Critical Remarks on a Paper by G. Lemaitre and M. S. Vallarta on Cosmic Radiation

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1. INTRODUCTION

IN a recent paper Lemaitre and Vallarta[1] have discussed the latitude effect in cosmic rays discovered by Clay and Berlage[2] and confirmed by Compton's extensive measurements.[3] LV make use of my mathematical theory of the motion of an electrical particle in the field of a magnetic doublet. Their paper is very difficult to understand even for those who are familiar with the subject, due partly to real errors and partly to misprints. I think therefore it serves a useful purpose to give a detailed criticism of the paper, and to put it into historical perspective as far as the theory of electronic motions in the field of a magnetic doublet is concerned. In a later paper I hope to show how far my earlier researches in this field may be taken over in the theory of cosmic rays.

2. ON THE FUNDAMENTAL PRINCIPLES USED BY LEMAITRE AND VALLARTA

As to the fundamental principles enumerated in Section II of their paper LV say: "If now we assume that the intensity distribution of cosmic radiation at infinity is homogeneous and isotropic, the intensity in all allowed directions at any point in the earth's magnetic field is, by Liouville's theorem the same." This statement is given without further proof as if it were a direct consequence of Liouville's theorem. It has been pointed out by W. Swann[4] that Liouville's theorem cannot be applied in the manner of LV because they neglected the fact that the canonical momenta entering Liouville's theorem contain the magnetic vector potential; but Swann says that in a modified form Liouville's theorem can still be applied to an infinitely small tube of orbits. But this correction does not give the full story. The essential difficulty met with in the application of Liouville's theorem to this problem in the manner of LV is that the question, which directions are allowed and which are forbidden, is not decided by Eq. (14) and by the LV studies of asymptotic orbits only, but requires a detailed study of the shape of all the orbits from infinity, as function of their initial conditions.

If one remembers the immensely great variety of complicated orbits which may occur,[5] spiral orbits and nearly periodic ones and so on, one is warned to be very cautious in drawing general conclusions which are not well founded.

They say further: "Thus the question of calculating the intensity at any point at the earth's surface reduces to that of finding out in which directions particles coming from infinity can reach that point. There are as we shall see, three possibilities, either all directions are forbidden, or all directions are allowed, or only certain directions are allowed and the rest forbidden." The occurrence of forbidden directions was pointed out in my paper[6] many years ago. It was deduced from a formula found in 1904,[7] which is the same formula (14) given by LV. As is well known, the allowed and forbidden directions have also played an important part in the application to the aurora borealis.

3. THE EQUATION OF MOTION

The first error in LV comes from the sign of the earth's magnetic field. In fact the magnetic

[1] G. Lemaitre and M. S. Vallarta, Phys. Rev. **43**, 87 (1933). In the following we shall refer to this paper or to the authors as LV simply.

[2] J. Clay and H. P. Berlage, Proc. Roy. Acad. (Amsterdam) **30**, 1115 (1927); **31**, 1091 (1928); **33**, 711 (1930) and Naturwiss. **20**, 687 (1932).

[3] A. H. Compton, Phys. Rev. **41**, 111, 681 (1932) and a paper presented at the Chicago Meeting of the American Physical Society, November 25, 1932.

[4] W. F. G. Swann, Phys. Rev. **44**, 224 (1933).

[5] Carl Störmer, Vid. Selsk. Skr. Christiania (Oslo) 1913; Zeits. f. Astrophysik **1**, 237 (1930) and *How the horseshoe-formed auroral curtains can be explained by the corpuscular theory*, Terr. Magn. and Atm. Elec. **37**, 375 (1932).

[6] Carl Störmer, Vid. Selsk. Skr. (1904).

[7] Reference 6, formula VI.

pole in the northern hemisphere is not a north pole, but a south pole, which gives for H_r and H_z the *opposite* sign of that in their paper. This has the consequence that the equations of motions (1), (2) and (3) come out with wrong signs for the second members.

Another less important thing is the fact that LV consider the axis of the earth's magnetic field to coincide with the axis through the *magnetic poles of the earth*. It is well known, however, and beautifully illustrated by the distribution of aurorae, that in the first approximation due to Gauss's series for the magnetical potential, the magnetic axis cuts the earth's surface in the neighborhood of Smith's Sound in northwestern Greenland, halfway between the magnetic and geographical poles.

As to the deduction of the equations of motion, which is not given in LV's paper, it can be done by general principles or by specializing the general equations I have published in a paper in 1912.[8]

As the Eq. (4) is deduced from the equations of motion the same wrong sign comes out here. As to the splitting up of the motion and the Eq. (5) references to my papers are given.

