## On the Mass Defect of Helium

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If one assumes that the potential energy between protons and neutrons has the shape of a simple potential hole, it is possible from the experimental value of the mass defect of the  $H^2$ , to derive a connection between the mean width and the depth of this curve. This connection proves to be, to a large extent, independent of the finer details of the potential curve. By assuming a certain probable value, obtained from scattering experiments, for the width of the potential hole, it is possible to make calculations on the mass defects of other nuclei. Such computations were carried out for He and yield values which are greater than the mass defect of  $H^2$  by a rather large factor. This agrees with experiment. For the higher elements, the Pauli principle has to be taken into account and the structure of higher nuclei is discussed on this basis.

## I.

THE discovery of the neutron by Chadwick,<sup>1</sup> and by Curie and Joliot<sup>2</sup> has made possible a more detailed picture of the constitution of the nuclei. As far as can be seen at present, there are three different assumptions possible concerning the elementary particles.

(a) The only elementary particles are the proton and the electron. This point of view has been emphasized by Heisenberg and treated by him in a series of papers.<sup>3</sup>

(b) The neutrons are elementary particles and the nuclei are built up by protons, electrons and neutrons. This point of view was proposed by Dirac and adopted by Bartlett<sup>4</sup> in his discussion of the constituents of the light elements.

(c) It may be assumed furthermore that in addition to the neutrons, discovered by Chadwick ("heavy neutrons") there are "light neutrons" of electronic mass, as first proposed by Pauli.<sup>5</sup> The number of light neutrons should be equal to the number of electrons in every nucleus and they leave the nucleus simultaneously with the  $\beta$ -rays. The number of electrons (and light

neutrons) should be, just as in (b), equal to the number of "free electrons," as proposed by Beck.<sup>6</sup> Some arguments in favor of this assumption were given by the present author.<sup>7</sup>

For the present purpose (the comparison of the mass-defects of the first few elements) it does not make any difference whether we adopt the hypothesis contained in (b) or (c), because the first elements, even up to Cl, do not contain any free electrons. The calculations will probably hold, even if the hypothesis (a) is adopted.

There seem to be three alternative possible assumptions concerning the nature of the forces acting between protons and neutrons. (The forces between two protons or between two neutrons are always neglected.) Heisenberg assumed that these forces are of the exchange type, similar to those of the  $H_2^+$  molecule. If we suppose, however, that the neutrons have to be treated as elementary particles, one must either assume a certain potential energy V(r) between a proton and a neutron, or a three-body force. The present calculations will be made on the basis of the former assumption. The other possibility is to calculate with a potential energy which is a function of the mutual distance of three particles. Forces of this kind<sup>8</sup> must be assumed in the hypothesis (c) for

<sup>6</sup>G. Beck, Zeits. f. Physik 47, 407 (1928); 50, 548 (1928).

<sup>7</sup> E. Wigner, Proc. Hung. Acad., 1932.

<sup>8</sup> An example for such a potential is  $cE^2(1+e^{r/\rho})^{-1}$ , where *c* is a constant, *r* the distance of the neutron from one proton and *E* the electric field strength produced by the other protons.

<sup>&</sup>lt;sup>1</sup> Chadwick, Nature 129, 469 (1932).

<sup>&</sup>lt;sup>2</sup> Curie and Joliot, Comptes Rendus 193, 1412 and 1415 (1931).

<sup>&</sup>lt;sup>8</sup> W. Heisenberg, Zeits. f. Physik 77, 1 (1932); 78, 156 (1932).

<sup>&</sup>lt;sup>4</sup> Cf. W. Bartlett, Phys. Rev. 42, 145 (1932), (Letter to the Editor).

<sup>&</sup>lt;sup>5</sup> Cf. Carlson and Oppenheimer, Phys. Rev. 38, 1787 (1931).

the light neutrons, so it does not seem unnatural to allow them for the heavy neutrons as well.

