless number depending on initial momenta and spin and the polarizations of the quanta. δW is the difference in energy of the initial and the final states. Thus if $W=-|W|$ is the energy of the electron in its initial state and if $h\nu_1$, $h\nu_2$ are the energies of the quanta, then

$$\delta W=c(p_2^2+m^2c^2)^{\frac{1}{2}}+W_1-h\nu_1-h\nu_2;\qquad W_1=-W, \quad (3)$$

where

$$\mathbf{p}_2=\mathbf{p}_1+\mathbf{P}_1+\mathbf{P}_2 \quad (4)$$

is the final momentum of the electron and $\mathbf{P}_1$, $\mathbf{P}_2$ are the momenta of the quanta. The total electron density due to the two light quanta is obtained by summing expression (2) over all possible states of negative energy. The equal and opposite spin directions for every $\mathbf{p}_1$ contribute to the density. One is thus only interested in the average for B over the different directions σ of the positron spin. This average will be called $\bar{B}^\sigma$. There are $2p_1^2dp_1d\omega_1\cdot V/h^3$ electronic states of negative energy in the fundamental volume V for which the momentum is $\mathbf{p}_1$ and the direction is within the solid angle $d\omega_1$. Each of these has a density $1/V$. The number of positron electron pairs produced per cm^3 corresponding to the absolute value of positron momentum being between p_1 and p_1+dp_1 in the direction $-\mathbf{p}_1$ and in solid angle $d\omega_1$ is thus obtained from (2) by multiplying it by $2p_1^2dp_1d\omega_1/h^3$. Integrating over dp_1, and making use of

$$d(\delta W)=c^2[p_1/W_1+\mathbf{p}_1\mathbf{p}_2/p_1W_2]dp_1, \quad (5)$$

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* Now at Department of Physics, University of Wisconsin.

** National Research Fellow now at Copenhagen.

which follows from (3) and (4), and substituting

$$|a|^2=(c/2\pi\nu^2)I=(hc/2\pi\nu)N, \tag{6}$$

where I is the intensity of each beam and N is the number of light quanta per cm^2 per second, one obtains the probability of pair formation per unit solid angle of the positron and per unit volume of the space in which the light quanta collide, as

$$\frac{2p_1^2(e^2/mc^2)^2N_1N_2\bar{B}^\sigma t}{h\nu_1h\nu_2[p_1/W_1+\mathbf{p}_1\mathbf{p}_2/p_1W_2]}. \tag{7}$$

Here $(-\mathbf{p}_1)$, $\mathbf{p}_2$ are the vectors representing the momenta of the positron and electron, p_1, p_2 are their absolute values and W_1, W_2 are, respectively, the energies of the positron and electron. It is convenient to express the above probability in terms of an effective collision area σ. A convenient definition is to express (7) as $tN_1N_2\sigma/c$. The effective collision area σ thus defined corresponds to a picture of N_1 light quanta per unit area per second traveling through space in which the quanta $h\nu_2$ are thought of as being distributed with their density N_2/c. This definition is arbitrary but convenient for transformations to other frames of reference. It should be noted, however, that if two beams of quanta are shot against each other head-on then the number of pairs produced is

$$A(\sigma/2)\int N_1dt\cdot\int N_2dt, \tag{8}$$

where A is the common cross-section area of the two beams, the factor 1/2 arising from the fact that both beams travel against each other with the velocity of light. The effective collision area for head-on collisions when expressed in terms of the numbers of quanta shot at each other rather than in terms of the density of one of them is thus $\sigma/2$. We have

$$\sigma=\frac{2cp_1^2(e^2/mc^2)^2\bar{B}^\sigma}{h\nu_1h\nu_2[p_1/W_1+\mathbf{p}_1\mathbf{p}_2/p_1W_2]}. \tag{9}$$

