Note on Kowalewski's Top in Quantum Mechanics

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(Received February 14, 1933)

It is well known that in classical mechanics algebraic integrals of the top equations only exist in the cases of Euler (asymmetric top, no electric moment), of Lagrange (symmetric top, electric moment parallel to the axis of figure) and of Kowalewski. The latter case is that of a symmetric top whose two equal moments of inertia are twice as large as the third ($A=B=2C$) with an electric moment perpendicular to the axis of figure. The quantum mechanical analogue to Euler's and Lagrange's cases being well known, Kowalewski's case was tried. If ϑ, ψ, φ are the Euler angles, $Q_{\rm I}=Q_1+iQ_2$, $Q_{\rm II}=Q_1-iQ_2$ linear combinations of the momenta around the principal axes and

$$U=Q_{\rm I}^2+4C\mu\sin\vartheta e^{-i\psi},$$

then Kowalewski's integral becomes:

$$UU^*+U^*U+4h^2(Q_{\rm I}Q_{\rm II}+Q_{\rm II}Q_{\rm I})=\text{Diag. Matrix},$$

which differs from the classical result by the symmetrization and by the last term proportional to h^2.

I. INTRODUCTION

THERE are two cases where the classical equations for the rotation of a rigid heavy body around a fixed point may be integrated readily by separation of variable of the Hamiltonian in question: Firstly the case of Lagrange, the case of a symmetric top ($A=B\neq C$) with the center of gravity on the axis of figure. Secondly the case of Euler, the case of a totally asymmetric top ($A\neq B\neq C$) with the center of gravity coinciding with the fixed point of rotation. However in 1888 Sonya Kowalewski[1] succeeded in solving another case of top motion, namely that of a symmetric top whose two equal moments of inertia are twice as large as the third ($A=B=2C$) and whose center of gravity lies in the plane of the two equal moments, or perpendicular to the axis of figure. Kowalewski only was able to integrate this case because she discovered (in addition to the energy and momentum integrals) a third algebraic integral which brought the number of first integrals up to three, as in the cases of Euler and Lagrange.

It is of interest to note that in 1907 it was proved by Husson[2] that a third algebraic integral only exists in the aforementioned cases of Euler, Lagrange and Kowalewski or in special cases of these.

The quantum mechanical analogue of Lagrange's case, namely that of a symmetric top molecule with the electric moment parallel to the axis of figure was treated by Reiche and Rademacher,[3] by Kronig and Rabi,[4] and by Manneback.[5] Euler's case, the asymmetric top, was treated quantum-mechanically by Wang,[6] by Kramers and Ittmann,[7] and by Klein.[8] In the present paper we show that the quantum mechanical analogue of the Kowalewski top also possesses a third algebraic integral. Thus it is to be expected that a complete solution of the problem is possible in spite of the inseparability of Hamiltonian of the problem.

II. THE FUNDAMENTAL EQUATIONS OF KOWALEWSKI'S TOP

For the sequel the reader is referred to the Leiden dissertation of H. B. G. Casimir[9] where more detailed derivations of the equations of rotation in quantum mechanics are given.

The independent variables are the Euler angles.

[1] S. Kowalewski, Acta Mathematica **12**, 177 (1888).

[2] E. Husson, Ann. Fac. Science Toulouse **8**, 73 (1906).

[3] F. Reiche and H. Rademacher, Zeits. f. Physik **39**, 444 (1926); **41**, 453 (1927).

[4] R. deL. Kronig and J. J. Rabi, Phys. Rev. **29**, 262 (1927).

[5] C. Manneback, Phys. Zeits. **28**, 72 (1927).

[6] S. C. Wang, Phys. Rev. **34**, 243 (1929).

[7] H. A. Kramers and G. P. Ittmann, Zeits. f. Physik **53**, 553 (1929); **58**, 217 (1929); **60**, 663 (1930).

[8] O. Klein, Zeits. f. Physik **58**, 730 (1929).

[9] H. B. G. Casimir, *Leiden Dissertation*, 1931.

