Application of Liouville's Theorem to Electron Orbits in the Earth's Magnetic Field

W. F. G. SWANN, *Bartol Research Foundation of the Franklin Institute*

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It is pointed out that in the application of Liouville's theorem to the problem of cosmic-ray intensities, Lemaitre and Vallarta have implicitly taken the electron momentum as that corresponding to a free particle. Calling this momentum p' the particle momentum, we have to realize that Liouville's theorem is usually based upon the Hamiltonian equations in which the momentum p associated with an electron is not the same as p', but is connected with it by the relation $p=p'+eU/c$, where U is the vector potential determining the magnetic field. The Hamiltonian equations are not valid in terms of momenta of the type p', and, it is not, therefore, clear that Liouville's theorem is valid when expressed in terms of these momenta. The object of the paper is to show that a theorem the equivalent of Liouville's theorem is, in fact, true in terms of the coordinates and the momenta p', so that the ultimate validity of the use of the theorem by Lemaitre and Vallarta is substantiated. It is to be observed, moreover, that the validity of this extended form of Liouville's theorem is true even in the presence of an electric field.

LIOUVILLE'S theorem states that, if q's refer to coordinates and p's to momenta of a system satisfying the Hamiltonian equations

$$\dot{p}_r = -\partial\mathcal{H}/\partial q_r, \quad \dot{q}_r = \partial\mathcal{H}/\partial p_r \tag{1}$$

then

$$\int\int\int\cdots\int \delta q_1 \delta q_2 \cdots \delta p_1 \delta p_2 \cdots = \text{constant with the time}, \tag{2}$$

where the δq's and δp's refer to variations in the coordinates and momenta between two different dynamical paths, and the integral is taken over some region conveniently thought of as infinitesimal in its practical application. In an analytically equivalent form, the theorem states that if in a generalized space representing as coordinates the coordinates and momenta of a dynamical system, we plot a group of points representing different possible states of the system and forming a density τ of such points in a certain vicinity of the space and if we follow the motion of the points in the generalized space, the density τ will remain constant with the time, i.e.,

$$D\tau/Dt = 0. \tag{3}$$

This theorem has received a very beautiful application in the hands of Lemaitre and Vallarta[1] to the problem of electron motion in the earth's magnetic field. The system considered is a single electron and the generalized (phase) space is one of six dimensions. The different points in the phase space represent different possible states of the electron; and they may be extended in their meaning to apply to different electrons having different positions and velocities so long as the electrons are mutually non-interacting.

Let us mark out in the actual three dimensional space an elementary cone of solid angle $d\omega$ and of length dl, parallel to the direction of motion of the electron at that point, with its apex pointing in the direction of the resulting velocity of the particle. Let us truncate the apex of this cone by an element of area dS (perpendicular to dl) so as to cut off a length near the apex, small compared with dl. Consider the number δn of electrons which lie within this cone, which have component momenta between p_1 and p_1+dp_1 parallel to the axis of the cone and whose direction of motion is such that they travel through the base of the cone and through its truncated apex dS. Let us divide δn by the volume of the truncated cone, also by dS and by the product of dp_1 and dp_2dp_3 where dp_2 and dp_3 represent the mutually perpendicular infinitesimal ranges of momenta perpendicular to dl, which serve to specify the limits of momenta specified by the cone.[2] The result obtained is the density of the

[1] G. Lemaitre and M. S. Vallarta, Phys. Rev. **43**, 87 (1933).

[2] For purposes of description it is convenient to think of a cone with a rectangular cross section.

points in the phase space corresponding to the position of dS and to the vectorial magnitude of the momenta specified by the absolute magnitude thereof and by the direction of the cone. This is the quantity which, when multiplied by $dp_2dp_3/d\omega$ and also by the velocity corresponding to the momentum p_1, determines the intensity of electron flux per unit range of p_1 in the direction of p_1, i.e., the number of electrons per unit range of momentum p_1 passing per second through unit area perpendicular to dl, per unit solid angle.