The effect of the first kind of forces was fully discussed by Heisenberg and the discussion of the effect of the forces of the second kind can be carried out in a very similar way. One interesting feature of the second kind of forces is that it is probable that if a nucleus with  $n_p$  protons and  $n_h$  neutrons is stable and if  $n_p$  is odd then there is also a stable nucleus with  $n_p + 1$  protons and  $n_h$ neutrons. Also if  $n_h$  is odd there probably exists a stable nucleus with  $n_p$  protons and  $n_h+1$ neutrons. The reason for this is that if  $n_p$  is odd, the next proton may have the same wavefunction as this one, which is in conflict with the Pauli principle if  $n_p$  is even. From O up to Cl the nuclei predicted in this way are all known. Below oxygen, however, there are some nuclei lacking, namely those with the  $(n_p, n_h)$ values (1, 2), (2, 1), (4, 3), (4, 6), (6, 5), (6, 8), (8, 7). A possible reason that these nuclei have so far escaped detection, together with a more exact proof of the above-mentioned rule, will be given in Section III, a different explanation of their constitution was put forward by Jones.<sup>9</sup>

It may be seen furthermore, that just as in the theory of Heisenberg, the energies of the nuclei (n, n') and (n', n) are equal. Consequently among all nuclei with the same mass n+n' that with the charge  $n_p = (n_p + n_h)/2$  will be the most stable, having the largest number  $(n_p + n_h)^2/4$ of attracting terms. The formation of the nuclei after O<sup>16</sup> may be imagined like this<sup>4</sup>: Assuming that the addition of a heavy neutron to O<sup>16</sup> is connected with an energy gain, we get O<sup>17</sup>. Then according to the preceding rule, the capture of another neutron is possible, giving O<sup>18</sup>. By this process the number of neutrons is increased so much in the nucleus, that it may capture a new proton giving F and then another, giving Ne<sup>20</sup>. Now by the increased number of protons the capture of a new neutron is possible, giving Ne<sup>21</sup>, and with another one Ne<sup>22</sup>, and so on.

In addition to the difficulty connected with the apparent non-existence of the above-mentioned nuclei, it seems rather surprising that the nuclei between O and Cl adhere so very closely to the condition  $n_p = n_h$ . This difficulty can be avoided, of course, by assuming a repulsive force between the neutrons and between the protons at small distances.

The potential, as suggested in reference eight is also capable of explaining the qualitative features of the series of existing elements in some respects even better than that just discussed. It does not seem, however, to be easy to make simple assumptions as to the general shape of such a potential.

II.

One of the remarkable facts about the mass defects in the very first elements is the very great binding energy of the He nucleus. The binding energy of the H<sup>2</sup> nucleus is  $only^{10}$  three times the rest energy  $mc^2$  of the electron, the binding energy of the He is<sup>11</sup>  $52mc^2$ , if we assume the mass of the neutron equal to the mass of the proton 1.00724 (referred to the mass of neutral O<sup>16</sup>). The masses of the H<sup>2</sup> and He nuclei are taken to be 2.01297 and 4.00108, respectively. The binding energy of He is around 17 times larger than that of H<sup>2</sup>.

This would rather indicate an attraction between the neutrons or between the protons, which is very unlikely on the basis of the previous discussion. The purpose of the subsequent calculation is to see how far it is possible to explain the large mass defect of He without such an assumption, or even to reconcile it with the existence of some repulsive forces between the different neutrons and also the different protons.

First we consider the  $H^2$  nucleus. There are several indications that the first energy value depends only on the rough shape of the potential curve. For the  $H^2$  nucleus, therefore, the potential energy was assumed for the purpose of the calculation to be

$$V(r) = 4v_0/(1 + e^{r/\rho})(1 + e^{-r/\rho})$$
(1)

in units of  $mc^2$ , where  $v_0$  and  $\rho$  are constants. The Schrödinger equation becomes in this case

$$\left[-10\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V\right]\psi(xyz) = \epsilon\psi(xyz), \quad (2)$$

where x, y, z are the components of the distance between the two particles, and the energy and

<sup>&</sup>lt;sup>9</sup> E. G. Jones, Nature 130, 580 (1932).

