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Confirmation of Crystal Wave-Length Measurements and Determination of $h/e^{4/3}$

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By observation of x-ray continuous spectrum isochromats the potential necessary for the production of general radiation capable of reflection from calcite at a known Bragg angle θ has been determined. By using a double spectrometer and tungsten source, and reflecting at the Bragg angle for Ag $K\alpha_1$ the value $V' \sin \theta = 2035.3 \pm 0.2$ international volts has been obtained. This is very close to Feder's value, 2036.0. In agreement with Feder it is concluded that Wagner's "low" value is in error because of experimental conditions and that this error can not now be rectified. It is shown that the "high" value of Duane, Palmer and Yeh is subject to certain calculable corrections which lower it to 2034.6, in much closer agreement with present results. The wave-length of Ag $K\alpha_1$ as given by the law of Duane and Hunt is found to be in agreement with the value obtained from crystal measurements and at variance with ruled grating results. Utilizing crystal wave-length measurements by Bragg's law the present results yield $h/e^{4/3} = (1.7559 \pm 0.0002) \times 10^{-14}$ if Birge's (1929) auxiliary constants are employed, and 1.7563 with more recently determined values of five of these constants. Assuming the 1930 oil drop value $e = (4.770 \pm 0.005) \times 10^{-10}$ e.s.u. this investigation gives $h = (6.546 \pm 0.006) \times 10^{-27}$ erg·sec.

INTRODUCTION

SINCE the equations which are useful for determining the values of the more fundamental physical constants frequently contain more than one such constant it is advantageous to evaluate the several constants together, as in the method of Bond[1] for e, h and m, by using numerical data from a variety of experiments, each concerned with the investigation of some one of the several available relations which connect these constants. In this situation it is very desirable that all such relations which are to participate in a final evaluation be known with something like comparable accuracy, as otherwise the superior accuracy of some will not be fully effective.

Among the relations connecting h and e the value of $h/e^{5/3}$, obtained from the Rydberg constant, is known with outstanding accuracy, a fact which makes it very desirable to improve the precision of the other accessible relations. The x-ray experiment presently to be described offers this possibility and has been performed to this end several times in the past, but unfortunately these past investigations (to be discussed in a later section) are not in good agreement. Kirchner[2] has recently emphasized the desirability of new work upon the relation between h, and e by this method.

The minimum wave-length of the continuous x-ray spectrum is related to the cathode electron potential fall by the equation $Ve = hc/\lambda$. Upon measuring corresponding values of V and λ one obtains a numerical value of a relation between h and e. If λ has been measured by an absolute method, for example with a ruled grating, this relation is simply h/e. However, if the value of λ is got by crystal reflection it is dependent upon the crystal grating constant which, at present, can be evaluated only through a process involving e in such a way that one obtains finally a numerical value of the ratio $h/e^{4/3}$. The existing discrepancy between wave-length measurements by grating and crystal methods makes a premature commitment to either method inadvisable and we shall defer this decision and state experimental results in terms of the more directly measured quantity $V' \sin \theta$, the product of electron potential fall (in international volts) by the sine of the Bragg angle employed in our experiment. Since the V' and θ thus combined are associated with vanishing x-ray intensity we cannot realize them by a direct observation but only by extrapolating to zero intensity a curve of intensity against either θ or V'. We have employed the latter method, that is, θ has been maintained at a single fixed value and curves of

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[1] Bond, Phil. Mag. **10**, 994 (1930); **12**, 632 (1931).

[2] Kirchner, Ann. d. Physik **13**, 59 (1932).

reflected intensity against tube potential (isochromats) have been observed through a voltage range including V'.

Experimental Procedure

A double spectrometer with large polished calcite crystals was used. Very little incoherent scattering comes through the double spectrometer so the backgrounds of the isochromats were kept quite faint. The x-ray tube used in preliminary runs had not been outgassed by heating, and was operated in continuous connection with its pumps. It was found impossible to keep the target of this tube sufficiently clean to avoid electron retardation in the contaminating surface layer. Upon starting with a clean target this layer steadily increased in thickness, as evidenced by a steady shift of successive isochromats toward higher voltages. It is true that the initial voltage for a given wave-length should be independent of the target material, whether metal or grease residue. The *slopes* of the isochromats are very different in these two cases, however, so that the effect of a dirty target is to produce a shifted isochromat from the underlying metal with a small foot of very much reduced slope, vanishing at the true critical voltage for the wave-length received. Since in all cases isochromats possess feet due to other causes it is out of the question to try to interpret quantitatively the effect of the target uncleanliness and the only alternative is to eliminate it. We have therefore obtained all final isochromats from a G.E. deep therapy tube. These tubes are baked out for several hours while under load in the process of manufacture and even old tubes show no visible contamination on the focal spot. The tungsten target also furnishes a relatively strong continuous spectrum.