The recombination probability of electrons and positrons may also be expressed in terms of the quantity B used in (2). One starts with an electron being in a positive energy state and considers the terms in a_1a_2 according to the method of quantizing light waves. The probability for the electron to jump into the hole may then be calculated for this pair of light waves. Using electron and positron states normalized to 1, i.e., of density $1/V$, a result is obtained which is similar to Eq. (2). It differs from (2) only through the replacement of every $|a|^2$ by the initial expectation of a^+a. Thus the contribution to the emission probability due to the possible cooperation of a pair of light waves is obtained by substituting

$$|a|^2=(c/2\pi\nu^2)I$$

and then letting $\quad I=h\nu c/V. \qquad (10)$

Consider a positron with a definite spin and momentum going through an electron distribution also having a definite spin and momentum. In order to obtain the emission probability we must sum expression (2) modified in accordance with (10) over all possible pairs of light quanta. Thus for given momenta of the light quanta 1, 2 each light quantum can have two perpendicular and independent directions of polarization s_1, s_1' and s_2, s_2', respectively, and one obtains contributions to the emission probability due to every possible combination of polarizations. Thus one has the emission probability as a sum of terms

$$(h^2c^4/4\pi^2\nu_1\nu_2)(e^2/mc^2)^2V^{-2}\beta(1,2)\times|1-\exp(-it\delta W/\hbar|^2/(\delta W)^2, \tag{11}$$

where $\beta(1, 2)$ should be made to take in turn the values $\beta(s_1, s_2)$, $\beta(s_1, s_2')$, $\beta(s_1', s_2)$, $\beta(s_1', s_2')$. If it happens that the pair production calculation for quanta 1, 2 gives for the assigned positron spin also the assigned electron spin then $\beta(1, 2)=B(1, 2)$. In general, however,

$$\beta(1,2)=\frac{|(\psi_\sigma^*,\psi_{\sigma'})|^2}{(\psi_{\sigma'}^*\psi_{\sigma'})(\psi_\sigma^*\psi_\sigma)}B(1,2), \tag{12}$$

where ψ_σ, $\psi_{\sigma'}$ are, respectively, the electron state with specified spin and the electron state which arises from the positron state under the action of the light quanta 1, 2. The probability of having recombination in which light quantum 1 has the polarization s_1 while the polarization of light quantum 2 is subject to no restriction is obtained by omitting $\beta(s_1', s_2)$, $\beta(s_1', s_2')$ from the summation. The result depends then only on the average value

$$\beta(s_1, \bar{s}_2) = \tfrac{1}{2}[\beta(s_1, s_2) + \beta(s_1, s_2')]. \quad (12')$$

Performing the integration over P_1 and allowing $\mathbf{P}_1$ to lie within a small solid angle of $d\Omega_1$ one obtains the number of emissions per second. This number refers to electron and positron densities $1/V$. To reduce to unit electron and positron density the probability must be multiplied by V^2 and it must again be divided by V to reduce to probability of emission in unit volume. The probability of emission per second per cm^3 due to unit electron and positron densities restricting the emission processes to those in which one of the light quantum pair has a direction in $d\Omega_1$ is

$$2c\left(\frac{e^2}{mc^2}\right)^2 \beta(s_1, \bar{s}_2)\frac{\nu_1}{\nu_2}\left[1 - \frac{\mathbf{P}_1\mathbf{P}_2}{P_1P_2}\right]^{-1} d\Omega_1. \quad (13)$$

In obtaining the total emission probability from this expression one should be careful not to sum over s_1 and then to integrate over $d\Omega_1$ without dividing the result by two. Otherwise one counts every pair of quanta twice: one time in the order 1, 2 and another time in the order 2, 1. Letting

$$\beta(\bar{s}_1, \bar{s}_2) = \tfrac{1}{2}[\beta(s_1, \bar{s}_2) + \beta(s_1', \bar{s}_2)] \quad (14)$$

the total emission probability is

$$2c\left(\frac{e^2}{mc^2}\right)^2 \int \beta(\bar{s}_1, \bar{s}_2)\frac{\nu_1}{\nu_2}\left(1 - \frac{\mathbf{P}_1\mathbf{P}_2}{P_1P_2}\right)^{-1} d\Omega_1. \quad (15)$$

One is usually interested in the passage of positrons with random spins through electron distributions having random spins, and in this case $\beta(\bar{s}_1, \bar{s}_2)$ should be averaged over these spins. Similarly, in the production of pairs one is usually interested in the average of (9) for quanta having random polarizations. One should be careful not to suppose that the average thus entering in (9) $[\bar{\bar{B}}^\sigma]$ is equal to the average of $\beta(\bar{s}_1, \bar{s}_2)$ which we call $\bar{\beta}(\bar{s}_1, \bar{s}_2)$. In fact, using (12)

$$\bar{\beta}(\bar{s}_1, \bar{s}_2) = \tfrac{1}{2}\bar{\bar{B}}^\sigma. \quad (16)$$