The notation of Whittaker's Analytical Dynamics is used so that if $OXYZ$ is a right-hand system fixed in space and $Oxyz$ a right-hand system fixed in the body and if OK is the nodal line then

$$\sphericalangle zOZ=\vartheta;\quad \sphericalangle YOK=\varphi;\quad \sphericalangle yOK=\psi.$$

It is not necessary to write down all nine direction cosines (cf. Whittaker, page 10), all we need are the cosines of the angles formed by the vertical Z

$$\begin{aligned}\text{with } x:\ \gamma_1&=-\sin\vartheta\cos\psi,\\ \text{with } y:\ \gamma_2&=\ \ \sin\vartheta\sin\psi,\\ \text{with } z:\ \gamma_3&=\ \ \cos\vartheta.\end{aligned}\tag{1}$$

As axis of figure we choose the z axis. Let the electric moment be in the direction of the x axis, so that the potential energy is

$$V=\mu\gamma_1=-\mu\sin\vartheta\cos\psi.\tag{2}$$

As momenta we have Q_1, Q_2, Q_3, the momenta around the axes of the principal moments of inertia. Now it is essential for the following, that both for the Q and for the γ we always use the following complex combinations:

$$\begin{aligned}Q_{\mathrm{I}}&=Q_1+iQ_2;\quad \gamma_{\mathrm{I}}=\gamma_1+i\gamma_2=-\sin\vartheta e^{-i\psi},\\ Q_{\mathrm{II}}&=Q_1-iQ_2;\quad \gamma_{\mathrm{II}}=\gamma_1-i\gamma_2=-\sin\vartheta e^{+i\psi}.\end{aligned}\tag{3}$$

As operators upon functions of the Euler angles the Q look as follows[10]:

$$\begin{aligned}Q_{\mathrm{I}}&=-ihe^{-i\psi}\left\{\frac{1}{\sin\vartheta}\left(\cos\vartheta\frac{\partial}{\partial\psi}-\frac{\partial}{\partial\varphi}\right)+i\frac{\partial}{\partial\vartheta}\right\},\\ Q_{\mathrm{II}}&=-ihe^{+i\psi}\left\{\frac{1}{\sin\vartheta}\left(\cos\vartheta\frac{\partial}{\partial\psi}-\frac{\partial}{\partial\varphi}\right)-i\frac{\partial}{\partial\vartheta}\right\},\\ Q_3&=-ih\frac{\partial}{\partial\psi}.\end{aligned}\tag{4}$$

From these formulae the exchange relations of the Q with each other and with the γ may be derived; or they may be obtained by generalizing, in the well-known way their Poisson brackets of classical mechanics. Using the abbreviation of Born and Jordan[11]

$$(i/h)(xy-yx)=[x,y]\tag{5}$$

we put down, for future reference, the following table of exchange relations:[12]

$$\begin{aligned}
&[Q_{\mathrm{I}},\ Q_3]=iQ_{\mathrm{I}}, &&\text{(a)}\quad &&[Q_k,\ \gamma_k]=0, &&\text{(h)}\\
&[Q_{\mathrm{II}},Q_3]=-iQ_{\mathrm{II}}, &&\text{(b)} &&[Q_{\mathrm{I}},\ \gamma_3]=i\gamma_{\mathrm{I}}, &&\text{(i)}\\
&[Q_{\mathrm{I}},Q_{\mathrm{II}}]=-2iQ_3, &&\text{(c)} &&[Q_{\mathrm{II}},\gamma_3]=-i\gamma_{\mathrm{II}}, &&\text{(k)}\\
&[Q_{\mathrm{I}}^2,Q_3]=2iQ_{\mathrm{I}}^2, &&\text{(d)} &&[Q_3,\ \gamma_{\mathrm{I}}]=-i\gamma_{\mathrm{I}}, &&\text{(l)}\\
&[Q_{\mathrm{II}}^2,Q_3]=-2iQ_{\mathrm{II}}^2, &&\text{(e)} &&[Q_3,\gamma_{\mathrm{II}}]=i\gamma_{\mathrm{II}}, &&\text{(m)}\\
&[Q_{\mathrm{I}}Q_{\mathrm{II}},Q_3]=0, &&\text{(f)} &&[Q_{\mathrm{I}},\gamma_{\mathrm{II}}]=-2i\gamma_3, &&\text{(n)}\\
&[\gamma_k,\gamma_l]=0, &&\text{(g)} &&[Q_{\mathrm{II}},\gamma_{\mathrm{I}}]=2i\gamma_3, &&\text{(o)}
\end{aligned}$$

$$[Q_{\mathrm{I}}^2,\gamma_{\mathrm{II}}]=-2i(Q_{\mathrm{I}}\gamma_3+\gamma_3Q_{\mathrm{I}}),\tag{p}$$

$$[Q_{\mathrm{II}}^2,\gamma_{\mathrm{I}}]=+2i(Q_{\mathrm{II}}\gamma_3+\gamma_3Q_{\mathrm{II}}).\tag{q}$$