Tube potential was supplied by rectified and filtered 580 cycle a.c. with an approximately sinusoidal ripple of less than 5 volts under our conditions of operation. Voltage was controlled by a rheostat in the field circuit of the 580 cycle generator. For voltage measurement a resistance was connected in parallel with the x-ray tube consisting of about 6 megohms of "copnic" wire (R_M) with a manganin-wound decade box (R_B), cut in at its center. The resistance of this central box was adjusted to be a known fraction (about 1/10,000) of that of the remaining 6 megohms and the potential drop across the central box was measured with a potentiometer. Whenever a spectrometer observation was in progress an assistant kept the potentiometer in balance at a preselected setting by manipulating the generator field rheostat. The magnitude of the fluctuations of the potentiometer galvanometer indicated that the high potential fluctuations were kept within 2 volts of the desired value throughout any one spectrometer observation.

This method of high potential measurement does not require knowledge of the absolute values of any resistances but does call for precise adjustment of the ratio of the resistances R_M and R_B. This adjustment was accomplished in two steps by Wheatstone bridge operations, making use of a pair of standard resistances as ratio arms whose nominal values were 100 and 10,000 ohms and whose more exact values were given by Bureau of Standards certificates. The purpose of the first bridge was to adjust a box resistance R_S, to a value one rth that of R_M, where r is the ratio of the larger standard resistance to the smaller. Similarly the second bridge permitted R_B to be adjusted to one rth of R_S or $1/r^2$ times R_M. The sensitivity of the bridge was such that a change of one part in 50,000 in the variable arm of the balanced bridge resulted in a readable deflection of the potentiometer galvanometer from its null position. Due precautions against thermal e.m.f.'s and resistance changes in the bridges were observed. The bridge balancing operations were performed before and after each set of isochromat observations. The dial readings of the boxes making up R_B were usually identical to one part in 50,000 for the two balances taken before and after a run, but from day to day these readings varied from their mean as much as one part in 10,000. If this resulted from a real change in R_M it does not impair the accuracy of the voltage measurement, and the fact that no significant correlation exists between the residuals of R and of V' for the several isochromats tends to indicate that this is the case. Nevertheless this variation presents the largest single source of uncertainty in our final voltage determination.

Corrections

The potential difference V' is not identical with the measured drop across R_M and R_B, and the several essential corrections will now be described. As Fig. 1 shows, a choke coil R_C, provided for protective purposes is inserted in such a position that it carries the target current of 22.0 m.a. The resistance of the choke coil and milliammeter is 3270 ohms so it is necessary to *subtract* from V_M, the drop across R_M, the 71.9 volt drop across the choke and meter. All other high tension line drops were shown by direct resistance measurements to be negligibly small.

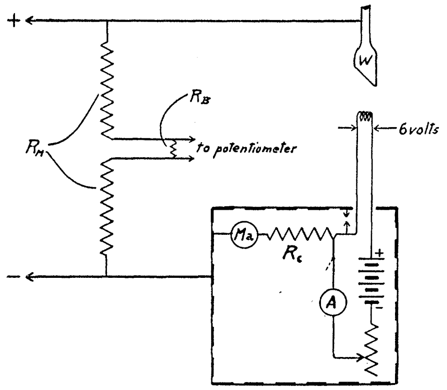

FIG. 1. Wiring diagram. The potentiometer is approximately at ground potential.