Thus the total emission probability in all directions per unit volume per second due to unit positron and electron densities is

$$c\left(\frac{e^2}{mc^2}\right)^2 \int \bar{\bar{B}}^\sigma \frac{\nu_1}{\nu_2}\left(1 - \frac{\mathbf{P}_1\mathbf{P}_2}{P_1P_2}\right)^{-1} d\Omega_1 \quad (17)$$

and for light quanta with random polarizations the "density collision cross section" is

$$\bar{\sigma} = 2\left(\frac{e^2}{mc^2}\right)^2 \int \frac{cp_1^2}{h^2\nu_1\nu_2} \bar{\bar{B}}^\sigma \left[\frac{p_1}{W_1} + \frac{\mathbf{p}_1\mathbf{p}_2}{p_1W_2}\right]^{-1} d\omega_1. \quad (18)$$

If the momenta of the electron and positron are equal and opposite the probability (17) is

$$\tfrac{1}{2}c(e^2/mc^2)^2 \int \bar{\bar{B}}^\sigma d\Omega_1 \quad (17')$$

and

$$\bar{\sigma} = (e^2/mc^2)^2(cp/W) \int \bar{\bar{B}}^\sigma d\omega_1. \quad (18')$$

The quantity B was calculated as in Dirac's paper. The method used by him can be applied to any initial state of a free electron. The initial wave functions for an electron having an energy $mc^2 \cosh\theta$ a momentum $\mathbf{p} = mc(\mathbf{p}/p)\sinh\theta$ and two independent spin directions are given by the first two columns of $[-\cosh\theta + (\mathbf{p}/p)\boldsymbol{\alpha}\sinh\theta + \rho_3](1+\rho_3)$. Following the procedure of Dirac[1] the calculation of $\psi^{(2)}$ with such an initial wave function reduces to mechanical operations. The answer comes out as a four row four column matrix, the last two columns of which are automatically zero. The first two columns are the wave functions arising under the influence of (1) from the wave functions represented by the first two columns of $[-\cosh\theta + (\mathbf{p}/p)\boldsymbol{\alpha}\sinh\theta + \rho_3](1+\rho_3)$. In this manner one can obtain the spin properties of $\psi^{(2)}$ as well as its density. At present there seem to be no applications for the spin relations between the initial and final states, and we give therefore only the results for the density. The results take a simple form in the reference system in which the total momentum is zero. In the results listed below the common line of the two quanta is taken to be the x-axis. The direction cosines of the produced positron are called λ, μ, ν. We let the energy of the

[1] P. A. M. Dirac, Proc. Camb. Phil. Soc. **24**, 361 (1930). Although Eq. (28) of Dirac's paper appears to refer to the problem here discussed it does not actually do so, because the negative energy state is kept fixed and the frequency of the quantum is varied.

electron as well as that of the positron be $W = mc^2 \cosh\theta$ and

$$C = \cosh\theta, \quad S = \sinh\theta. \tag{19}$$

We obtain

$(\mathcal{E}_1 \| y,\ \mathcal{E}_2 \| y)$

$$B_{\|} = [S^2 - 2S^2(\nu^2 - \mu^2) - S^4(\nu^2 - \mu^2)^2 + S^2C^2(1 - \lambda^2)]C^{-2}(C^2 - \lambda^2 S^2)^{-2}, \tag{20.1}$$

$(\mathcal{E}_1 \| y,\ \mathcal{E}_2 \| z)$

$$B_{\perp} = [C^2 - 4S^4\mu^2\nu^2 + S^2C^2(1 - \lambda^2)]C^{-2}(C^2 - \lambda^2 S^2)^{-2}. \tag{20.2}$$

Averaging over the angle which one of the electric intensities makes with the y-axis

$$\bar{B}_{\|}{}^{\varphi} = (S^2/2)[C^2 + 3 - 2\lambda^2 - \lambda^4 S^2]C^{-2}(C^2 - \lambda^2 S^2)^{-2};$$

$$\bar{B}_{\perp}{}^{\varphi} = \tfrac{1}{2}[C^4 + 2C^2 - 1 - 2\lambda^2 S^2 - \lambda^4 S^4]C^{-2} \times (C^2 - \lambda^2 S^2)^{-2}. \tag{20.3}$$