If we put $A=B=2C$ we obtain as Hamiltonian[13]:

$$H=(1/8C)(Q_{\mathrm{I}}Q_{\mathrm{II}}+Q_{\mathrm{II}}Q_{\mathrm{I}})+(1/2C)Q_3^2+\tfrac{1}{2}\mu(\gamma_{\mathrm{I}}+\gamma_{\mathrm{II}}).\tag{6}$$

H put equal to the diagonal matric E is the first of the three algebraic integrals. The second is due to the fact that H does not contain the Euler angle φ explicitly: thus

$$p_\varphi=\tfrac{1}{2}(\gamma_{\mathrm{II}}Q_{\mathrm{I}}+\gamma_{\mathrm{I}}Q_{\mathrm{II}})+\gamma_3Q_3=\text{Diag. Matrix.}\tag{7}$$

III. Kowalewski's Integral

We now have to work with the Euler equations of the top which are obtained from the consideration, that the time derivative of any matric function f is given by $\dot{f}=[H,f]$. We then obtain, using repeatedly the exchange relations:

$$\begin{aligned}4C\dot{Q}_{\mathrm{I}}&=-i(Q_3Q_{\mathrm{I}}+Q_{\mathrm{I}}Q_3)+4iC\mu\gamma_3,\\ 4C\dot{Q}_{\mathrm{II}}&=+i(Q_3Q_{\mathrm{II}}+Q_{\mathrm{II}}Q_3)-4iC\mu\gamma_3,\\ 4C\dot{Q}_3&=2iC\mu(\gamma_{\mathrm{I}}-\gamma_{\mathrm{II}}),\end{aligned}\tag{8}$$

$$\begin{aligned}4C\dot{\gamma}_{\mathrm{I}}&=-2i(Q_3\gamma_{\mathrm{I}}+\gamma_{\mathrm{I}}Q_3)+i(Q_{\mathrm{I}}\gamma_3+\gamma_3Q_{\mathrm{I}}),\\ 4C\dot{\gamma}_{\mathrm{II}}&=+2i(Q_3\gamma_{\mathrm{II}}+\gamma_{\mathrm{II}}Q_3)-i(Q_{\mathrm{II}}\gamma_3+\gamma_3Q_{\mathrm{II}}),\\ 4C\dot{\gamma}_3&=-\tfrac{1}{2}i\{(Q_{\mathrm{I}}\gamma_{\mathrm{II}}+\gamma_{\mathrm{II}}Q_{\mathrm{I}})-(Q_{\mathrm{II}}\gamma_{\mathrm{I}}+\gamma_{\mathrm{I}}Q_{\mathrm{II}})\}.\end{aligned}\tag{9}$$

[10] Casimir, p. 57, Eq. (18).

[11] M. Born and P. Jordan, *Elementare Quantenmechanik*, Berlin, 1930, p. 23, Eq. (7).

[12] Casimir, p. 44, Eqs. (1) to (6).

[13] Casimir, p. 45, Eqs. (10).