The electrons effective in producing x-rays may be emitted from any part of the filament, and since a potential drop (6 volts) occurs along the filament the potential fall from filament to target will not have a common value for all electrons. Assuming uniform emission along the filament Feder[3] has shown that the straight part of the isochromat obtained under these conditions is characteristic of the electrons from the center of the filament. Taking account of the current direction in our filament this means that 3 more volts must be *subtracted* from V_M. The adoption of an even 3 volts for this correction depends upon the assumption that the effective center of electron emission coincides with the potential center of the filament. In view of possible temperature non-uniformity along the filament and differential space charge limitations it is impossible to claim that this assumption is entirely correct. It seems unlikely, however, that the error which results is greater than a few tenths of a volt.

The electrons leaving the filament are emitted with velocities which correspond on the average to about 0.2 volt. This voltage is to be *added* to V_M.

Upon arriving at the target and entering the metal the electrons are subject to attractive forces which confer upon them an increment of energy corresponding to the work function of the target material. We therefore *add* to V_M the work function[4] of tungsten or 4.5 volts.

The potential applied to the tube was, as stated, slightly rippled. It can be shown that the isochromat produced by a potential with a small symmetrical ripple is, in its straight portion, identical with that produced by constant potential having the same average value. Since the potential measuring system measures average voltage the existence of ripple may be ignored.

A similar conclusion may be drawn regarding the effect of the finite wave-length band reflected by the spectrometer crystals. In the double spectrometer the width of this band is not determined by the slit system but is characteristic of the nature and treatment of the crystals themselves. The width and shape of the band of radiation passed by our spectrometer is known, and we have analyzed its effect upon the isochromat (a term which strictly speaking is here a misnomer) in a thorough study which is withheld here only for reasons of economy. The conclusion obtained is that the straight part of the isochromat so produced is indistinguishable (except perhaps in absolute ordinate scale) from the isochromat produced by an indefinitely narrow wave-length band at the peak (center) of the spectrometer transmission band. While as stated above this is not a slit problem the argument and conclusions are closely analogous to those which apply to slit effects, a subject to which we shall return in a later section of this paper.

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[3] Feder, Ann. d. Physik [5] **1**, 497 (1929).

[4] We are indebted to Mr. R. H. Varian for bringing to our attention the necessity for including this correction.

DETERMINATION OF θ

Errors in the measurement of the angle of reflection, θ, have been all but eliminated through the use of the Bragg angle of reflection of Ag $K\alpha_1$. This angle has been determined with great care in several different laboratories and is certainly known with higher precision than we are at present able to utilize. The mean of determinations by Leide,[5] Lang,[6] Kellström,[7] Cooksey,[8] Bearden[9] and Ross[10] when adjusted to the working temperature of this investigation (20°C) gives $\theta = 5° \ 17' \ 13.7'' \pm 0.4''$, or $\sin\theta = 0.092147 \pm 0.000002$, the index of precision being the statistical probable error of the six determinations.

To obtain this angle a silver target tube was mounted before the spectrometer, the $K\alpha_1$ line found and a setting made on its peak. The symmetry of the line was excellent and the peak could readily be located within 0.3″. After this setting the silver tube was replaced by the tungsten target therapy tube and the spectrometer was left untouched during the observation of isochromats. When these data were complete the silver line was observed again and found to be unchanged in position by any amount as large as the uncertainty of location of the peak.

DETERMINATION OF V' AND $V' \sin\theta$

Eleven isochromats (see Fig. 2) were obtained in a series of runs extending over five days. Room temperatures during these observations varied between 19 and 21°C. Observations were taken at intervals of ten or in a few cases twenty volts. The background, the curved foot and the linear isochromatic portion are sufficiently evident in the figure. In obtaining the critical voltage corresponding to the angle θ the linear portion was produced downward to its point of intersection (P) with the extrapolated background. The abscissa of P is the desired potential. Both extrapolations (for all isochromats) were performed by least squares assuming the background to be parabolic and the upper segment linear. It is unprofitable to extend these curves upward to higher voltages since the isochromat undergoes a rather abrupt decrease in slope at about 150 volts above its excitation potential, as Wagner[11] and others have shown.

