An Application of Probabilities to the Counting of Alpha-Particles

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The Poisson law of probability is directly applicable to radioactive emissions only in certain simple cases. In general, when a radioactive series such as the thorium series is present the probability function is complex. It can be found readily, however, if the interval of time for which probabilities are calculated is chosen so as to be either very short or very long compared with the mean life of each radioactive substance involved. The probability function for α-particles from the thorium series is determined on the basis of an interval of five minutes and is illustrated graphically in a special case.

WHEN events are distributed individually and collectively at random, the probability of any number of events occurring within a chosen interval is given by the Poisson law $p(n)=\epsilon^n e^{-\epsilon}/n!$.[1] Here n is the number of events whose probability is required, ϵ is the mean or expected number for the given interval and $e=2.71828$, as usual. The rays emitted during the disintegration of a radioactive substance or of a mixture of effectively independent radioactive substances constitute a set of events distributed individually and collectively at random in time, at least in the case where the amount of radioactive material does not vary appreciably over a period of time sufficiently great to contain a large number of the basic intervals for which the probability of events is desired. Experimental verification of the Poisson law as applied to simple radioactive disintegrations has been obtained in the case of α-particles by Rutherford and Geiger[2] and in the case of β-particles by Kovarik.[3]

Unfortunately, in many cases of radioactive disintegration the events are not distributed individually and collectively at random. Such a case has arisen in experiments now being conducted by Kovarik and the writer to make an accurate determination of the disintegration constant of thorium. Here α-particles are counted by the same method as in the recent new determination of the disintegration constant of uranium.[4] Since the α-particles counted do not come from thorium alone but from the entire thorium series containing six α-particle emitters, each dependent on the preceding, the events are not individually at random and the Poisson law does not apply.

The general derivation of the probability law for the case just quoted appears to be very difficult. We can, however, deduce the form of the probability function if a simple restriction is imposed. Let us suppose that the interval of time for which the probability of n α-particles must be found is either very short or very long compared with the mean life of each of the six α-particle emitters in the thorium series. We assume that the thorium and its following products are in radioactive equilibrium, so that the amount of each substance present is effectively constant, the mean life of thorium being so great. Evidently the number of α-particles coming from any product x, say, whose mean life is long compared to the chosen interval, is a small fraction of the number of atoms of x present or produced during the interval and the distribution of these α-particles in time is in accord with the Poisson law. On the other hand, the number of α-particles coming from a product y whose mean life is short compared to the interval is equal simply to the number of atoms of y produced during the interval from the preceding substance in the radioactive series. Thus, if y follows x the

[1] For an explanation of the criterion "individually and collectively at random" and a complete discussion of the Poisson law the reader is referred to Fry, *Probability and Its Engineering Uses*, Chapter VIII.

[2] Rutherford and Geiger, Phil. Mag. [6] **20**, 698–707 (1910).

[3] Kovarik, Phys. Rev. **13**, 272–280 (1919).

[4] Kovarik and Adams, Phys. Rev. **40**, 718–726 (1932).

α-particles are associated in groups of two, the distribution of the groups in time following the Poisson law. Similarly, if another product z of short life follows y in turn, the α-particles are associated in groups of three and so on.

The substances in the thorium series which give rise to α-particles are shown in Table I. As five minutes is a convenient period for which to record counts experimentally, let us use this length of time as the basic interval in calculating probabilities. Examination of the table shows

TABLE I.

Substance	*Mean life*
Thorium	$1.8(10)^{10}$ yr.
Radiothorium	2.74 yr.
Thorium X	5.25 day
Thoron	78.6 sec.
Thorium A	0.21 sec.
Thorium C	87.4 min.

that thorium, radiothorium and thorium C emit α-particles distributed individually and collectively at random, while thorium X, thoron and thorium A give rise to α-particles in groups of three similarly distributed. Thus, if ϵ is the mean rate of emission for the entire thorium series, the probability functions for the α-particles coming from the "singles" class and from the "triples" class are, respectively,

$$p_s(n_s)=\frac{(\epsilon/2)^{n_s}e^{-\epsilon/2}}{n_s!},\quad p_t(n_t)=\tfrac{1}{3}\frac{(\epsilon/6)^{n_t/3}e^{-\epsilon/6}}{(n_t/3)!},$$

where $n_s=0$, 1, 2, 3, etc., and $n_t=0$, 3, 6, 9, etc. The factor $\frac{1}{3}$ in the second expression is introduced to normalize the coordinates in the usual manner.