If one of the quanta is polarized along y and the other at random

$$B(\mathcal{E}_y, \bar{\mathcal{E}}) = \tfrac{1}{2}[C^4 + 2C^2 - 2 - 2\lambda^2 S^2 - \lambda^4 S^4 + 2(\mu^2 - \nu^2)S^2]C^{-2}(C^2 - \lambda^2 S^2)^{-2}. \tag{20.4}$$

If both quanta are polarized at random

$$B(\bar{\mathcal{E}}, \bar{\mathcal{E}}) = \tfrac{1}{2}[C^4 + 2C^2 - 2 - 2\lambda^2 S^2 - \lambda^4 S^4]C^{-2}(C^2 - \lambda^2 S^2)^{-2}. \tag{20.5}$$

For all of the above six conditions Eq. (9) reduces to

$$\sigma = (e^2/mc^2)^2 B \tanh\theta \tag{9'}$$

and the number of pairs in which the positron has the direction (λ, μ, ν) produced per unit solid angle per unit volume per second is $N_1N_2\sigma/c$. By integration one obtains the number of pairs in all solid angles. The values of σ which correspond to the two cases (20.3) are

$$\bar{\sigma}_{\|} = 2\pi(e^2/mc^2)^2[-SC^{-3} - \tfrac{3}{2}SC^{-5} - \tfrac{3}{2}\theta C^{-6} + 2\theta C^{-4} + 2\theta C^{-2}], \tag{21.1}$$

$$\bar{\sigma}_{\perp} = 2\pi(e^2/mc^2)^2[-SC^{-3} - \tfrac{1}{2}SC^{-5} - \tfrac{1}{2}\theta C^{-6} + 2\theta C^{-4} + 2\theta C^{-2}], \tag{21.2}$$

and for random polarizations

$$\bar{\sigma} = 2\pi(e^2/mc^2)^2[-SC^{-3} - SC^{-5} - \theta C^{-6} + 2\theta C^{-4} + 2\theta C^{-2}]. \tag{21.3}$$

Formulas (21.1), (21.2) apply also to (20.1), (20.2). It should be remembered that Eqs. (21) apply to the "density collision cross section" defined in terms of $N_1N_2\bar{\sigma}/c$ and that the total number of pair producing collisions for two beams of quanta each containing n quanta is $n_1n_2\bar{\sigma}/(2A)$ in accordance with Eq. (8).[2]

If the pair production conditions described by Eqs. (20), (21) are viewed from a moving frame of reference, the number of pairs produced per unit volume per unit time remains the same because pair production is an event in four dimensional space and the density of events is Lorentz invariant. Thus

$$N_1'N_2'\sigma' d\omega_1 = N_1N_2\sigma d\omega_1, \tag{22}$$

where N' is the number of quanta per cm^2 per sec. in the new frame of reference and σ' is the "density collision cross section" per unit positron solid angle in that frame. For any two light quanta it is possible to perform a Lorentz transformation so that they appear to be equal and opposite. The only case in which this is impossible is that of quanta travelling in the same direction. However, this case is of no interest because it leads to no pair production. If the frequencies of the quanta are ν_1', ν_2' and the angle between their directions is φ then the frequency of either in the frames of zero total momentum is $\nu_1 = \nu_2 = \nu = (\nu_1'\nu_2')^{1/2}\sin(\varphi/2)$ and $N_1' = (\nu_1'/\nu_1)N_1$, $N_2' = (\nu_2'/\nu_2)N_2$. Hence

$$\sigma' d\omega_1' = \sin^2(\varphi/2)\sigma((\nu_1'\nu_2')^{\frac{1}{2}}\sin(\varphi/2))d\omega_1. \tag{23}$$

The velocity with which the primed system of reference is moving with respect to the unprimed system is $-c^2(\mathbf{P}_1' + \mathbf{P}_2')/(h\nu_1' + h\nu_2')$. From (23) and (20) one can obtain the angular distribution of positrons and electrons by calculating $d\omega_1'/d\omega_1$ by means of the usual formulas

[2] In a forthcoming issue of the Proc. Roy. Soc. the pair production due to collisions of γ-rays and electrons with the electric fields of atoms is treated successfully by Williams by means of Eq. (21.3) which was published in an abstract of a paper read at the Washington meeting of the Physical Society (Phys. Rev. **45**, 766(A) (1934)). The formula in the abstract gives an incorrect value of σ which is twice as large as that given here. Correct results are obtained by Williams using $n_1n_2\sigma/(2A)$ and σ as in (21.3).