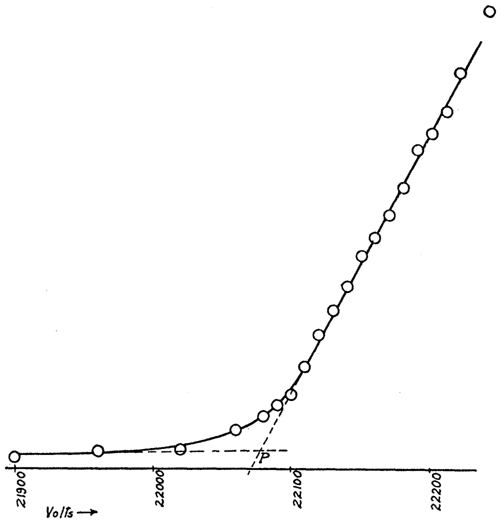

FIG. 2. One of eleven isochromats used in this investigation. Ordinates are ionization currents in arbitrary units.

The weighted mean value of V' from all data is $V' = 22087.5 \pm 1.5$ volts. The probable error attached includes the statistical probable error of the eleven values (0.005 percent) and the probable errors of all quantities entering into the calculation of the voltage and voltage corrections. Combining this result with the mean value of θ one obtains $V' \sin\theta = 2035.3 \pm 0.2$ international volts.

Before proceeding to discuss the effect of this determination upon the values of e or h it will be of interest to note (Table I) that the present

TABLE I. *Results of determinations of $V' \sin\theta$.*

Determined by	As determined	Adjusted
Duane, Palmer and Yeh	2039.9 ± 0.9	2034.6 ± 0.9
Wagner	2031.5 ± 3.0	
Feder	2036.0 ± 1.0	2036.0 ± 1.0
Kirkpatrick and Ross	2035.3 ± 0.2	

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[5] Leide, Diss. Lund. (1925).

[6] Lang, Ann. d. Physik **75**, 489 (1924).

[7] Kellström, Zeits. f. Physik **41**, 516 (1927).

[8] Cooksey, Phys. Rev. **36**, 85 (1930).

[9] Bearden, Phys. Rev. **43**, 92 (1933).

[10] Ross, Phys. Rev. **44**, 977 (1933).

[11] Wagner, Ann. d. Physik **57**, 401 (1918).

value is in rather good agreement with that of Feder and stands about midway between the widely discrepant values given by Wagner[12] and by Duane, Palmer and Yeh.[13]

CRITIQUE OF FORMER DETERMINATIONS

All previous determinations are, we believe, subject to certain corrections and fortunately they can in some cases still be applied. In the work of Duane, Palmer and Yeh the effect of work function was not considered, the potential drop was measured to one end rather than to the center of the filament, thermal emission velocities were omitted from consideration, and the effect of finite slit widths was incorrectly treated. In applying Feder's conclusions to Duane, Palmer and Yeh's cathode potential drop the magnitude of this drop should be known. Since it is stated that a Coolidge cathode was used it is probably safe to assume a drop of six volts, and this assumption in connection with Fig. 1 of Duane, Palmer and Yeh permits the correction (3 volts) to be applied. The effect of electron emission energy is much smaller than the probable error in V' and need not be considered. Analysis[14] of the effect of spectrometer slits upon the isochromat shows that the effective angle θ determining the position of the upper or linear part of the isochromat is the angle between the crystal face and the *center line* of the slit system. The argument in this reference refers explicitly to a system in which the two slits are used to define the *incident* beam, but with trifling modifications it will apply to a system like that of Duane, Palmer and Yeh in which one slit is on the side of incidence and the other on the side of reflection.

One finds therefore that the angle θ employed in calculation by Duane, Palmer and Yeh is too large by 46″, and that their stated value of V' is too small by 1.5 volts. Recalculating we obtain $V' \sin\theta = 2034.6 \pm 0.9$ international volts.

The low value obtained by Wagner has been discussed by Feder who ascribes the result to a shift of the isochromat caused by wide slits but does not attempt to explain the effect. We have performed special experiments to test this effect and have obtained single crystal isochromats like Feder's Fig. 4, confirming his statement that the wide slit isochromat apparently meets the voltage axis (or the background) at a smaller V' than does the corresponding curve obtained with narrow slits. We do not, however, find the shift proportional to slit width (which Feder assumes), and over a range of sufficiently narrow slits no shift appears. Our observations lead to the conclusion that the effect depends upon the existence of the knee of the isochromat. The point V' is found by extrapolating downward the apparently straight segment between foot and knee. As the narrow slits are gradually widened this segment does not shift but becomes shorter due to the upward extension of the enlarging foot. When the foot has broadened and risen until it reaches the knee the straight segment upon which V' depends no longer exists, and one makes the natural mistake of extrapolating to the voltage axis the segment of the isochromat *above* the knee, thus obtaining too small a value. The stages in this process are illustrated in a somewhat formalized manner in Fig. 3.