It is now a simple matter to find the probability function for any number of α-particles in the chosen interval for the entire series. Consider, for example, five. This number may be composed of one triple and two singles, or of five singles, the total probability function for which is

$$p_t(3)\cdot p_s(2)+p_t(0)\cdot p_s(5).$$

Similarly, the probability function for six α-particles is

$$p_t(6)\cdot p_s(0)+p_t(3)\cdot p_s(3)+p_t(0)\cdot p_s(6),$$

and that for seven is

$$p_t(6)\cdot p_s(1)+p_t(3)\cdot p_s(4)+p_t(0)\cdot p_s(7).$$

Evidently the general form of the probability function depends on whether the number of α-particles is one less than, equal to or one greater than an integral multiple of three. If we denote by ν a number of α-particles which is an integral multiple of three, we have, for the three cases just mentioned,

$$p_{-1}(\nu-1)=\tfrac{1}{3}e^{-2\epsilon/3}\left[\frac{(\epsilon/6)^{\nu/3-1}(\epsilon/2)^2}{(\nu/3-1)!2!}+\frac{(\epsilon/6)^{\nu/3-2}(\epsilon/2)^5}{(\nu/3-2)!5!}+\cdots+\frac{(\epsilon/6)^1(\epsilon/2)^{\nu-4}}{1!(\nu-4)!}+\frac{(\epsilon/6)^0(\epsilon/2)^{\nu-1}}{0!(\nu-1)!}\right]$$

$$p_0(\nu)=\tfrac{1}{3}e^{-2\epsilon/3}\left[\frac{(\epsilon/6)^{\nu/3}(\epsilon/2)^0}{(\nu/3)!0!}+\frac{(\epsilon/6)^{\nu/3-1}(\epsilon/2)^3}{(\nu/3-1)!3!}+\frac{(\epsilon/6)^{\nu/3-2}(\epsilon/2)^6}{(\nu/3-2)!6!}+\cdots+\frac{(\epsilon/6)^1(\epsilon/2)^{\nu-3}}{1!(\nu-3)!}+\frac{(\epsilon/6)^0(\epsilon/2)^{\nu}}{0!\nu!}\right]$$

$$p_{+1}(\nu+1)=\tfrac{1}{3}e^{-2\epsilon/3}\left[\frac{(\epsilon/6)^{\nu/3}(\epsilon/2)^1}{(\nu/3)!1!}+\frac{(\epsilon/6)^{\nu/3-1}(\epsilon/2)^4}{(\nu/3-1)!4!}+\frac{(\epsilon/6)^{\nu/3-2}(\epsilon/2)^7}{(\nu/3-2)!7!}+\cdots+\frac{(\epsilon/6)^1(\epsilon/2)^{\nu-2}}{1!(\nu-2)!}+\frac{(\epsilon/6)^0(\epsilon/2)^{\nu+1}}{0!(\nu+1)!}\right].$$

We may use these expressions separately if we choose, but it is simpler to employ the composite function $P(\nu)=p_{-1}(\nu-1)+p_0(\nu)+p_{+1}(\nu+1)$

which applies to the coordinate interval $\nu-1$ to $\nu+1$, inclusive. After some manipulation we find

$$P(\nu)=\tfrac{1}{3}e^{-2\epsilon/3}\sum_{x=0}^{\nu/3}\frac{(\epsilon/6)^{\nu/3-x}(\epsilon/2)^{3x}}{(\nu/3-x)!(3x)!}\times\left(\frac{3x}{\epsilon/2}+1+\frac{\epsilon/2}{3x+1}\right)$$

$$=\tfrac{1}{3}(\epsilon/6)^{\nu/3}e^{-2\epsilon/3}\sum_{x=0}^{\nu/3}\frac{(3^{\frac{1}{2}}\epsilon/2)^{2x}}{(\nu/3-x)!(3x)!}\times\left(\frac{3x}{\epsilon/2}+1+\frac{\epsilon/2}{3x+1}\right).$$

Since the coordinates have been normalized, the actual probability of a number of α-particles within the coordinate interval $\nu-1$ to $\nu+1$ inclusive is $3P(\nu)$ and the probability for an extended interval ν_1-1 to ν_2+1, say, is $3\sum_{\nu_1}^{\nu_2}P(\nu)$. When ϵ is large we may replace $P(\nu)$ by a continuous function $P(n)$ and express the probability as the integral $\int_{n_1}^{n_2}P(n)dn$ in the usual way.

The probability curve $P(n)$ for $\epsilon=210$ is shown in Fig. 1. For the sake of comparison the curve $p(n)=\epsilon^n e^{-\epsilon}/n!$ is indicated by the broken line. This curve applies to any case where α-particles are produced by a single substance or by a mixture of effectively independent substances, such as the uranium series. It is evident that the standard deviation σ for $P(n)$ is considerably greater than for $p(n)$. In fact, in the first case the value of σ, obtained from the curve, is approximately 21, while in the second case it is known to be $210^{\frac{1}{2}}=14.5$. This difference is easily detected experimentally.

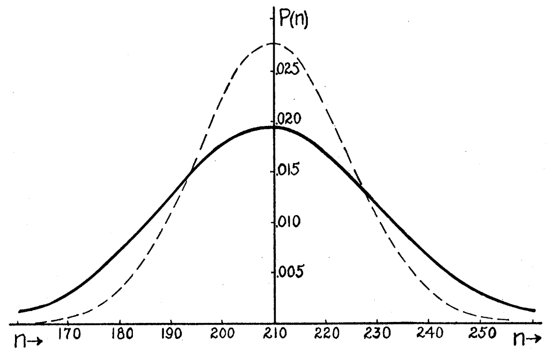

FIG. 1.