In the Eq. (7) there is a misprint. In fact the factor 1/2 on the left side has to be dropped. This equation is not different however from the equation VIII,7 which I published in 1907,[9]

$$r_1^4[\tfrac{1}{2}d^2(r_1^2)/ds_1^2-1]=\cos^2\lambda-\alpha^2 r_1$$

where r_1 and the arc of the trajectory s_1 are measured with

$c_1=(M|e|/mv)^{\frac{1}{2}}$ centimeters

($|e|$ the absolute value of e)

as unit of length and where α^2 is a constant of integration. In fact if we introduce here $r=c_1r_1$, $t=c_1s_1/v$ we get

$$\frac{d(r\dot{r})}{dt}-v^2=\frac{e^2M^2}{m^2}\frac{1}{r^4}\left(-\frac{\alpha^2}{c_1}r+\cos^2\lambda\right)$$

and if we call with LV 2γ the constant $-\alpha^2/c_1$ their Eq. (7) with correct first member comes out. It is to be pointed out that the constant 2γ used by LV is not the same as the constant 2γ used in my papers.

The next step, the deduction of the Eq. (8) and the resulting Eqs. (9) and (10) is an original contribution of value in LV's paper. In Eq. (9) there is, however, a new misprint: the sign for the integral in the numerator has to be changed. Furthermore the constant of integration C in the Eq. (10) is not necessarily identical with the constant C in Eq. (9); that depends on the limits adopted for the indefinite integrals.

4. The Normalized Coordinates and Formula (14)

The introduction of what LV call normalized coordinates is the same transformation I used in my paper from 1907 and what they call x is thus the length of the radius vector measured with the unit of length c_1 introduced in the foregoing section.

On account of the wrong sign for the magnetic components the *formula (14) in LV's paper is also wrong*. The sign must be chosen in such a manner that + corresponds to positive and − to negative charges. The constant γ_1 in their formula is identical with the constant γ_1 in my papers (equal to $-\gamma$ where γ is the constant I have introduced).[10]

It is useful to compare the corrected LV formula with my formula of 1907.[10]

My formula is

$$\sin\theta=-2\gamma_1/R+R/r^3$$

and if we introduce $R=r\cos\lambda$:

$$\sin\theta=-2\gamma_1/r\cos\lambda+\cos\lambda/r^2.$$

The corrected LV formula is for negative particles

$$-\sin\theta=-2\gamma_1/x\cos\lambda+\cos\lambda/x^2,$$

which coincide with my formula if we remark that x is the same as r and the positive direction of θ is the opposite one (Figs. 1 and 2).

5. Remarks on Section IV

This section is very difficult to understand even for readers who are familiar with my papers

[8] Carl Störmer, Vid. Skr. Christiania 1912, Eqs. (V).

[9] Carl Störmer, Archives des sciences phys. et natur. **24** (1907).

[10] Reference 9, formula (3).

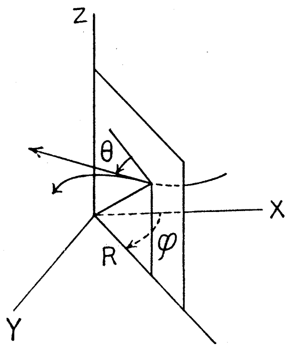

FIG. 1. Positive directions of θ and φ in my papers.

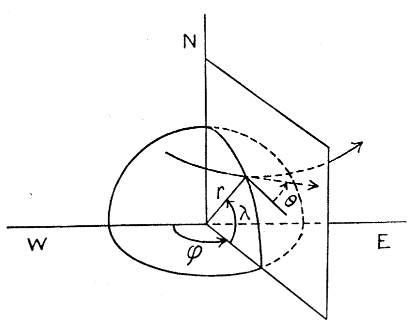

FIG. 2. Positive directions of θ and φ in LV.

on the subject. There is a great lack of precision and the exposition is very obscure.

The first part, the discussion of the regions of the earth's surface into which no particles can enter, is a repetition of a discussion which I have already given in my paper from 1907. A formula for the limiting value λ_1 is also given explicitly in my paper on the problems of the aurora.[11]

Further they say: "For latitudes greater than λ_1 and values $x_0<1$ there are trajectories coming from infinity but they have a limit and this limiting trajectory must be asymptotic to a periodic orbit." It can scarcely be maintained that this sentence conveys a definite meaning. It looks as if the authors mean that an asymptotic trajectory forms a limit between the trajectories reaching the earth and those not doing it. This, however, is not the case. Such questions were discussed by me in a paper published in 1911,[12] where I have found the conditions which they suggest. I content myself with that reference, hoping to come back to the question in a later communication.[15]

Then comes a reference to my paper on periodic orbits where the following explanation is given: "A good approximation to the mean value of x for a periodic orbit can be found directly from (9) in agreement with the numerical computations of Störmer. By neglecting the integral and using a mean value for λ, the constant C in the denominator of Eq. (9) must be so chosen that this denominator has a double root."