In order that equality of density in the phase space along a dynamical path shall imply equality of intensity in the above sense, it is necessary that $dp_2dp_3/d\omega$ shall be constant. If du_2 and du_3 are the velocity ranges corresponding to $d\omega$ and u_1 is the velocity corresponding to p_1, constancy of $dp_2dp_3/d\omega$ implies constancy of $u_1^2dp_2dp_3/du_2du_3$. For convenience of discussion we shall use the term, particle momentum, to designate the quantity p' defined as

$$p'=m_0u/(1-u^2/c^2)^{\frac{1}{2}} \tag{4}$$

where u is the velocity. Now if in the foregoing discussion we could regard the momentum as p', the constancy of $u_1^2dp_2dp_3/du_2du_3$ would be provided for in the case of motion in a pure magnetic field, since in that case u^2 is constant, and so p' is vectorially proportional to u. It is to be observed, however, that the p's which occur in (2) are the p's which occur in (1); and, in its application in (1) p is not the same as p', but is given by

$$p=m_0\mathbf{u}/(1-u^2/c^2)^{\frac{1}{2}}+(e/c)\mathbf{U}, \tag{5}$$

where $\mathbf{U}$ is the vector potential which determines the magnetic field. For, the relativistic Lagrangian function for the electron is

$$L=-m_0c^2(1-u^2/c^2)^{\frac{1}{2}}+(e/c)(\mathbf{U}\cdot\mathbf{u})-e\varphi, \tag{6}$$

where φ is the scalar potential and p is defined as

$$p=\partial L/\partial u,$$

which leads to (5). This is the p which occurs in the Hamiltonian function $\mathcal{H}$, given by

$$\mathcal{H}=m_0c^2\{1+(1/m_0^2c^2)[(p_x-(e/c)U_x)^2 +(p_y-(e/c)U_y)^2+(p_z-(e/c)U_z)^2]^{\frac{1}{2}}\}+e\varphi. \tag{7}$$

The quantity $(e/c)\mathbf{U}$ in (5) is not a small term. It is comparable with the whole magnitude of p and, of course, varies with the position of the electron in its path, since it is the quantity whose curl determines the magnetic field. Thus although $|u|$ remains constant in the path, $|p|$ does not. It is the presence of $\mathbf{U}$ in (6) which through the Lagrangian equations leads to the very term $\mathbf{u}\times\mathbf{H}/c$ which in the equation of motion

$$\frac{d}{dt}\left[\frac{m_0\mathbf{u}/e}{(1-u^2/c^2)^{\frac{1}{2}}}\right]=E+\frac{\mathbf{u}\times\mathbf{H}}{c} \tag{8}$$

represents the action of the magnetic field on the electron.[3]

The point here raised is not something concerned merely with the theory of relativity. For the only effect of neglecting relativity is to replace (5) by

$$p=m_0\mathbf{u}+e\mathbf{U}/c.$$

It is *only* with p understood in this sense that even the nonrelativistic equations assume the Hamiltonian form in their story of the action of the magnetic field on the electron.

In order therefore that the conclusions of Lemaitre and Vallarta may be substantiated, it is necessary to show that although (1) is most assuredly not true for the p''s, nevertheless (2) is true with the p's replaced by p''s. To this end we make, through (4) and (5) the mathematical transformation

$$p=p'+e\mathbf{U}/c, \tag{9}$$

$$q=q'. \tag{10}$$

Expression (2) then yields

$$\int\int\int\cdots\int J\delta q_1\delta q_2\cdots\delta p_1'\delta p_2'\cdots =\text{constant with time}, \tag{11}$$

[3] Or if we speak in terms of the Hamiltonian equations, it is the presence of $\mathbf{U}$ in (7) which leads through (1) to the $\mathbf{u}\times\mathbf{H}/c$ in (8).

