Variational Principles in Electromagnetism

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The familiar relation between the potentials in electromagnetic theory is regarded as a consequence of the principles of angular momentum and center of mass expressed by the symmetry of the stress-energy tensor. The electromagnetic equations are derived from a Lagrangian function L equal to an arbitrary function of the invariants E^2-H^2 and $E\cdot H$ of the electromagnetic field and rules different from that of Schrödinger are given for the construction of a stress-energy-tensor.

IN a recent paper Fock and Podolsky[1] have derived the equations of electromagnetism from the Lagrangian function

$$L=\tfrac{1}{2}(E^2-H^2)-\tfrac{1}{2}\left(\operatorname{div} A+\frac{1}{c}\frac{\partial\Phi}{\partial t}\right)^2 \qquad (1)$$

and have added the equation

$$E_t\equiv\operatorname{div} A+(1/c)(\partial\Phi/\partial t)=0 \qquad (2)$$

after the equations

$$\operatorname{curl} H=\frac{1}{c}\frac{\partial E}{\partial t}+\nabla E_t,\quad \operatorname{div} E+\frac{1}{c}\frac{\partial E_t}{\partial t}=0 \qquad (3)'$$

have been derived by assuming the relations $H=\operatorname{curl} A$, $E=-(1/c)(\partial A/\partial t)-\nabla\Phi$, which imply that

$$\operatorname{curl} E=-(1/c)(\partial H/\partial t),\quad \operatorname{div} H=0. \qquad (3)''$$

A physical reason is needed for Eq. (2) and it has occurred to the author that this may be furnished by the requirement of symmetry of the stress-energy-tensor associated with Eqs. (3).

In forming this tensor we shall adopt the principle that the components of the tensor should involve only the quantities E_x, E_y, E_z, H_x, H_y, H_z, E_t occurring in the electromagnetic equations. A tensor satisfying this requirement and the laws of conservation of energy and linear momentum has already been constructed,[2] its components are

$$\begin{aligned}
X_x&=E_x^2+H_x^2+\tfrac{1}{2}(E_t^2-E^2-H^2),\\
X_y&=E_xE_y+H_xH_y-E_tH_z,\\
Y_x&=E_xE_y+H_xH_y+E_tH_z,\\
X_z&=E_xE_z+H_xH_z+E_tH_y,\\
Z_x&=E_xE_z+H_xH_z-E_tH_y,\\
G_x&=E_yH_z-E_zH_y-E_xE_t,\\
S_x&=c(E_yH_z-E_zH_y+E_xE_t),\\
W&=\tfrac{1}{2}(E^2+H^2+E_t^2),\quad\cdots.
\end{aligned} \qquad (4)$$

The conservation laws

$$\begin{aligned}
&\partial X_x/\partial x+\partial X_y/\partial y+\partial X_z/\partial z=(1/c)(\partial G_x/\partial t)\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\partial S_x/\partial x+\partial S_y/\partial y+\partial S_z/\partial z+\partial W/\partial t=0
\end{aligned} \qquad (5)$$

are readily seen to be a consequence of Eqs. (3) but the law of symmetry

$$Y_z=Z_y,\ Z_x=X_z,\ X_y=Y_x,\ S_x=cG_x,\ S_y=cG_y,\ S_z=cG_z \qquad (6)$$

associated with the principle of the conservation of angular momentum and the property of the center of mass, is satisfied only when $E_t=0$ unless there is no electromagnetic field. It should be mentioned that a symmetrical tensor

$$\begin{aligned}
X_x&=E_x^2+H_x^2-\tfrac{1}{2}(E^2+H^2-E_t^2)-2E_t\frac{\partial A_x}{\partial x},\\
X_y&=E_xE_y+H_xH_y-E_t\left(\frac{\partial A_x}{\partial y}+\frac{\partial A_y}{\partial x}\right)=Y_x,\\
G_x&=E_yH_z-E_zH_y-E_t\left(\frac{1}{c}\frac{\partial A_x}{\partial t}-\frac{\partial\Phi}{\partial x}\right)=\frac{1}{c}S_x,\\
W&=\tfrac{1}{2}(E^2+H^2+E_t^2)-2E_t\frac{1}{c}\frac{\partial\Phi}{\partial t}
\end{aligned} \qquad (7)$$

[1] V. Fock and B. Podolsky, Phys. Zeits. d. Sow. 1, 801 (1932).

