Sidelights on Electromagnetic Theory

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The electromagnetic equations may be derived from a variational principle by expressing the field in terms of the scalar parameters of a number of constituent fields each of which can be described by means of moving lines of magnetic induction. One set of field equations is then satisfied identically and the other set is derived from the variational principle by varying independently the parameters of all the constituent fields. The advantage of the use of scalar parameters is that the stress tensor occurring in the laws of conservation may be obtained by means of a simplified form of Schrödinger's rule. The method is extended to the case in which the Lagrangian function is an arbitrary function of two relativistic invariants and some remarks are made on Born's new electrodynamics.

§1.

THE electromagnetic equations may be derived from a variational principle[1]

$$\delta\int Ldx=0,\quad L=\tfrac{1}{2}(B^2-E^2)$$

in many ways. The usual plan is to express the antisymmetric 6-vector f_{kl}, which specifies the field, in terms of a 4-vector ϕ_k so that one set of 4 electromagnetic equations is a result of the expressions chosen. The other set of 4 electromagnetic equations is then obtained from the Eulerian equations of the variation problem when ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are varied independently. Another plan is to regard the field as the result of superposing a number of elementary fields each of which can be described by means of moving lines of magnetic induction and their associated parameters α, β, which are scalar quantities.[2] One set of field equations is then a consequence of the expressions

$$f_{kl}=\sum\partial(\alpha,\beta)/\partial(x_k,x_l),\tag{1}$$

in which the summation extends over the parameters of the different constituent fields. By varying these parameters independently we obtain a number of pairs of equations of the type

$$\sum_{s=1}^{4}(\partial\alpha/\partial x_s)\sum_{r=1}^{4}(\partial f_{rs}/\partial x_r)=0,$$
$$\sum_{s=1}^{4}(\partial\beta/\partial x_s)\sum_{r=1}^{4}(\partial f_{rs}/\partial x_r)=0.$$

If there are at least two pairs of parameters α, β; α', β' which are all independent so that their Jacobian is not zero, the set of Eulerian equations can only hold simultaneously if the field equations

$$\sum_{r=1}^{4}(\partial f_{rs}/\partial x_r)=0\quad(s=1,2,3,4)\tag{2}$$

are satisfied. When, however, the Jacobian is zero for each pair of parameters it is possible to find a 4-vector w_s such that

$$\sum_{r=1}^{4}(\partial f_{rs}/\partial x_r)=w_s\quad(s=1,2,3,4)\tag{3}$$

and

$$\sum_{s=1}^{4}w_s(\partial\alpha/\partial x_s)=0,\quad \sum_{s=1}^{4}w_s(\partial\beta/\partial x_s)=0\tag{4}$$

for each pair of parameters. These equations give the relations

$$\sum_{s=1}^{4}w_sf_{rs}=0\qquad(r=1,2,3,4),\tag{5}$$

which can be interpreted to mean that the total force on each element of electric charge is null.

The particular form of electromagnetic theory in which the field vectors satisfy Eqs. (3), (5) and the relations deduced from Eq. (1) has been discussed by only a few writers.[3] Though some interesting solutions have been obtained a con-

[1] The notation is that of M. Born in his paper on the quantum theory of the electromagnetic field, Proc. Roy. Soc. (London) **A143**, 410 (1934). dx is an abbreviation for $dx_0dx_1dx_2dx_3$ and $x_4=ix_0=ict$, $B=(f_{23},f_{31},f_{12})$, $E=i(f_{14},f_{24},f_{34})$.

[2] The general theory of elementary fields of this type is given in the author's paper cited later. The idea that moving lines of magnetic induction may represent quanta has been developed by H. Stanley Allen, Phil. Mag. **48**, 429 (1924).

[3] T. Levi Civita, Comptes Rendus **145**, 417 (1907), Ven. Ist. Atti **67**, 995 (1908), Rend. Lincei **18**, 83 (190

nection with the ideas of modern physics has not been established. We shall be content here to indicate the extension of the theory in which a more general Lagrangian function is used. Before making this extension, however, we shall say a few words about the laws of conservation and the derivation of the stress-energy-tensor in the classical case.

The familiar stress-energy tensor of electromagnetic theory is obtained by applying Schrödinger's rule in the form[4]

$$T_{kl}=L\delta_{kl}-\sum[\alpha_k(\partial L/\partial\alpha_l)+\beta_k(\partial L/\partial\beta_l)], \quad (6)$$

where $\delta_{kl}=0$ when $k\neq 1$ and $\delta_{kk}=1$. The summation extends over all pairs of parameters of type α, β. It is noteworthy that this rule gives Mie's formula[5]

$$T_{kl}=L\delta_{kl}+\sum f_{kn}p_{nl}=T_{lk},$$

wherein

$$p_{nl}=\partial L/\partial f_{nl}.$$

It should be noticed that our expression differs in the sign of the second term from that given by Born and that when a similar correction is made in his second tensor it becomes[6]

$$S_{kl}=H\delta_{kl}+\sum p_{kn}{}^*f_{nl}{}^*=S_{lk}.$$

The two tensors differ only in sign. Born's plan of taking half the difference of the two tensors as the correct tensor means that either tensor may be used, with the appropriate sign. It is noteworthy that the form of the tensor is independent of the number of elementary fields from which the total field is supposed to be built up. This is true also if we regard the potentials ϕ_k as sums of elementary potentials which can be varied arbitrarily; the tensor can be calculated then either by Schrödinger's rule or by Mie's rule. So far as the tensor is concerned it is immaterial how the first set of electromagnetic equations is satisfied in the formulation of the variation problem.