We will try to explain what they mean: They first presuppose that the Eq. (11) is brought to a reduced form by introducing normalized coordinates and the value γ_1. We have $r=c_1x$, $\gamma=-\gamma_1/c_1$, which gives

$$\frac{x^2d\lambda^2}{dx^2}=\frac{-\dfrac{4\gamma_1^2}{\cos^2\lambda}-\dfrac{\cos^2\lambda}{x^2}+C_1-2\displaystyle\int^{\circ}\frac{\cos^2\lambda}{x^3}dx}{x^2+\dfrac{4\gamma_1}{x}-C_1+2\displaystyle\int\frac{\cos^2\lambda}{x^3}dx},$$

where $C_1=Cc_1^2$, C_1 is called C in their paper.

If we now, as they propose, neglect, that is, drop, the integral, the denominator will be $x^2+4\gamma_1/x-C_1$.

But this cannot be their meaning, for the condition of a double root here gives a condition different from their Eq. (17).

If we however use a *constant* mean value of $\cos^2\lambda$ given by the Eqs. (15) and (16) we get:

$$2\int(\cos^2\lambda/x^3)dx=-\cos^2\lambda/x$$

and

$$\frac{x^2d\lambda^2}{dx^2}=\frac{C_1-4\gamma_1^2/\cos^2\lambda}{x^2+4\gamma_1/x-C_1-\cos^2\lambda/x^2}$$

and the condition for a double root x_p in the denominator is now

$$x_p-2\gamma_1/x_p^2+\cos^2\lambda/x_p^3=0,$$

which reduces to the Eq. (17) if we use the Eqs. (15) and (16).

In this connection it may be pointed out that a more natural way to this and to more general results is given in my paper of 1931.[13] It is sufficient to specialize the results in the third part of my

[11] Carl Störmer, *Ergebnisse der kosmischen Physik,* Akad. Verlagsbuch. 1931, p. 56.
[12] Carl Störmer, Archiv f. Math. og Naturv. **31**, No. 11, (1911).
[13] Carl Störmer, Zeits. f. Astrophysik **3**, 31 (1931).

paper[14] which gives the approximate system:

$$(dx/d\sigma)^2 = x^4 - C_1 x^2 + 4\gamma^2{}_1 x - \alpha^2,$$
$$(d\lambda/d\sigma)^2 = C_1 - 4\gamma_1{}^2/\cos^2\lambda,$$

where α^2 is just such a mean value of $\cos^2\lambda$ and where σ is related to the arc s_1 of the trajectory by the equation $ds_1 = x^2 d\sigma$.

This system has great advantages for the study of the asymptotic trajectories and of the periodic orbits to which they approach. We will come back to this question in a later paper.[15]

As to the Eq. (16) in LV it is already given in my general paper from 1907 §6, as formula for $\cos\psi'$ for $k_1 = 1$.

The formula (18) seems to be wrong. The correct formula should be

$$C_1 = 2x_p{}^2 + 2\gamma_1 x_p{}^{-1}.$$

As to the value of γ_1 for which the periodic orbits disappear their estimate $\gamma_1 = 0.783$ is very good. See my paper.[15]

[14] Carl Störmer, Zeits. f. Astrophysik **4**, 290 (1932).

[15] Carl Störmer, *On the Trajectories of Electric Particles in the Field of a Magnetic Dipole with Applications to the Theory of Cosmic Radiation*, first and second communication, Publications from Oslo University Observatory No. 10 and No. 12. Oslo, 1934.

On the next page LV say: "We have carried out the numerical integration of Eq. (10) for $\gamma_1 = 0.911$ corresponding to $\lambda_m = 20°$, beginning at $\log x = -0.04$ and $\lambda = 14.14°$ and decreasing values of λ. From the results of this integration we have made estimates of the inclination with the radius vector of the asymptotic family of trajectories passing through points of coordinates $x = 0.5$, 0.6, 0.7, 0.8, and 0.9 both for $\lambda = 0$ (equator) and 10°. For each pair of values x_0 and λ we know three points of the cone, i.e., the value of θ_1 for $\gamma_1 = 1$ for which the angle $\eta = 0$, for $\gamma_1 = 0.911$ for which we have estimated values of η as described above, and θ_3 for $\gamma_1 = 0.783$, for which $\eta = 90°$."

Here the exposition is so obscure that it is very difficult to guess their meaning. It would be a delusion to believe it possible from the numerical integration of a few orbits to be able to make any trustworthy estimates in the way attempted by LV.

It seems, on the whole that the numerical results on the intensity distribution of cosmic radiation given by LV must be illusory.