On the Interpretation of Heat in Relativistic Thermodynamics

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This article investigates the interpretation of the right-hand side of the relativistic second law of thermodynamics

$$\left(\phi_0 \frac{dx^\mu}{ds}\right)_\mu (-g)^{\frac{1}{2}} dx^1 dx^2 dx^3 dx^4 \geqq \frac{dQ_0}{T_0}$$

and shows that the quantity dQ_0 can be interpreted as the heat—measured by a local observer at rest in the fluid at the point of interest—which flows relative to the fluid into an element of the fluid having the instantaneous proper volume dV_0 during the proper time dt_0, where these quantities are chosen so as to satisfy the numerical equality $dV_0 dt_0 = (-g)^{\frac{1}{2}} dx^1 dx^2 dx^3 dx^4$ and the quantity T_0 is taken as the temperature ascribed to this heat by the local observer.

§1. INTRODUCTION

THE analogue in relativistic thermodynamics of the usual second law of thermodynamics may be conveniently written in either of the two equivalent forms[1]

$$\left(\phi_0 \frac{dx^\mu}{ds}\right)_\mu (-g)^{\frac{1}{2}} dx^1 dx^2 dx^3 dx^4 \geqq \frac{dQ_0}{T_0} \quad (1)$$

or

$$\frac{\partial}{\partial x^\mu}\left(\phi_0 \frac{dx^\mu}{ds}(-g)^{\frac{1}{2}}\right) dx^1 dx^2 dx^3 dx^4 \geqq \frac{dQ_0}{T_0}, \quad (2)$$

where the sign "is greater than" ($>$) applies to irreversible processes and the sign "is equal to" ($=$) applies to reversible processes.

The quantity ϕ_0 occurring in these expressions is the proper density of entropy as measured by a local observer at rest at the point of interest with respect to the thermodynamic fluid or working substance which is under consideration. The quantities dx^μ/ds are the components of the macroscopic "velocity" of this fluid at the point of interest. And the quantity dQ_0/T_0 may be described as the heat which flows into the element of the fluid and during the time denoted by $dx^1 dx^2 dx^3 dx^4$, divided by its temperature—both of these quantities being measured in proper coordinates.

In the applications of relativistic thermodynamics which have so far been made, there has been no flow of heat relative to the fluid under consideration, so that the term dQ_0/T_0 has actually been zero. These applications have been in the first place to cases of static *thermodynamic equilibrium*[2] where the heat flow was zero on account of the condition of static equilibrium. In the second place, applications have been made to the *reversible* expansion and contraction of certain models of the universe[3] in which there has been no flow of heat from one portion of the material filling the model to another owing to uniformity of conditions throughout the models considered. In the third place, applications have also been made to the *irreversible* expansion and contraction of models of the universe[4] in which, however, there was still no flow of heat from one portion of the material to another,—again on account of the assumed homogeneity of the models.

In general, nevertheless, the expression of the relativistic second law given by (1) and (2) should be valid when the heat flow does not happen to be equal to zero. And the purpose of the present note is to make as clear as may be a correct method in the general case for the specification of the quantity dQ_0/T_0.

[1] Tolman, Proc. Nat. Acad. 14, 268 (1928); ibid. 14, 701 (1928); Phys. Rev. 35, 896 (1930).

[2] Tolman, Proc. Nat. Acad. 14, 353 (1928); ibid. 17, 153 (1931); Phys. Rev. 35, 904 (1930); Tolman and Ehrenfest, Phys. Rev. 36, 1791 (1930).

[3] Tolman, Phys. Rev. 37, 1639 (1931); ibid. 38, 797 (1931); ibid. 38, 1758 (1931).

[4] Tolman, Phys. Rev. 39, 320 (1932); Tolman and Ward, Phys. Rev. 39, 835 (1932).

§2. INVARIANCE OF dQ_0/T_0 WITH RESPECT TO COORDINATE TRANSFORMATIONS

Examining the expression for the relativistic second law in the form given by (1), it will be noted that the two factors $(\phi_0 \ dx^\mu/ds)_\mu$ and $(-g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4$, which are multiplied together to give the left-hand side of the expression, are both of them invariant scalars. The first of these factors $(\phi_0 \ dx^\mu/ds)_\mu$ is a scalar since it is the *contracted covariant derivative* of a vector and it will have a numerical value which is the same in all coordinate systems. The second of the factors $(-g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4$ is the known expression in *natural measure* for the four-dimensional volume specified by the range $dx^1dx^2dx^3dx^4$, and hence is also a scalar, having a value proportional to the designated range, but otherwise independent of the coordinate system.