<sup>&</sup>lt;sup>10</sup> K. T. Bainbridge, Phys. Rev. **42**, 1 (1932). J. D. Hardy, E. F. Barker and D. U. Dennison, Phys. Rev. **42**, 279 (1932).

<sup>&</sup>lt;sup>11</sup> F. W. Aston, Proc. Roy. Soc. A115, 502 (1927).

distance are always measured in units  $mc^2$  and  $e^2/mc^2$ , respectively, and we have for convenience set  $h^2mc^2/4\pi^2M=10$ . The characteristic numbers and functions of (2) with the potential (1) are known from the work of Eckart.<sup>12</sup> The lowest energy level is

$$-\epsilon = 50/8\rho^2 + v_0 - (30/8\rho^2)(1 + 8v_0\rho^2/5)^{1/2}, \quad (3)$$

while the corresponding (unnormalized) characteristic function is

$$\psi = \frac{\rho}{r} \frac{e^{r/\rho} - 1}{e^{r/\rho} + 1} \frac{1}{(1 + e^{r/\rho})^{\nu} (1 + e^{-r/\rho})^{\nu}}$$
(4)

with  $\nu = (-\epsilon \rho_2/10)^{1/2}$ . The function V(r) is graphically given in Fig. 1 (heavy line). The constants  $\rho$  and  $v_0$  must be chosen such that  $\epsilon = -3$  should give the observed binding energy of  $H^2$ . This gives an equation between  $v_0$ , the potential for r=0, and  $\rho$ , the mean thickness of the potential hole, which is given in Fig. 2 (heavy line). In order to have a better insight into the conditions governing the behavior of the characteristic values and characteristic function, the characteristic function (4) for  $\rho = 0.22$ ,  $v_0 = 140$  is given by the broken line in Fig. 1. One sees that it extends over a much wider region than V(r)and in consequence the mean potential energy is much smaller than  $v_0$ . In Fig. 3, the mean negative potential energy -P, the mean kinetic energy K and the negative total energy  $-\epsilon = 3$ are plotted against the parameter  $\rho$  for the case in which  $v_0$  is taken from Fig. 2, yielding (by Eq. (3))  $\epsilon = -3$ . For small values of  $\rho$ , the negative mean potential energy is much larger than  $-\epsilon$ , and is almost totally compensated by the kinetic energy. Thus the value of  $\epsilon$  is very sensitive to small variations of  $v_0$ , because these latter increase the mean potential energy without affecting the kinetic.

In order to have a check on the relative independence of the  $(v_0, \rho)$  curve on the exact shape of the potential function, another two-parameter family  $ae^{-br}$  of such functions was taken (light line in Fig. 1) and the parameters  $a=1.4v_0$ ,  $b=0.63/\rho$  have been chosen in such a way that this new potential be as similar to (1) as possible. The lowest energy-value was calculated then by a simple variational method (taking  $\psi = e^{-\beta r}$ ) and then a and b adjusted in such a way that the lowest energy value be again -3. The light line in Fig. 2 gives the relation obtained in this way between  $v_0 = a/1.4$  and  $\rho = 0.63/b$ . It runs very near to the line obtained for the potentialfunction (1). A more exact calculation would show that it runs yet a little lower than shown in Fig. 2.



<sup>&</sup>lt;sup>12</sup> C. Eckart, Phys. Rev. 35, 1303 (1930).



It is clear now what the cause of the large

the He consists of four potential energies (the attraction of both protons on both neutrons) and only four kinetic energies, as contrasted to one potential energy and two kinetic energies in the H<sup>2</sup>. In He therefore, the former will over-compensate the latter much more than in H<sup>2</sup>. A similar phenomenon exists also in atomic spectra: the lowest energy value of the He is four times larger than that of H, because the ratio of the terms of potential to kinetic energy is 2:1 instead of 1:1 in H. The conditions are still more pronounced in the nucleus.