This property of the tensor seems to hold also when L is an arbitrary function of the two relativistic invariants

$$\sigma=\tfrac{1}{2}(\mathbf{B}^2-\mathbf{E}^2), \quad \tau=\mathbf{E}\cdot\mathbf{B}.$$

The analysis for this case is exactly the same as that already given except that Eqs. (2) and (3) are replaced by

$$\textstyle\sum_{r=1}^4(\partial p_{rs}/\partial x_r)=0 \quad (s=1, 2, 3, 4) \quad (2')$$

and

$$\textstyle\sum_{r=1}^4(\partial p_{rs}/\partial x_r)=w_s \quad (s=1, 2, 3, 4) \quad (3')$$

respectively. The type of electromagnetic field which corresponds to the kind which Levi-Civita calls pure is now given by Eqs. (1), (3) and (5). If, moreover, the Lagrangian function is that adopted by Born

$$L=(1/a^2)[(1+a^2F)^{\frac{1}{2}}-1],$$

where a is a constant and

$$F=\textstyle\sum_{k>l}f_{kl}{}^2=2\sigma,$$

we have $\tau=0$ and in attempting to solve the equations we need only use one pair of parameters α, β. Writing u for $\partial L/\partial\sigma$ we have to solve the equations

$$\textstyle\sum_{s=1}^4 w_s\alpha_s=0, \quad \sum_{s=1}^4 w_s\beta_s=0,$$

which, when written in full are

$$\textstyle\sum_{s=1}^4\alpha_s\sum_{r=1}^4(\partial/\partial x_r)[u(\alpha_r\beta_s-\alpha_s\beta_r)]=0,$$
$$\textstyle\sum_{s=1}^4\beta_s\sum_{r=1}^4(\partial/\partial x_r)[u(\alpha_r\beta_s-\alpha_s\beta_r)]=0.$$

It should be noticed that if we write $h_{mn}=f_{mn}+ig_{mn}{}^*$ where $g_{mn}{}^*$ is the dual of g_{mn},

$$f_{m,\,n}=\partial(\alpha,\beta)/\partial(x_m, x_n),$$
$$g_{m,\,n}=\partial(\alpha',\beta')/\partial(x_m, x_n)$$

and consider a variational principle in which

$L=\sum_{m>n}h_{mn}{}^2$

and α, β, α', β' are varied independently

L. Caffarati Nuove Cimento (5) **15**, 369; **16**, 376 (1907–1908); V. Cisotti, Ven. Ist. Atti **68**, 361 (1909), Nuovo Cimento (5) **17**, 103 (1909); H. Bateman, Mess. of Math. **46**, 136 (1917); W. F. G. Swann, Bull. Nat. Res. Council, Vol. 4, No. 24, p. 27 (1922).

[4] We use α_k to denote $\partial\alpha/\partial x_k$. The rule was given in Ann. d. Physik **82**, 265 (1927). A special application of this form of the rule was made by M. Abraham, Phys. Zeits. **13**, 1 (1912), but the connection with a variation problem was not indicated. It is easy to verify that when L is the Lagrangian function of a variation problem the laws of conservation are satisfied by this tensor on account of the Eulerian equations of the variation problem.

[5] G. Mie, Ann. d. Physik **40**, 1 (1913). The explicit expressions in terms of L are given by Born but are easily derived from those on p. 34 of Mie's paper.

[6] f^* is the dual tensor of f, obtained by changing the suffixes 23, 31, 12 with 14, 24, 34, respectively; similarly p^* is the dual of p and $H=-L+\Sigma_{k>l}p_{kl}f_{kl}$.

we are led to a set of equations which are satisfied when the field equations

$$\sum(\partial h_{m,\,n}/\partial x_m)=0,\quad \sum(dh_{mn}{}^*/\partial x_m)=0$$

are satisfied but which generally imply that

$$\sum(\partial h_{mn}/\partial x_m)=u_n,\quad \sum(\partial h_{mn}{}^*/\partial x_m)=v_n,$$

$$\sum u_n(\partial\alpha/\partial x_n)=0,\quad \sum v_n(\partial\alpha'/\partial x_n)=0,$$

$$\sum u_n(\partial\beta/\partial x_n)=0,\quad \sum v_n(\partial\beta'/\partial x_n)=0.$$

This means that both electricity and magnetism are present at the same time and move in such a way that a line $\alpha=$ constant $\beta=$ constant moves with the electricity while a line $\alpha'=$ constant, $\beta'=$ constant moves with the magnetism. Schrödinger's rule fails to give a tensor depending only on h just as it did in the analogous cases considered before.[7] The failure of the rule can, perhaps, be used as a reason for excluding fields in which both electricity and magnetism occur. It may mean, however, that the addition of the components of two tensors f_{mn} and $ig_{mn}{}^*$ is to be ruled out for mathematical reasons.