Except for the symmetrizations Eqs. (8) and (9) are exactly the same as in classical mechanics. We multiply the first Eq. (8) with Q_{I} from the left and then again with Q_{I} from the right and thus obtain the time derivative of the square of Q_{I}:

$$4C(Q_{\mathrm{I}}^2)^{\cdot}=-i(Q_{\mathrm{I}}^2Q_3+2Q_{\mathrm{I}}Q_3Q_{\mathrm{I}}+Q_3Q_{\mathrm{I}}^2)+4iC\mu(Q_{\mathrm{I}}\gamma_3+\gamma_3Q_{\mathrm{I}}).$$

The last term of this equation is eliminated by means of (9.1). Applying exchange rule (a), (d) and (l) we have:

$$\begin{aligned}C(Q_{\mathrm{I}}^2-4\mu C\gamma_{\mathrm{I}})^{\cdot}&=-iQ_3(Q_{\mathrm{I}}^2-4\mu C\gamma_{\mathrm{I}})-ih(Q_{\mathrm{I}}^2-2\mu C\gamma_{\mathrm{I}}) &(10.1)\\&=-i(Q_{\mathrm{I}}^2-4\mu C\gamma_{\mathrm{I}})Q_3+ih(Q_{\mathrm{I}}^2-2\mu C\gamma_{\mathrm{I}}). &(10.2)\end{aligned}$$

We shall use both forms later on. We now treat the second Eq. (8) and the second Eq. (9) in a completely analogous manner and obtain:

$$\begin{aligned}C(Q_{\mathrm{II}}^2-4\mu C\gamma_{\mathrm{II}})^{\cdot}&=iQ_3(Q_{\mathrm{II}}^2-4\mu C\gamma_{\mathrm{II}})-ih(Q_{\mathrm{II}}^2-2\mu C\gamma_{\mathrm{II}}) &(11.1)\\&=i(Q_{\mathrm{II}}^2-4\mu C\gamma_{\mathrm{II}})Q_3+ih(Q_{\mathrm{II}}^2-2\mu C\gamma_{\mathrm{II}}). &(11.2)\end{aligned}$$

We introduce the abbreviations:

$$\begin{aligned}Q_{\mathrm{I}}^2-4\mu C\gamma_{\mathrm{I}}&=U,\\Q_{\mathrm{II}}^2-4\mu C\gamma_{\mathrm{II}}&=U^*,\\UU^*+U^*U&=T.\end{aligned}\qquad(12)$$

We compute

$$\dot{T}=\dot{U}U^*+U\dot{U}^*+\dot{U}^*U+U^*\dot{U}. \qquad (13)$$

It now simplifies the calculation of $\dot{T}$ considerably if in the first term of this expression one uses (10.2) for $\dot{U}$ and in the last term (10.1) for $\dot{U}$. Similarly in the second term of (13) we will use (11.1) for U^* and in the third term (11.2) for U^*. We then have:

$$\begin{aligned}-iC\dot{T}=&\{-(Q_{\mathrm{I}}^2-4\mu C\gamma_{\mathrm{I}})Q_3+h(Q_{\mathrm{I}}^2-2\mu C\gamma_{\mathrm{I}})\}(Q_{\mathrm{II}}^2-4\mu C\gamma_{\mathrm{II}})\\&+(Q_{\mathrm{I}}^2-4\mu C\gamma_{\mathrm{I}})\{Q_3(Q_{\mathrm{II}}^2-4\mu C\gamma_{\mathrm{II}})-h(Q_{\mathrm{II}}^2-2\mu C\gamma_{\mathrm{II}})\}\\&+\{(Q_{\mathrm{II}}^2-4\mu C\gamma_{\mathrm{II}})Q_3+h(Q_{\mathrm{II}}^2-2\mu C\gamma_{\mathrm{II}})\}(Q_{\mathrm{I}}^2-4\mu C\gamma_{\mathrm{I}})\\&+(Q_{\mathrm{II}}^2-4\mu C\gamma_{\mathrm{II}})\{-Q_3(Q_{\mathrm{I}}^2-4\mu C\gamma_{\mathrm{I}})-h(Q_{\mathrm{I}}^2-2\mu C\gamma_{\mathrm{I}})\}.\end{aligned}$$

Evidently all terms which are not proportional to h cancel. We are left with:

$$C\dot{T}=h^2\{[Q_{\mathrm{I}}^2-2\mu C\gamma_{\mathrm{I}},\,Q_{\mathrm{II}}^2-4\mu C\gamma_{\mathrm{II}}]+[Q_{\mathrm{II}}^2-2\mu C\gamma_{\mathrm{II}},\,Q_{\mathrm{I}}^2-4\mu C\gamma_{\mathrm{I}}]\}.$$