where J, the Jacobian of the transformation is given by

$$J=\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\partial U_x'}{\partial q_x} & \frac{\partial U_x'}{\partial q_y} & \frac{\partial U_x'}{\partial q_z} & 1 & 0 & 0 \\ \frac{\partial U_y'}{\partial q_x} & \frac{\partial U_y'}{\partial q_y} & \frac{\partial U_y'}{\partial q_z} & 0 & 1 & 0 \\ \frac{\partial U_z'}{\partial q_x} & \frac{\partial U_z'}{\partial q_y} & \frac{\partial U_z'}{\partial q_z} & 0 & 0 & 1 \end{vmatrix}$$

where $\mathbf{U}'$ is written in short for $e\mathbf{U}/c$. Hence the Jacobian J is unity, and a theorem analogous to Liouville's theorem also holds for the q's and p''s, which justifies the ultimate conclusions of Lemaitre and Vallarta. It will be seen, moreover, that the validity of this extended form of Liouville's theorem is not limited to the case where an electric field is absent but is quite general. Absence of electric field is, of course, necessary in the application to the cosmic-ray problem.

The foregoing argument tends perhaps to conceal the mathematical mechanism of the process. The establishment of the result at one blow by utilization of the properties of the Jacobian seems to imply that it should be obvious. For this reason and in view of the importance of a clear realization of what is happening, we shall establish the result in a way in which the $\mathbf{U}$ in (5) plays a more conspicuous rôle, a rôle comparable with its importance in determining the dynamical paths, for example, and in which it only disappears finally by a sort of accidental cancellation to yield in terms of the q's and p''s a theorem which is the equivalent of Liouville's.

We shall discuss the method along the line which seeks to prove that the density of the points in the phase space does not change with the time.

Following precisely the usual method, as, for example, that given by Jeans[4] we arrive at

$$(D\tau/Dt)+\tau\sum_1^3\{(\partial\dot{p}_r/\partial p_r)+(\partial\dot{q}_r/\partial q_r)\}=0, \quad (12)$$

where τ is the density of the points in the multidimensional space, which in our problem is a six dimensional space, with coordinates q_x, q_y, q_z, p_x, p_y, p_z.

Up to this point no dynamics has been used. If at this stage, however, we assume (1), the curly bracket in (12) vanishes and we find that τ is constant with the time. If we use the quantities p' and q' defined by (9) and (10), the Hamiltonian equations do not hold, however, and it is necessary to find what equations do hold. We have

$$d\mathcal{H}=\frac{\partial\mathcal{H}}{\partial q_x'}dq_x'+\frac{\partial\mathcal{H}}{\partial q_y'}dq_y'+\frac{\partial\mathcal{H}}{\partial q_z'}dq_z'+\frac{\partial\mathcal{H}}{\partial p_x'}dp_x'+\frac{\partial\mathcal{H}}{\partial p_y'}dp_y'+\frac{\partial\mathcal{H}}{\partial p_z'}dp_z'.$$

Hence, by using (9) and (10)

$$\frac{\partial\mathcal{H}}{\partial q_x}=\frac{\partial\mathcal{H}}{\partial q_x'}-\frac{e}{c}\left(\frac{\partial U_x}{\partial q_x}\frac{\partial\mathcal{H}}{\partial p_x'}+\frac{\partial U_y}{\partial q_x}\frac{\partial\mathcal{H}}{\partial p_y'}+\frac{\partial U_z}{\partial q_x}\frac{\partial\mathcal{H}}{\partial p_z'}\right),$$

$$\frac{\partial\mathcal{H}}{\partial p_x}=\frac{\partial\mathcal{H}}{\partial p_x'},$$

with similar expressions for the y and z coordinates and momenta. Hence, substituting in (1), and again using (9) and (10) we find

$$\dot{p}_x'+\frac{e}{c}\left(\frac{\partial U_x}{\partial t}+\dot{q}_x\frac{\partial U_x}{\partial q_x}+\dot{q}_y\frac{\partial U_x}{\partial q_y}+\dot{q}_z\frac{\partial U_x}{\partial q_z}\right)=-\frac{\partial\mathcal{H}}{\partial q_x'}+\frac{e}{c}\left(\frac{\partial U_x}{\partial q_x'}\frac{\partial\mathcal{H}}{\partial p_x'}+\frac{\partial U_y}{\partial q_x'}\frac{\partial\mathcal{H}}{\partial p_y'}+\frac{\partial U_z}{\partial q_x'}\frac{\partial\mathcal{H}}{\partial p_z'}\right); \quad (13)$$

also

$$\dot{q}_x'=\partial\mathcal{H}/\partial p_x', \quad (14)$$

with similar expressions for the other directions.