[2] H. Bateman, Phys. Rev. 12, 459 (1918).

may be derived from the Lagrangian function (1) by using Schrödinger's rule[3] but if $E_t \neq 0$ the conservation laws are not generally a consequence of (3) and, moreover, this tensor involves quantities which do not enter into the electromagnetic Eqs. (3) and so is ruled out by our requirement. It should be mentioned that Schrödinger's rule does not always associate a symmetrical stress-energy tensor with a Lagrangian function[4] so even if this rule were regarded as of universal application the principle of the conservation of angular momentum could not be regarded as a direct consequence of the variational principle alone. There are cases, however, in which Schrödinger's rule does not seem to be applicable. To construct such a case let us write[5]

$$\begin{aligned} &h_x=\gamma_y-\beta_z-a_t-w_x, \quad e_x=b_z-c_y-\alpha_t-\phi_x,\\ &h_y=\alpha_z-\gamma_x-b_t-w_y, \quad e_y=c_x-a_z-\beta_t-\phi_y,\\ &h_z=\beta_x-\alpha_y-c_t-w_z, \quad e_z=a_y-b_x-\gamma_t-\phi_z,\\ &e_t=\alpha_x+\beta_y+\gamma_z+\phi_t, \quad h_t=a_x+b_y+c_z+w_t,\\ &\sigma=\tfrac{1}{2}(h_x^2+h_y^2+h_z^2-h_t^2)-\tfrac{1}{2}(e_x^2+e_y^2+e_z^2-e_t^2),\\ &\tau=h_xe_x+h_ye_y+h_ze_z-h_te_t \end{aligned} \tag{8}$$

and consider the Lagrangian function

$$L=f(\sigma, \tau).$$

It is readily seen that

$$\begin{aligned} &\frac{\partial L}{\partial \alpha_t}=\frac{\partial L}{\partial \phi_x}=\frac{\partial L}{\partial c_y}=-\frac{\partial L}{\partial b_z}=e_x\frac{\partial L}{\partial \sigma}-h_x\frac{\partial L}{\partial \tau}=E_x, \quad \text{say},\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ &\frac{\partial L}{\partial \alpha_x}=\frac{\partial L}{\partial \beta_y}=\frac{\partial L}{\partial \gamma_z}=\frac{\partial L}{\partial \phi_t}=e_t\frac{\partial L}{\partial \sigma}-h_t\frac{\partial L}{\partial \tau}=E_t, \quad \text{say},\\ &-\frac{\partial L}{\partial \alpha_t}=-\frac{\partial L}{\partial w_x}=\frac{\partial L}{\partial \gamma_y}=-\frac{\partial L}{\partial \beta_z}=h_x\frac{\partial L}{\partial \sigma}+e_x\frac{\partial L}{\partial \tau}=H_x, \quad \text{say},\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ &-\frac{\partial L}{\partial w_t}=-\frac{\partial L}{\partial a_x}=-\frac{\partial L}{\partial b_y}=-\frac{\partial L}{\partial c_z}=h_t\frac{\partial L}{\partial \sigma}+e_t\frac{\partial L}{\partial \tau}=H_t, \quad \text{say}, \end{aligned} \tag{9}$$

consequently the equations of Euler and Lagrange for the variation problem

$$\delta\int Ldxdydzdt=0$$

take the form[6]

$$\begin{aligned} &\nabla E_t+\partial E/\partial t=\text{curl } H, \quad \text{div } E+\partial E_t/\partial t=0,\\ &\nabla H_t+\partial H/\partial t=-\text{curl } E, \quad \text{div } H+\partial H_t/\partial t=0, \end{aligned} \tag{10}$$

and so reduce to the equations of Maxwell when $E_t=H_t=0$. When Schrödinger's rule

$$\begin{aligned} T_{xy}=&a_x\frac{\partial L}{\partial a_y}+b_x\frac{\partial L}{\partial b_y}+c_x\frac{\partial L}{\partial c_y}+w_x\frac{\partial L}{\partial w_y}\\ &+a_x\frac{\partial L}{\partial b_x}+a_y\frac{\partial L}{\partial b_y}+a_z\frac{\partial L}{\partial b_z}+a_t\frac{\partial L}{\partial b_t}\\ &+\alpha_x\frac{\partial L}{\partial \alpha_y}+\beta_x\frac{\partial L}{\partial \beta_y}+\gamma_x\frac{\partial L}{\partial \gamma_y}+\phi_x\frac{\partial L}{\partial \phi_y}\\ &+\alpha_x\frac{\partial L}{\partial \beta_x}+\alpha_y\frac{\partial L}{\partial \beta_y}+\alpha_2\frac{\partial L}{\partial \beta_z}+\alpha_t\frac{\partial L}{\partial \beta_t}-\delta_{xy}L,\\ &\delta_{xy}=0 \quad x\neq y\\ &\phantom{\delta_{xy}}=1 \quad x=y \end{aligned} \tag{11}$$

is applied to this function L it gives a tensor T whose components cannot be expressed simply in

[3] E. Schrödinger, Ann. d. Physik **82**, 265 (1927).