The terms containing Q in the fourth order cancel each other in the two exchange brackets. So do the terms containing products of γ. We then have:

$$\dot{T}=-2h^2\mu\{[Q_{\mathrm{I}}^2,\,\gamma_{\mathrm{II}}]+[Q_{\mathrm{II}}^2,\,\gamma_{\mathrm{I}}]\}.$$

Applying exchange rule (p) and (q) we finally get:

$$\dot{T}=4ih^2\mu(Q_{\mathrm{I}}\gamma_3+\gamma_3Q_{\mathrm{I}}-Q_{\mathrm{II}}\gamma_3-\gamma_3Q_{\mathrm{II}}). \qquad (14)$$

Leaving this equation for the moment we now return to the Euler Eqs. (8). We propose to multiply (8.1) with Q_{II} first from the left and then from the right and similarly (8.2) with Q_{I} both from the left and right sides. These four equations are then added:

$$\left.\begin{aligned}4C(Q_{\mathrm{I}}Q_{\mathrm{II}}+Q_{\mathrm{II}}Q_{\mathrm{I}})^{\cdot}&\\=i(-Q_{\mathrm{II}}Q_3Q_{\mathrm{I}}-Q_{\mathrm{II}}Q_{\mathrm{I}}Q_3-Q_3Q_{\mathrm{I}}Q_{\mathrm{II}}&\\-Q_{\mathrm{I}}Q_3Q_{\mathrm{II}}+Q_{\mathrm{I}}Q_3Q_{\mathrm{II}}+Q_{\mathrm{I}}Q_{\mathrm{II}}Q_3&\\+Q_3Q_{\mathrm{II}}Q_{\mathrm{I}}+Q_{\mathrm{II}}Q_3Q_{\mathrm{I}})&\\+4i\mu C(Q_{\mathrm{II}}\gamma_3+\gamma_3Q_{\mathrm{II}}-Q_{\mathrm{I}}\gamma_3-\gamma_3Q_{\mathrm{I}}).&\end{aligned}\right\}\qquad(15)$$

In the first parenthesis on the right side the first and last as well as the fourth and fifth term cancel. The remaining four terms of the first parenthesis may be written:

$$[[Q_{\mathrm{I}},\,Q_{\mathrm{II}}],\,Q_3]$$

which vanishes because of (c). Combining (14) and (15) we find:

$$\dot{T}=-4h^2(Q_{\mathrm{I}}Q_{\mathrm{II}}+Q_{\mathrm{II}}Q_{\mathrm{I}})^{\cdot}$$

or integrating and using (13):

$$UU^*+U^*U+4h^2(Q_{\rm I}Q_{\rm II}+Q_{\rm II}Q_{\rm I})=\text{Diag. Matrix.} \tag{16.1}$$

Taking into account (12) and (3) one may write this:

$$(Q_{\rm I}^2-4\mu C\gamma_{\rm I})(Q_{\rm II}^2-4\mu C\gamma_{\rm II})+(Q_{\rm II}^2-4\mu C\gamma_{\rm II})(Q_{\rm I}^2-4\mu C\gamma_{\rm I})+4h^2(Q_{\rm I}Q_{\rm II}+Q_{\rm II}Q_{\rm I})=\text{Diag. Matrix.} \tag{16.2}$$

or

$$(Q_1^2-Q_2^2+4\mu C\sin\vartheta\cos\psi)^2+(Q_1Q_2+Q_2Q_1-4\mu C\sin\vartheta\sin\psi)^2+4h^2(Q_1^2+Q_2^2)=\text{Diag. Matrix.} \tag{16.3}$$

Eqs. (16) represent the quantum mechanical analogue of the Kowalewski Integral, which in classical mechanics simply is: $UU^*=\text{const.}$ It is interesting that the simplest quantum-like generalization, namely symmetrization of U and U^* is not sufficient to obtain an integral; the term proportional to h^2 has to be added. Nor is it possible to embody this correction term in the definition of U as one might attempt in order to preserve the classical form as closely as possible.

Eqs. (6), (7) and (16) form the three first integrals of the problem. By means of these (or rather their classical counterparts) Kowalewski showed that the problem could be completely solved and reduced to hyperelliptic integrals. It remains to be seen whether or not a similar reduction is possible also in quantum mechanics.