## III.

Before making the actual calculation for He, a remark on the existence of H<sup>3</sup> should be added. The Schrödinger equation  $H\psi = E\psi$  for two neutrons 1, 2 and a proton 3 is

mass defect of He may be. The total energy of 
$$|$$
 neutrons 1, 2 and a proton 3 is  

$$-10\left(\frac{\partial^{2}}{\partial r_{23}^{2}} + \frac{2}{r_{23}}\frac{\partial}{\partial r_{23}} + \frac{\partial^{2}}{\partial r_{13}^{2}} + \frac{2}{r_{13}}\frac{\partial}{\partial r_{13}} + \frac{\partial^{2}}{\partial r_{12}^{2}} + \frac{2}{r_{12}}\frac{\partial}{\partial r_{12}} + \cos\left(213\right)\frac{\partial^{2}}{\partial r_{12}\partial r_{13}} + \cos\left(123\right)\frac{\partial^{2}}{\partial r_{12}\partial r_{23}} + \cos\left(132\right)\frac{\partial^{2}}{\partial r_{13}\partial r_{23}}\right)\psi + (V(r_{13}) + V(r_{23}))\psi = E\psi(r_{23}, r_{13}, r_{12}), \quad (5)$$

where (213) is the angle with the vertex 1 and the sides through 2 and 3. Assuming that  $\psi(r_{13})$ is the solution of the Schrödinger Eq. (2) between the neutron 1 and the proton 3, it is reasonable to try the wave function

$$\psi_0 = \psi(r_{13})\psi(r_{23}) \tag{6}$$

for (5). Actually, by calculating the expectation value for the energy  $E_0 = (\psi_0, H\psi_0)$  of  $\psi_0$  we obtain  $-2\epsilon$ . Therefore the binding energy of the second neutron is certainly even larger than that of the first.<sup>13</sup> This is independent of the potential function. The conditions will remain similar if we complete the odd number of protons or neutrons to an even number.

In order to have a better value for the mass defect of H<sup>3</sup> than  $-2\epsilon$  the Hassé variational method<sup>14</sup> may be tried. We calculate

$$(H - E_0)\psi_0 = -10\cos(132)\psi'(r_{13})\psi'(r_{23}) \quad (7)$$

and choose  $\alpha$  in

$$\psi_1 = \psi_0 + \alpha (H - E_0) \psi_0 \tag{8}$$

in such a way that  $(\psi_1, H\psi_1)$  assumes its minimal value

$$E_1 = \frac{1}{2}E_0 + V'/2V_2 - \left[\frac{1}{4}(V'/V_2 - E_0)^2 + V_2\right]^{1/2}$$
(9)

where  $E_0 = (\psi_0, H\psi_0) = -2\epsilon$ ,  $V_2 = (\psi_0, (H-E_0)^2\psi_0)$ and  $V' = (\psi_0, (H - E_0)^2 H \psi_0)$ . In the present case we have  $V_2 = (1/3)K^2$  where K is the mean kinetic energy in H<sup>2</sup>. This is very large and shows that  $\psi_0$  is certainly rather far from the correct wave function. As to  $E_1$ , however, as V' is even larger than  $(V_2)^{3/2}$  it turns out that it is not far from the value  $E_0 = -2\epsilon$ . It might be therefore that the second neutron is only somewhat (perhaps twice) as strongly bound as the first. The relative occurrence of H<sup>3</sup> would be therefore much rarer even than that of  $H^2$ , as usually an isotope with a mass number one larger than the other is very rare if the mass defect is so small. The magnitude of the mass defect and even the existence of the H<sup>3</sup> becomes uncertain of course, if we assume repulsive forces between the neutrons.

<sup>&</sup>lt;sup>13</sup> This is, of course, not true for the third neutron as a wave function like (6) is not allowable for more than two neutrons in consequence of Pauli's principle. Actually if  $\rho$  is not too large, the third neutron has no positive binding energy.

<sup>&</sup>lt;sup>14</sup> H. R. Hassé, Proc. Camb. Phil. Soc. 26, 542 (1930).