[4] H. Bateman, Proc. Nat. Acad. Sci. **13**, 326 (1927).

[5] The suffixes added to α, β, γ, ϕ, a, b, c, w are used to denote partial derivatives of the quantities. This rule does not extend to the suffixes added to e and h. For convenience the unit of time has been chosen so that the velocity of light is represented by unity.

[6] These are the equations considered by the author in Phys. Rev. **12**, 459 (1918).

terms of the quantities E_x, E_y, E_z, E_t, H_x, H_y, H_z, H_t, consequently we shall endeavor to replace it by some other rule. It is readily seen that if

$$
\begin{aligned}
2X_y &= \frac{\partial L}{\partial \alpha_t}\frac{\partial L}{\partial \phi_y}+\frac{\partial L}{\partial \alpha_x}\frac{\partial L}{\partial \alpha_y}+\frac{\partial L}{\partial \alpha_y}\frac{\partial L}{\partial \beta_y}+\frac{\partial L}{\partial \alpha_z}\frac{\partial L}{\partial \gamma_y}+\frac{\partial L}{\partial a_t}\frac{\partial L}{\partial w_y}+\frac{\partial L}{\partial a_x}\frac{\partial L}{\partial a_y}+\frac{\partial L}{\partial a_y}\frac{\partial L}{\partial b_y}+\frac{\partial L}{\partial a_z}\frac{\partial L}{\partial c_y},\\
2Y_x &= \frac{\partial L}{\partial \beta_t}\frac{\partial L}{\partial \phi_x}+\frac{\partial L}{\partial \beta_x}\frac{\partial L}{\partial \alpha_x}+\frac{\partial L}{\partial \beta_y}\frac{\partial L}{\partial \beta_x}+\frac{\partial L}{\partial \beta_z}\frac{\partial L}{\partial \gamma_x}+\frac{\partial L}{\partial b_t}\frac{\partial L}{\partial w_x}+\frac{\partial L}{\partial b_x}\frac{\partial L}{\partial a_x}+\frac{\partial L}{\partial b_y}\frac{\partial L}{\partial b_x}+\frac{\partial L}{\partial b_z}\frac{\partial L}{\partial c_x},\\
2X_x &= \frac{\partial L}{\partial \alpha_t}\frac{\partial L}{\partial \phi_x}+\left(\frac{\partial L}{\partial \alpha_x}\right)^2+\frac{\partial L}{\partial \alpha_y}\frac{\partial L}{\partial \beta_x}+\frac{\partial L}{\partial \alpha_z}\frac{\partial L}{\partial \gamma_x}+\frac{\partial L}{\partial a_t}\frac{\partial L}{\partial w_x}+\left(\frac{\partial L}{\partial a_x}\right)^2+\frac{\partial L}{\partial a_y}\frac{\partial L}{\partial b_x}+\frac{\partial L}{\partial a_z}\frac{\partial L}{\partial c_x},
\end{aligned}
\tag{12}
$$

and the other components are defined by corresponding equations, we obtain a tensor for which Eqs. (5) are satisfied. This tensor includes the tensor (4) as a particular case, indeed

$$
\begin{aligned}
X_x &= E_x^2+H_x^2-\tfrac{1}{2}(E^2+H^2-E_t^2-H_t^2),\\
X_y &= E_xE_y+H_xH_y-E_tH_z+H_tE_z,\\
Y_x &= E_xE_y+H_xH_y+E_tH_z-H_tE_z,
\end{aligned}
\tag{13}
$$

and so the components of (4) are obtained by putting $H_t=0$. On the other hand, a symmetrical tensor is obtained by writing