§2.

When Born's solution of Eqs. (2) is combined with the corrected form of the stress-energy-tensor[8] it is found that the energy of the elementary charge is equal to $4\pi m_0c^2$. The expression for the energy density is in fact

$$W=L+2E^2(\partial L/\partial F),\quad F=B^2-E^2.$$

Now in Born's solution

$$B=0,\quad E=(a^2+r^4e^{-2})^{-\frac{1}{2}},$$

$$2(\partial L/\partial F)=(1+a^2F)^{-\frac{1}{2}}.$$

Hence

$$(1-a^2E^2)^{\frac{1}{2}}=r^2(r^4+a^2e^2)^{-\frac{1}{2}},\quad Wa^2=(1-a^2E^2)^{-\frac{1}{2}}-1$$

and so

$$\int_0^\infty r^2Wdr=a^{-2}\int_0^\infty[(r^4+a^2e^2)^{\frac{1}{2}}-r^2]dr$$

$$=(e^3/a)^{\frac{1}{2}}\int_0^\infty[(1+x^4)^{\frac{1}{2}}-x^2]dx$$

$$=(e^3/a)^{\frac{1}{2}}[\{x(1+x^4)^{\frac{1}{2}}-x^3\}_0{}^\infty$$

$$+2\int_0^\infty x^2dx[1-x^2(1+x^4)^{-\frac{1}{2}}]].$$

On the other hand, Born has shown that

$$m_0c^2=(e^3/a)^{\frac{1}{2}}\int_0^\infty x^2dx[1-x^2(1+x^4)^{-\frac{1}{2}}].$$

Hence

$$\int_0^\infty r^2Wdr=m_0c^2.$$

Allowing for the factor 4π, which occurs in Born's definition of m_0 we can say that the energy is connected with the mass by Einstein's relation.

When the Lagrangian function has Born's form the solution for the case of homogeneous plane waves travelling with the speed of light is just the same as in the classical case.[9] Another solution, which is easily obtained, is that of an electric current along an infinite straight wire which we take for the axis of z. In this case we may write $E=0$, $B_x=-yf(\rho)$, $B_y=xf(\rho)$, $B_z=0$, where $\rho^2=x^2+y^2$. Since $B_xdx+B_ydy+B_zdz=\rho^2f(\rho)d\theta$, where $x\tan\theta=y$, we see that curl $(uB)=0$ if $u\rho^2f(\rho)$ is a constant or a function of θ. Now B^2 depends only on ρ and so the quantity u, which denotes $(\partial L/\partial\sigma)$, is also a function of ρ only, hence $u\rho^2f(\rho)$ is a constant, C, and we have the equation

$$\rho^2f(\rho)(1+a^2\rho^2[f(\rho)]^2)^{-\frac{1}{2}}=C;$$

$$\therefore\quad \rho^4[f(\rho)]^2=C^2[1+a^2\rho^2\{f(\rho)\}^2]$$

or

$$\rho f(\rho)=C[\rho^2-a^2C^2]^{-\frac{1}{2}}.$$

It should be noticed that the magnetic induction becomes infinite on the cylinder $\rho=aC$ while the magnetic force does not. The magnetic force, however, is infinite when $\rho=0$.

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[7] H. Bateman, Proc. Nat. Acad. Sci. **13**, 326 (1927). Phys. Rev. **43**, 481 (1933).

[8] The need of a correction was observed independently by Professor W. V. Houston. The formulae have been corrected and the expression for m_0 revised in a recent paper by M. Born and L. Infeld, Proc. Roy. Soc. (London) **A144**, 425 (1934).

[9] Since $B^2=E^2$ we have $F=0$ and so $u=2\partial L/\partial F=1$. We have then $\mathbf{H}\equiv\mathbf{B}$, $\mathbf{D}\equiv\mathbf{E}$. In a recent letter, Nature, Jan. 13, 1934, Born states that $\mathbf{H}\equiv\mathbf{B}$, $\mathbf{D}\equiv\mathbf{E}$ for very weak fields only. A plane electromagnetic wave would, presumably, be regarded from the physical standpoint as a weak field but from a mathematical standpoint B^2 and E^2 can be arbitrarily large. Perhaps quantum conditions would limit their magnitude in a plane wave.