As a result of this invariant scalar character of the left-hand side of (1), it is evident from the principle of covariance that the right-hand side of the expression must also be a similar invariant scalar, since otherwise the postulated law would not lead to the same results when applied in different coordinate systems. Hence the quantity dQ_0/T_0 must itself have a value, which is of course proportional to the four-dimensional volume $(-g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4$, but is otherwise independent of the coordinate system. This result is very useful, since it permits us to obtain a general determination of the quantity dQ_0/T_0 by first employing a specially chosen coordinate system in which the interpretation will be quite simple and clear.

§3. DETERMINATION OF dQ_0/T_0 BY USING "CO-MOVING" COORDINATES

To obtain a coordinate system which will make the interpretation easy, we may take for the space-like coordinates x^1, x^2, x^3 a three-dimensional network of lines permanently connecting adjacent macroscopically identifiable points of the fluid under consideration and moving with the fluid, and take for the time-like coordinate x^4 the readings of a set of natural clocks which have been distributed to a sufficient number of different points throughout the fluid and are then allowed to move therewith. The possibility of obtaining such a coordinate system

would appear guaranteed by the specifications for setting it up, and such systems which might be called "co-moving coordinates" are often very useful.

With such coordinates, it is evident that the fluid will always be permanently everywhere at rest with respect to the space-like coordinates, so that we can write

$$dx^1/ds = dx^2/ds = dx^3/ds = 0 \qquad (3)$$

for the spatial components of its macroscopic "velocity." Furthermore, for the temporal component we can write

$$dx^4/ds = 1 \qquad (4)$$

since increments in coordinate time and proper time have been made the same by our specification of x^4 with the help of natural clocks moving with the fluid.

Substituting (3) and (4) in the relativistic second law as given in the form (2), we then find that this reduces to the simple expression

$$\frac{\partial}{\partial x^4} (\phi_0(-g)^{\frac{1}{2}})dx^1dx^2dx^3dx^4 \geqq \frac{dQ_0}{T_0} \qquad (5)$$

and because of the mutual independence of the coordinates this can be rewritten in the form

$$\frac{\partial}{\partial x^4} (\phi_0(-g)^{\frac{1}{2}}dx^1dx^2dx^3)dx^4 \geqq \frac{dQ_0}{T_0}. \qquad (6)$$

To see more clearly, moreover, the significance of this result we may now introduce the evidently valid substitutions

$$dt_0 = dx^4; \quad dV_0 = (-g)^{\frac{1}{2}}dx^1dx^2dx^3, \qquad (7)$$

where dt_0 is an element of proper time as measured by a local observer at rest in the fluid at the point of interest, and dV_0 is the proper spatial volume, associated with the four-dimensional region $dx^1dx^2dx^3dx^4$, as measured by this same local observer. Introducing (7) in (6), we can then write

$$\frac{\partial}{\partial t_0} (\phi_0 dV_0)dt_0 \geqq \frac{dQ_0}{T_0}. \qquad (8)$$

The interpretation of dQ_0/T_0 with the help of this result is then quite simple. Since ϕ_0 is the density of entropy as measured by a local ob-

server at rest in the fluid, and dV_0 is the spatial volume as measured by this observer of a small region through the boundaries of which no fluid is passing, the left-hand side of the expression gives the change which the local observer finds in the time dt_0 in the entropy of a definite small element of the fluid. In accordance with the principle of equivalence, however, this change which a local observer finds in the entropy of a system small enough so that gravitational curvature can be neglected must be related to the heat that flows into the system in the way given by the ordinary second law of thermodynamics. Hence we can now conclude that dQ_0 is the heat as measured by the local observer that flows in the time dt_0 into the element of fluid instantaneously contained in the proper volume dV_0, and T_0 is the temperature associated with this heat. Furthermore, as a result of our previous discussion in §2 of the invariance of dQ_0/T_0 to coordinate transformations, we are thus furnished with a correct value to use for this quantity in any coordinate system, provided of course that we allow for the proportionality between the magnitude of dQ_0/T_0 and the magnitude $(-g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4$ of the four-dimensional volume associated with the specified coordinate range.

Hence, returning to our general expression for the relativistic second law, which is valid in any coordinate system

$$\left(\phi_0 \frac{dx^\mu}{ds}\right)_\mu (-g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4 \geqq \frac{dQ_0}{T_0} \qquad (9)$$

we may now state that the quantity dQ_0 occurring in this expression can be taken as the heat—measured by a local observer at rest in the fluid at the point of interest—which flows into an element of the fluid having the instantaneous proper volume dV_0 during the proper time dt_0, where these quantities are chosen so as to satisfy the numerical equality

$$dV_0 dt_0 = (-g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4 \qquad (10)$$

and the quantity T_0 is taken as the temperature ascribed to this heat by the local observer. We are thus provided with a perfectly definite interpretation of all the quantities occurring in the relativistic second law which will satisfy the principle of covariance by leading to the same

results no matter what coordinate system is used, and which satisfies the principle of equivalence by reducing, as we have seen with the help of a specially convenient coordinate system, to the ordinary second law for an infinitesimal portion of the fluid.