$$
\begin{aligned}
2X_y &= \frac{\partial L}{\partial \phi_x}\frac{\partial L}{\partial \phi_y}-\frac{\partial L}{\partial \alpha_x}\frac{\partial L}{\partial \alpha_y}-\frac{\partial L}{\partial \beta_x}\frac{\partial L}{\partial \beta_y}-\frac{\partial L}{\partial \gamma_x}\frac{\partial L}{\partial \gamma_y}+\frac{\partial L}{\partial w_x}\frac{\partial L}{\partial w_y}-\frac{\partial L}{\partial a_x}\frac{\partial L}{\partial a_y}-\frac{\partial L}{\partial b_x}\frac{\partial L}{\partial b_y}-\frac{\partial L}{\partial c_x}\frac{\partial L}{\partial c_y}=2Y_x,\\
2X_x &= \left(\frac{\partial L}{\partial \phi_x}\right)^2-\left(\frac{\partial L}{\partial \alpha_x}\right)^2-\left(\frac{\partial L}{\partial \beta_x}\right)^2-\left(\frac{\partial L}{\partial \gamma_x}\right)^2+\left(\frac{\partial L}{\partial w_x}\right)^2-\left(\frac{\partial L}{\partial a_x}\right)^2-\left(\frac{\partial L}{\partial b_x}\right)^2-\left(\frac{\partial L}{\partial c_x}\right)^2,
\end{aligned}
\tag{14}
$$

with similar equations for the other components. These equations give

$$
\begin{aligned}
X_x &= E_x^2+H_x^2-\tfrac{1}{2}(E^2+H^2+E_t^2+H_t^2),\\
X_y &= E_xE_y+H_xH_y,\\
X_z &= E_xE_z+H_xH_z,\\
G_x &= S_x=E_yH_z-E_zH_y,\\
W &= \tfrac{1}{2}(E^2+H^2-E_t^2-H_t^2)
\end{aligned}
\tag{15}
$$

but now the Eqs. (5) are satisfied only under certain conditions which are fulfilled when E_t and H_t are constants and there will be a density of negative energy at a great distance from all the electric and magnetic charges if these constants are different from zero. When $E_t=H_t=0$ the tensor reduces to the familiar electromagnetic tensor, also

$$
\tfrac{1}{2}(H^2-E^2)=\sigma\left\{\left(\frac{\partial L}{\partial \sigma}\right)^2-\left(\frac{\partial L}{\partial \tau}\right)^2\right\}+2\tau\frac{\partial L}{\partial \sigma}\frac{\partial L}{\partial \tau}=S,\text{ say,}
$$

$$
E\cdot H=\tau\left\{\left(\frac{\partial L}{\partial \sigma}\right)^2-\left(\frac{\partial L}{\partial \tau}\right)^2\right\}-2\sigma\frac{\partial L}{\partial \sigma}\frac{\partial L}{\partial \tau}=T,\text{ say,}
$$

so that L becomes a function of S and T. It should be mentioned that S. R. Milner[7] has already suggested that $(S^2+T^2)^{\frac{1}{2}}$ can be used as a Lagrangian function and on a previous occasion the author[4] has shown that T can be used in this way. The foregoing analysis shows that a very general function of S and T can also be used for the portion of space-time not occupied by electric

[7] S. R. Milner, Proc. Roy. Soc. **A120**, 483 (1928). In Milner's case $S=\sigma$, $T=-\tau$.

charges. The conditions at the outer boundary of this region may be fulfilled for all variations if the boundary is a moving wave-front beyond which there is no field. In the region occupied by electric charges additional terms must be added to the Lagrangian function. It should be mentioned that the tensor (15) can also be found by writing

$$2X_y=\frac{\partial L}{\partial\alpha_t}\frac{\partial L}{\partial\beta_t}-\frac{\partial L}{\partial\alpha_x}\frac{\partial L}{\partial\beta_x}-\frac{\partial L}{\partial\alpha_y}\frac{\partial L}{\partial\beta_y}-\frac{\partial L}{\partial\alpha_z}\frac{\partial L}{\partial\beta_z}+\frac{\partial L}{\partial a_t}\frac{\partial L}{\partial b_t}-\frac{\partial L}{\partial a_x}\frac{\partial L}{\partial b_x}-\frac{\partial L}{\partial a_y}\frac{\partial L}{\partial b_y}-\frac{\partial L}{\partial a_z}\frac{\partial L}{\partial b_z},$$

$$2X_x=\left(\frac{\partial L}{\partial\alpha_t}\right)^2-\left(\frac{\partial L}{\partial\alpha_x}\right)^2-\left(\frac{\partial L}{\partial\alpha_y}\right)^2-\left(\frac{\partial L}{\partial\alpha_z}\right)+\left(\frac{\partial L}{\partial a_t}\right)^2-\left(\frac{\partial L}{\partial a_x}\right)^2-\left(\frac{\partial L}{\partial a_y}\right)^2-\left(\frac{\partial L}{\partial a_z}\right)^2.$$

and similarly for the other components.