Hence, substituting in (13) from (14) and from the corresponding expressions for $\dot{q}_y'$ and $\dot{q}_z'$ and further observing that $q=q'$ and $\partial U/\partial q_z=\partial U/\partial q_z'$, we have

$$\dot{p}_x'=-\frac{\partial\mathcal{H}}{\partial q_x'}+\frac{e}{c}(\dot{q}_y'H_z-\dot{q}_z'H_y)-\frac{e}{c}\frac{\partial U_x}{\partial t}, \quad (15)$$

$$\dot{q}_x'=\partial\mathcal{H}/\partial p_x', \quad (16)$$

[4] J. H. Jeans, *The Dynamical Theory of Gases*, 4th Edition, pp. 70–71, Cambridge University Press.

in which we have observed that the magnetic field **H** is given by **H** = curl **U**.[5] Returning then to (12), when p' and q' are used instead of p and q and when (15) and (16) and their corresponding expressions in y and z are used, (12) becomes

$$-\frac{D\tau}{Dt}=\tau\sum_1^3\left(\frac{\partial\dot{p}_r'}{\partial p_r'}+\frac{\partial\dot{q}_r'}{\partial q_r'}\right)=\tau\sum_1^3\left(\frac{\partial^2\mathcal{H}}{\partial q_r'\partial p_r'}-\frac{\partial^2\mathcal{H}}{\partial p_r'\partial q_r'}\right)-\tau\frac{e}{c}\frac{\partial}{\partial t}\left(\frac{\partial U_x}{\partial p_x'}+\frac{\partial U_y}{\partial p_y'}+\frac{\partial U_z}{\partial p_z'}\right)$$

$$+\frac{e\tau}{c}\left\{\frac{\partial}{\partial p_x'}(\dot{q}_y'H_z'-\dot{q}_z'H_y')+\frac{\partial}{\partial p_y'}(\dot{q}_z'H_x-\dot{q}_x'H_z)+\frac{\partial}{\partial p_z'}(\dot{q}_x'H_y-\dot{q}_y'H_x)\right\}. \quad (17)$$

The first term on the right-hand side vanishes and the mechanism of its vanishing is representative of what is the whole story in the case of equations of Hamiltonian form. Here, however, we have the additional terms shown. Since **U** and **H** are expressible entirely in terms of the q's and so of the q''s, they are independent of the p''s, so that (17) becomes

$$-\frac{D\tau}{Dt}=\frac{e\tau}{c}\left\{H_x\left(\frac{\partial\dot{q}_z'}{\partial p_y'}-\frac{\partial\dot{q}_y'}{\partial p_z'}\right)+H_y\left(\frac{\partial\dot{q}_x'}{\partial p_z'}-\frac{\partial\dot{q}_z'}{\partial p_x'}\right)+H_z\left(\frac{\partial\dot{q}_y'}{\partial p_x'}-\frac{\partial\dot{q}_x'}{\partial p_y'}\right)\right\}.$$

In view of (16), each of the parentheses vanish, and we are left with

$$D\tau/Dt=0.$$

Hence, although the equations of electron motion are not of Hamiltonian form in terms of the spacial coordinates and the particle momenta p', Liouville's theorem extends to apply with these coordinates and momenta also. It will be observed that our proof is in no way limited to the case where there is no electric field.

[5] This relation is not fundamentally germane to the discussion and is introduced simply for convenience of notation.