§4. DETERMINATION OF dQ_0/T_0 BY USING NATURAL COORDINATES

In accordance with the above specifications for determining a correct value to use in the relativistic second law for the flow of heat dQ_0 into the element and during the time denoted by $dx^1dx^2dx^3dx^4$, it will be noted that this quantity is to be obtained from measurements by a local observer of the rate at which heat flows into a specified element of the fluid. Hence this quantity is to be regarded as corresponding to a flow of heat measured relative to boundaries at rest in the fluid, rather than in some way relative to boundaries at rest with respect to the spatial coordinates that are actually being employed, as might at first sight have seemed plausible. Since this conclusion was obtained, however, with the help of a coordinate system specially chosen so that boundaries at rest with respect to the fluid were also at rest with respect to the spatial coordinates, it will be illuminating to show that we should also be led to the same interpretation for dQ_0/T_0 using a system of coordinates with respect to which the fluid is not at rest.

To obtain such a demonstration, we shall now use a set of coordinates x, y, z, t with respect to which the fluid is not at rest, but which are so chosen as to be *natural coordinates* for the point of interest. In accordance with the principle of equivalence such natural coordinates for any desired spacetime point can always be found, and with respect to them the principles of special relativity as expressed in their usual form may be taken as valid in the immediate neighborhood of the point in question. Hence, by choosing these coordinates, we shall be able to employ the principles of thermodynamics as developed for the special theory of relativity by Planck and Einstein in arriving at the desired interpretation.

Making use of these natural coordinates x, y, z, t, we shall evidently have

$$(-g)^{\frac{1}{2}} = 1$$

and

$$\frac{\partial}{\partial x}(-g)^{\frac12}=\frac{\partial}{\partial y}(-g)^{\frac12}=\frac{\partial}{\partial z}(-g)^{\frac12}=\frac{\partial}{\partial t}(-g)^{\frac12}=0 \quad (11)$$

at the point of interest. Hence by substituting

into the relativistic second law of thermodynamics in the form given by (2), we can now write the left-hand side of that expression in our present coordinates in the form

$$\left[\frac{\partial}{\partial x}\left(\phi_0\frac{dx}{ds}\right)+\frac{\partial}{\partial y}\left(\phi_0\frac{dy}{ds}\right)+\frac{\partial}{\partial z}\left(\phi_0\frac{dz}{ds}\right)+\frac{\partial}{\partial t}\left(\phi_0\frac{dt}{ds}\right)\right]dxdydzdt \quad (12)$$

where dx/ds, dy/ds, and dz/ds are the components of the macroscopic velocity of the fluid at the point of interest with respect to proper time, and by a simple substitution we can rewrite this in the form

$$\left[\frac{\partial}{\partial x}\left(\phi_0\frac{dt}{ds}\frac{dx}{dt}\right)+\frac{\partial}{\partial y}\left(\phi_0\frac{dt}{ds}\frac{dy}{dt}\right)+\frac{\partial}{\partial z}\left(\phi_0\frac{dt}{ds}\frac{dz}{dt}\right)+\frac{\partial}{\partial t}\left(\phi_0\frac{dt}{ds}\right)\right]dxdydzdt, \quad (13)$$

where dx/dt, dy/dt and dz/dt give the components of the velocity of the fluid as ordinarily expressed in terms of the coordinate time t.

In accordance with the special theory of relativity, however, entropy is an invariant for the Lorentz transformation, and hence entropy density will be affected by the Lorentz factor of contraction ds/dt in such a way that we can substitute

$$\phi=\phi_0(dt/ds), \quad (14)$$

where ϕ is the density of entropy at the point of interest taken with respect to our present system of coordinates x, y, z, t.

Substituting (14) into (13), and writing for simplicity u, v, and w for the three components of velocity dx/dt, dy/dt and dz/dt, we then obtain

$$\left[\frac{\partial}{\partial x}(\phi u)+\frac{\partial}{\partial y}(\phi v)+\frac{\partial}{\partial z}(\phi w)+\frac{\partial\phi}{\partial t}\right]dxdydzdt \quad (15)$$

and by performing the indicated differentiations and rearranging the order, this can be rewritten in the form

$$\left[\frac{\partial\phi}{\partial t}+u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}+w\frac{\partial\phi}{\partial z}+\phi\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]dxdydzdt. \quad (16)$$