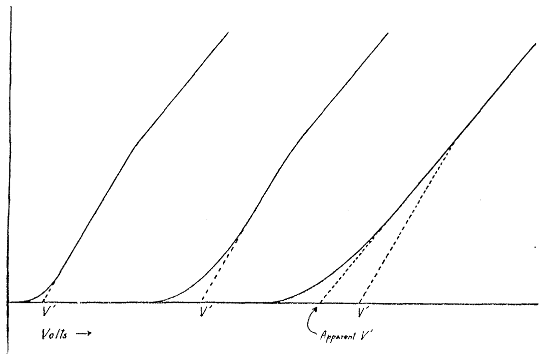

FIG. 3. Single crystal isochromats somewhat idealized. From left to right they represent curves taken with progressively wider slits.

In this situation there is no possibility of reinterpreting Wagner's data and obtaining a correct result, since the position of the knee and its associated change in slope are not known with sufficient exactness.

It remains to discuss Feder's own determination. This work was done with a rocksalt crystal

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[12] Wagner, Phys. Zeits. **21**, 621 (1920).

[13] Duane, Palmer and Yeh, J. O. S. A. **5**, 376 (1921).

[14] Kirkpatrick and Miyake, R. S. I. **3**, 1 (1932).

but since the ratio of the grating constants of rocksalt and calcite is accurately known[15] the results are readily placed upon a calcite basis. Having assumed the isochromat shift proportional to slit width Feder corrects for the supposed shift in his case by adding a small quantity to his measured potentials. His slits were so narrow as to make the curve foot quite small so, as shown above, the correction is not required. Upon removing this supposed correction and taking account of the work function of the tungsten target, Feder's original value is found unchanged, due to the cancellation of the two corrections.

The revised values are entered in the third column of Table I, and it will be seen that revision has resolved this long-standing uncertainty by bringing the usable values so close together that their probable errors all overlap. The mean of the three determinations is 2035.3 ±0.2, which is identical with the result of the present investigation. We have also retained a probable error for this mean which is the same as that of our own work since it is not apparent upon a basis of either external or internal consistency that the older determinations can be effective in reducing this figure. In spite of this, the agreement of these former determinations (as adjusted) with the present work is of the greatest service in indicating the probable freedom of our results from large undetected systematic errors.

Determination of X-Ray Wave-Lengths from the Law of Duane and Hunt

The application of the above results to a determination of the relation between h and e is obstructed by the systematic disagreement between crystal and grating wave-lengths which was mentioned above. Fortunately a reasonable choice between these two methods of wave-length measurement may now be effected by appeal to a third method which is potentially accurate and completely independent of the other two. We find such a method in the law of Duane and Hunt, according to which the minimum wave-length of the continuous spectrum $\lambda=(h/e)c^2/pqV'\times 10^8$. If a value of h/e derived independently of continuous spectrum observations be employed in this equation in combination with our experimental V' the wave-length of the $K\alpha_1$ line of silver is obtained.

Data collected and discussed by Birge[16] furnish values of h/e determined by five suitable independent methods. Assigning weights in accord with Birge's stated probable errors these values lead to the weighted average value $h/e=(1.3720 \pm 0.0004)\times 10^{-17}$ erg·sec./e.s.u. The resulting wave-length value for Ag $K\alpha_1$ as determined from the law of Duane and Hunt is 558.0±0.2 X.U. The most probable crystal value is obtained by using in Bragg's law the mean θ discussed in an earlier section of this paper and Bearden's[17] value of the grating constant of calcite. This gives a result which is slightly lower than the value adopted by Siegbahn (0.55828A) but the latter is based upon a grating constant which is in part frankly conventional. The $K\alpha_1$ line of silver has not yet been measured by the grating method but grating measurements[18] on other lines, when taken collectively, show a percentage difference between grating and crystal values which increases slightly with wave-length in such a way as to indicate that a good grating measurement of Ag $K\alpha_1$ would give a result 0.21±0.01 percent higher than the Siegbahn value. The various determinations (to four significant figures) are assembled in Table II.