for the transformation of momentum and energy. The collision cross section for pair production over all angles depends only on $\int \sigma' d\omega_1$ and one may thus use Eqs. (21) for the calculation of these cross sections for any angle between light quanta by applying

$$\bar{\sigma}' = \sin^2(\varphi/2)\bar{\sigma}((\nu_1'\nu_2')^{\frac{1}{2}} \sin(\varphi/2)). \quad (23')$$

The polarizations of the light quanta are changed by the Lorentz transformation and thus only (21.3) has in general a simple meaning.[3]

Using Eq. (17′), comparing it with Eq. (18′) and Eq. (21.3), one obtains the probability of recombination per unit volume per second as

$$c\rho_e\rho_p(\bar{\sigma}/2)(C/S)$$

in the frame of zero momentum in terms of the electron and positron densities ρ_e, ρ_p. Transforming to a frame in which the electrons are at rest one has

$$\rho_e' = (1-\beta^2)^{\frac{1}{2}}\rho_e, \quad \rho_p' = (1+\beta^2)(1-\beta^2)^{-\frac{1}{2}}\rho_p, \quad \beta = v/c$$

and $\quad \rho_e\rho_p = (1+\beta^2)^{-1}\rho_e'\rho_p' = [C^2/(C^2+S^2)]\rho_e'\rho_p'.$

As for pair productions the number of recombinations per unit volume per unit time is Lorentz invariant and thus in K' (system where electron is at rest) this number per unit electron and positron density is

$$c(\bar{\sigma}/2)C^3/[S(C^2+S^2)].$$

This is Dirac's recombination formula with Dirac's $\alpha = \cosh 2\theta$. One could also derive (21.3) from Dirac's recombination formula and the relations (17′), (18′). The other formulas (20), (21) require, however, the more detailed calculations, the results of which were reported above.

As has been reported at the Washington meeting, pair production due to collisions of cosmic rays with the temperature radiation of interstellar space is much too small to be of any interest. We do not give the explicit calculations, since the result is due to the orders of magnitude rather than exact relations. It is also hopeless to try to observe the pair formation in laboratory experiments with two beams of x-rays or γ-rays meeting each other on account of the smallness of σ and the insufficiently large available densities of quanta. In the considerations of Williams, however, the large nuclear electric fields lead to large densities of quanta in moving frames of reference. This, together with the large number of nucleii available in unit volume of ordinary materials, increases the effect to observable amounts. Analyzing the field of the nucleus into quanta by a procedure similar to that of v. Weizsäcker,[4] he finds that if one quantum $h\nu$ per cm^2 is incident on a nucleus of charge Ze then the number of pairs produced is[5]

$$\sigma(\xi) = (4/\pi)Z^2\alpha \int_1^{(\xi/2)^{\frac{1}{2}}} \sigma(C) \log \{(\xi/2)^{\frac{1}{2}}C^{-1}\} C^{-1}dC,$$

where $\sigma(C)$ is given by Eq. (21.3), $h\nu = mc^2\xi$ and $\alpha = 2\pi e^2/hc$. The evaluation of the integral shows that $\sigma(\xi)$ is in asymptotic agreement with the corresponding formula of Heitler and Sauter[6] for high ξ.

[3] Two light waves polarized parallel or perpendicular to each other retain their relative polarization when viewed from another frame of reference if they travel in the same direction. If, however, they travel in opposite directions the relative polarization is in general changed. On the other hand, an unpolarized beam remains unpolarized when viewed from any frame of reference. Thus Eq. (21.3) in conjunction with (23′) always applies to the collision of a quantum with quanta having random polarizations.

For quanta colliding head-on the relative polarizations are the same as in the frame of zero momentum, and for such quanta Eq. (23′) with σ as given by Eqs. (21.1), (21.2) may be applied directly to the calculation of collisions between quanta polarized parallel or perpendicular to each other whether the total momentum is zero or not.

[4] C. F. v. Weizsäcker, Zeits. f. Physik **88**, 612 (1934).

[5] We are very much indebted to Dr. E. J. Williams for permission to quote his results.

[6] W. Heitler and F. Sauter, Nature **132**, 892 (1933). Cf. also J. R. Oppenheimer and M. S. Plesset, Phys. Rev. **44**, 53 (1933).