Finally, noting the significance of the various terms in (16), we may now rewrite this expression for the left-hand side of the relativistic second law in the simple form

$$(d\phi/dt)dVdt+\phi(d/dt)(dV)dt, \quad (17)$$

where we have written dV for the volume of the element of fluid, instantaneously contained in the region $dxdydz$, and have written $d\phi/dt$ for the total rate of change in entropy density as we follow the moving fluid. Hence in natural coordinates the left-hand side of the relativistic second law is seen to be equal to the change that takes place in time dt in the entropy of the element of fluid instantaneously contained in the region $dxdydz$, and we can write for these coordinates

$$\left(\phi_0\frac{dx^\mu}{ds}\right)_\mu(-g)^{\frac12}dx^1dx^2dx^3dx^4=\frac{d}{dt}(\phi dV)dt. \quad (18)$$

In accordance, however, with special relativistic thermodynamic theory we can relate this change in the entropy, of the little thermodynamic system consisting of this element of fluid, to heat flow and temperature by the expression

$$d/dt\,(\phi dV)dt\geqq dQ/T, \quad (19)$$

where dQ is the heat absorbed by the element at temperature T and in the time dt referred to our present coordinates. Furthermore, since the ratio of heat to temperature is an invariant for the Lorentz transformation, we can also take

$$dQ/T=dQ_0/T_0, \quad (20)$$

where dQ_0 is the absorbed heat as measured in

proper coordinates and T_0 is the proper temperature of the element. Moreover, in accordance with the Lorentz contraction for volume elements and Lorentz dilation for time intervals we can write

$$dVdt = dV_0 dt_0, \qquad (21)$$

where dV_0 is the volume of this element as measured in proper coordinates and dt_0 is the proper time during which the heat absorption takes place.

Hence combining (18), (19) and (20), we have now obtained, also using our present coordinates, the result

$$\left(\phi_0 \frac{dx^\mu}{ds}\right)_\mu (-g)^{\frac{1}{2}} dx^1 dx^2 dx^3 dx^4 \geqq \frac{dQ_0}{T_0}, \qquad (22)$$

where, in accordance with (21), dQ_0 is to be taken as the heat—measured by a local observer at rest in the fluid at the point of interest—which flows into an element of the fluid having the instantaneous proper volume dV_0 during the proper time dt_0, these quantities being chosen so as to satisfy the numerical equality

$$dV_0 dt_0 = dVdt = (-g)^{\frac{1}{2}} dx^1 dx^2 dx^3 dx^4. \qquad (23)$$

Thus, using natural coordinates, we have satisfactorily obtained the same interpretation of the quantities occurring in the relativistic second law as was obtained in the preceding section using co-moving coordinates. And the treatment gives the desired illustration of the fact that our interpretation of dQ_0 as heat flowing through a boundary stationary in the fluid was not a result of our original special choice of coordinates in which the fluid was at rest.

§5. CONCLUSION

Before concluding, two further remarks may be made concerning the interpretation of the relativistic second law, which may be illuminating.

In accordance with the treatment given in the last section it is to be noted that the quantities ϕ, ϕu, ϕv and ϕw must be regarded as *total* densities of entropy and entropy flux at the point of interest, since otherwise, for example, the left-hand side of expression (19) could not have been

interpreted as the change in the total entropy of the specified element of fluid and correlated as was done with the influx of heat. This implies that we are to treat our thermodynamic fluid from a macroscopic point of view and take ϕ as the total entropy associated and moving along with unit volume of that material, rather than to try to employ a microscopic point of view and take ϕ as entropy belonging in some way solely to the molecules of the fluid with an additional quantity belonging to the radiation in the space between them. Similar remarks apply to ϕ_0 and $\phi_0 dx^\mu/ds$. It will be noted that the correct procedure is in entire agreement with ordinary thermodynamic practice and with the spirit of thermodynamics as a macroscopic phenomenological science.

In laying down a correct method for determining the magnitude of the heat flux divided by temperature dQ_0/T_0 which corresponds to the infinitesimal element of fluid and time denoted by $dx^1 dx^2 dx^3 dx^4$, it will have been noted that we have merely specified that the local observer should measure the proper heat dQ_0 entering an element of the fluid of instantaneous proper volume dV_0 during the proper time dt_0, where dV_0 and dt_0 are chosen so as to satisfy the numerical equality

$$dV_0 dt_0 = (-g)^{\frac{1}{2}} dx^1 dx^2 dx^3 dx^4 \qquad (21)$$

and have not introduced any specifications as to the shape of the four-dimensional region $dV_0 dt_0$. This, however, will be seen to be legitimate since the heat flux is directly proportional to the product of volume and time, independent of shape except for higher order differentials which can be neglected on shrinking the four-dimensional region down to the point of interest.

It is hoped that the treatment which we have given in this article will help to make the interpretation of the relativistic second law clear. The considerations presented are in agreement with the results obtained in previous applications of relativistic thermodynamics where there was no flow of heat relative to the fluid under consideration, and should be specially helpful when further applications are made to systems in which a flow of heat relative to the fluid does take place.