We now come to the calculation of the binding energy of the He nucleus, 1 and 2 are neutrons, 3 and 4 protons. We may try as the first approximation to  $\psi$  the following expression

$$\psi_1 = \frac{f(r_{13})f(r_{23})f(r_{14})f(r_{24})}{\left[\int f(r_{13})^2 f(r_{23})^2 d3\right]^{1/2}}$$
(10)

where  $\int \cdots d3$  indicates integration over all coordinates of the particle 3 and f is an as yet unknown function which will later be taken as

with a different V. The meaning of (10) is, that the probability  $\int \psi_1^2 d3$  of a certain position of 4 for a given position of 1 and 2 is  $f(r_{14})^2 f(r_{24})^2$ , in analogy to (6). Really,  $\psi$  will be symmetric with respect to the interchange of the pair 1, 2 with the pair, 3, 4.

the solution of an equation similar to (2) but

Upon calculating the expectation value of the potential energy for  $\psi_1$ , given by (10), one obtains  $4 \int f(r_{12})^2 V(r_{12}) d1 d2 = 4P_f$ , four times the mean potential energy of a nucleus H<sup>2</sup> in the state  $\psi = f$ . For the kinetic energy, however, one gets

$$4K_f - 10\int \left[\int \cos\left(314\right)f'(r_{14})f(r_{14})f(r_{13})f(r_{13})f(r_{23})^2f(r_{24})^2d3d4/\int f(r_{13})^2f(r_{23})^2d3\right]d1d2$$
(11)

with

where

$$K_f = 10 \int f'(r_{13})^2 d1 d3 \tag{12}$$

is the mean kinetic energy of the proton and neutron in H<sup>2</sup> in the state  $\psi = f$ . The kinetic energy for (10) is, in consequence of (11), smaller than  $4K_f$  as the integral in (11) is positive. One sees this by writing

$$\cos (314) = \cos (312) \cos (214) + \sin (312) \sin (314) \cos \alpha \quad (13)$$

where  $\alpha$  is the angle between the planes through 1, 3, 2 and through 1, 3, 4. After inserting (13) into (11) one sees that the integral arising from the second part of (13) will vanish upon integration over  $\alpha$  and the total integral in (11) becomes

$$\int \frac{(\int \cos{(312)}f'(r_{13})f(r_{13})f(r_{23})^2 d3)^2}{\int f(r_{13})^2 f(r_{23})^2 d3} d1 d2 \qquad (14)$$

which is clearly positive. This integral was estimated in the following way. The function f(r)was approximated by  $\alpha e^{-\beta r^2}$  with undetermined  $\alpha$  and  $\beta$ . Then (14) was calculated and compared with the integral occurring in the expression (12) of the kinetic energy. It was found—as might be expected—that the ratio of both is independent from  $\alpha$  and  $\beta$  and is equal to 0.5. For  $f = \alpha e^{-\beta r}$ the ratio is even 0.64, but 0.5 was adopted in the subsequent calculation in order to stay on the safer side. The total energy therefore is for  $\psi_1$ 

$$(\psi_1, H\psi_1) = 4P_f + 3.5K_f = 3.5(K_f + 1.14P_f).$$
 (15)

Now we can choose f so as to minimize (15), which is readily obtained, by assuming that f is the solution of a differential equation like (2),

only with the potential multiplied by 1.14. The total energy (15) is then 3.5 times the binding energy of such an imaginary nucleus, with 1.14 times the real attraction.

As was pointed out before, the characteristic value of (2) is under these conditions very sensitive to a small increase of the potential. For  $\rho = 0.22$ ,  $v_0 = 140$  we obtain in this way a mass defect for the He, which is 7.85 times larger than that of the H<sup>2</sup>.