TABLE II. *Wave-length of silver $K\alpha_1$.*

Most probable crystal value (mean θ and Bearden's d)	558.0 X.U.
As given by Siegbahn[19] (conventional d)	558.3
Ruled grating value (calculated)	559.5
By law of Duane and Hunt	558.0±0.2

The complete agreement of the results of the new method with the crystal measurements compels us to accept the latter as essentially correct.

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[15] Siegbahn, *Spektroskopie der Röntgenstrahlen*, p. 42, (1931).

[16] Birge, Phys. Rev. **40**, 228 (1932).

[17] Bearden, Phys. Rev. **38**, 2089 (1931).

[18] Cork, Phys. Rev. **35**, 1456 (1930); Bearden, Phys. Rev. **37**, 1210 (1931); Witmer and Cork, Phys. Rev. **42**, 743 (1932); Purdom and Cork, Phys. Rev. **44**, 974 (1933).

[19] Siegbahn, reference 15, p. 185.

Values of $h/e^{4/3}$ and h

Replacing λ in the equation $Ve=hc/\lambda$ by its Bragg's law expression (with the correcting factor 0.999865 for crystal refraction), expressing the grating constant (of calcite) in the form

$$d=\left\{\frac{\frac{1}{2}Me\ 0.0111800}{\rho\varphi Aqc}\right\}^{1/3}$$

and changing from electrostatic to international potential units we have

$$\frac{h}{e^{4/3}}=\frac{(2.\times 0.999865)pq^{2/3}10^{8}}{c^{7/3}}\left[\frac{M\ 0.0111800}{2\rho\varphi A}\right]^{1/3}\times V'\sin\theta, \quad (1)$$

in which p = international ohm/absolute ohm, q = international amp./absolute amp., c = speed of light, M = molecular weight of calcite, ρ = density of calcite, φ = a known function of the angle between the crystal edges, A = atomic weight of silver, θ = external Bragg angle (calcite cleavage face), V' = minimum electron potential fall (international volts) capable of exciting radiation reflectable at angle θ.

With the constants recommended by Birge[20] in 1929 and the new value $V'\sin\theta=2035.3\pm 0.2$ we obtain $h/e^{4/3}=(1.7559\pm 0.0002)\times 10^{-14}$.

Recent work on c, ρ, φ, p and q makes advisable certain slight changes in the adopted values of these constants. The new values, together with their sources, are as follows:

$c=2.99774\times 10^{10}$ cm/sec.,[21] $\rho=2.71030\pm 0.00003$ g/cm^3,[22] $\varphi=1.09594\pm 0.00001$,[17] $p=1.000460\pm 0.000020$,[23] $q=0.999900\pm 0.000040$.[23]

With these constants we obtain from (1) the result $h/e^{4/3}=(1.7563\pm 0.0002)\times 10^{-14}$.

The deduction of most probable individual values of e and h from the results of this paper in combination with other known relations between these constants is a critical task which we are content to leave to others. It is of much interest, however, to combine the above value of $h/e^{4/3}$ with an independent value of e and thus determine h. Since Birge's most probable values of e are based in part upon the x-ray determinations quoted above in support of our finally adopted $V'\sin\theta$ it appears more appropriate to employ here Millikan's[24] 1930 oil drop value $e=(4.770\pm 0.005)\times 10^{-10}$ e.s.u. Recent work upon c and pq is without effect upon this value as given. This calculation yields

$$h_{\text{crystal}}=(6.546\pm 0.006)\times 10^{-27}\ \text{erg}\cdot\text{sec.},$$

which is presented as the most probable present value of h as obtained by the x-ray method alone.

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[20] Birge, Phys. Rev. Suppl. **1**, 1 (1929).

[21] Work of Michelson, Pease and Pearson, reported by *Science Service* in Science **78**, Dec. 29 (1933).

[22] Bearden, Phys. Rev. **43**, 92 (1933).

[23] Curtis, Bull. Nat. Res. Council, No. 93 (1933).

[24] Millikan, Phys. Rev. **35**, 1231 (1930).