One more improvement was made in this calculation: instead of  $\psi_1$  the symmetrized function

 $\psi =$ 

$$\psi_1 + \psi_2 \tag{16}$$

$$\psi_2 = \frac{f(r_{13})f(r_{23})f(r_{14})f(r_{24})}{\left[\int f(r_{13})^2 f(r_{14})^2 dt\right]^{1/2}}$$
(10a)

was taken as wave function. Then the expectation value for the energy becomes

$$\frac{(\psi_1, H\psi_1) + (\psi_1, H\psi_2)}{1 + (\psi_1, \psi_2)} = \frac{A+B}{1+S}.$$
 (17)

A is, according to (15)  $4P_f+3.5K_f$ . In order to calculate S and B, the function f(r) was again approximated by  $\alpha e^{-\beta r^2}$ , which gave S=0.84 and  $B=3.82P_f+2.80K_f$  so that in the whole we have

$$E = 3.45(K_f + 1.24P_f). \tag{18}$$

Minimizing (18) in the same way we get 12 for the ratio of the mass defect of He and H<sup>2</sup> in contrast with the observed value of about 17. This again corresponds to  $\rho = 0.22$  or a halfwidth of about  $0.38\dot{e}^2/mc^2$  for the potential hole. For larger  $\rho$  the ratio becomes smaller, for

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smaller  $\rho$  larger. Another possibility is to take  $\psi = (\psi_1 \psi_2)^{1/2}$ . This gives on a similar calculation  $E = 3.2(K_f + 1.25P_f)$  for  $\rho = 0.22$  or a ratio of  $11\frac{1}{2}$ . Now one could take a linear combination of this  $\psi$  with that of (16), which would give a still somewhat lower value—not very much, however, because the two wave functions forming the linear combination do not differ very much from each other.<sup>15</sup>

There is, however, an other possibility to improve the wave function, namely to take advantage of the mixed differential coefficient terms in (5) as we did it for H<sup>3</sup> with the Hassé method. This would for the  $\rho$  under consideration, probably increase the ratio even somewhat over the experimental value.

It seems therefore that if the potential hole is thin, the attractive forces between the neutron and proton give even a too large mass defect for the He, so that a repulsion between the different neutrons and between the different protons may be assumed.

In conclusion one can state that if the basis of the present calculation should prove to be correct, the difference of the mass defects of He and  $H^2$  can be attributed to the great sensitivity of the total energy to a virtual increase of the potential—as is brought about by the fact that every particle in the He is under the influence of two attracting particles, instead of one as in the case of  $H^2$ . The reason for this sensitivity lies in the functional dependence of the lowest energy value on a multiplicative factor v of the potential, which is as follows. For very small values of v there is no negative energy value at all (provided that the potential goes more strongly to zero than  $1/r^2$ ). If v attains a critical value  $(5/\rho^2$  for the potential (1)), there arises one discrete energy value at zero, which becomes more negative on a further increase of v. In the neighborhood of the critical value, however, a very large relative change corresponds to a comparatively small relative change of v. A characteristic property in the neighborhood of the critical value of v is that the mean kinetic energy is almost oppositely equal to the mean potential energy, i.e., the total negative energy is much smaller than the kinetic. That this is actually the case can be simply shown by an application of Heisenberg's indetermination principle.<sup>16</sup>

No similar sensibility exists, of course, in one dimension as the critical value of v is 0 in this case.<sup>17</sup>

<sup>17</sup> R. Peierls, Zeits. f. Physik 58, 59 (1929).

<sup>&</sup>lt;sup>15</sup> The best wave function I could find was  $\alpha e^{-\frac{1}{2}\beta(r_{13}^2+r_{23}^2+r_{14}^2+r_{24}^2+\frac{1}{2}r_{12}^2+\frac{1}{2}r_{34}^2)}$ , it gave a ratio of about 14.

<sup>&</sup>lt;sup>16</sup> W. Heisenberg, reference 3, calculated in this way the kinetic energy of the electron in the neutron and inferred from the number obtained that it cannot obey the laws of quantum mechanics since the mean kinetic energy is much larger than the mass defect. This consideration, of course, cannot be applied for the free electrons in the higher nuclei (cf. reference 6) as the mass defects are much larger than that used by Heisenberg for the neutron, and the same holds also for the nuclear diameters.