as at that time there was probably no chance at all to verify the consequences of such negations. The principle of flexibility must be handled with great care, having always in mind the possibility of experimental verification. It seems to me that the time for an extended application of the principle is ripe now. The conceptional difficulties in quantum mechanics may be interpreted as due to the peculiar inconsistencies of this theory which in certain respects conforms with our principle of flexibility, whereas in other respects, some of which we mentioned in this paper, quantum mechanics and the relativity theory are based on very antiquated notions. It should also be clear from our discussion that the recent controversies regarding the absolute truth of uncertainty principle *versus* causality are quite futile, as scientific truth intrinsically cannot be absolute.

Finally I hope that the principle of flexibility of scientific truth is itself flexible enough so as not to annihilate itself through its own tools after the fashion of Epimenides, the Cretan.

I wish to thank Professor E. T. Bell for many discussions.

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California Institute of Technology,
Pasadena,
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Remarks on the Preceding Note on Many-Valued Truths

The new type of reasoning suggested in Professor Zwicky's note appears to be closely related to some of the projects which have occupied workers in the foundations of mathematics during the past two decades (since L. E. J. Brouwer's rejection of the law of excluded middle as a universally valid law of reasoning), and more particularly in the last five years. For this reason, Professor Zwicky's totally independent approach should be of interest to pure mathematicians. Conversely, some of the recent work in the foundations of mathematics may be of interest to those concerned with the foundations of theoretical physics. That Professor Zwicky arrived independently at his conclusions, gives a new interest to the mathematics and suggests further mathematical investigations. Professor Zwicky's principle of flexibility obviously has a wider scope than the new mathematics.

Many-valued logics have been created and studied in considerable detail by Tarski and Lukasiewicz, and their followers. There is a readable popular exposition of some of this work in a paper in the *Monist*, October, 1932, by Professor C. I. Lewis. The paper contains examples of such logics.

It will be noticed that one of Professor's Zwicky's suggestions challenges the universal applicability to physics of the law of identity. This law has also been scrutinized and rejected in some recent work in mathematics, and in quantum mechanics, where, however, only the identifiability of particles of the same type has been questioned. There thus remains of the Aristotelian system only the law of contradiction. This, so far as I know, has not been challenged outright, although its statement in a many-valued logic must be modified.

If many-valued logics become current, one statement of Professor Zwicky's is likely to receive prompt confirmation from the mathematical side. He predicts that description by "arithmetical numbers" will some day cease to be an adequate rule of scientific thinking. So far as those properties of integers which are independent of order relations are concerned, it may be reasonably doubted now whether "arithmetical numbers" are either necessary or sufficient in any reasoning. For (as shown by the present writer in a paper in the *Transactions of the American Mathematical Society for 1927*), common arithmetic is abstractly identical, except for order relations, with the common two-valued logic of classes. This extends to the logic of relations. By a remarkable coincidence, the problem of extending this to a many-valued logic, and hence getting a fundamentally new generalization of the concept of number, was already under way when I first heard of Professor Zwicky's similar idea. The generalization was proposed solely for its intrinsic interest as an extension of the classical theory of ideals in arithmetic, without any notion that it might have a scientific interpretation. Instead of the integers 0, 1 (for true, false, respectively), of Boolean algebra, we may have some or all of the rational numbers in the interval 0 to 1 as truth values of propositions, and a given truth value may be interpreted as a probability. The last, however, is merely one possibility of interpretation, and does not affect the abstract formulation of the generalized arithmetic.

There is another point in illustration of Professor Zwicky's principle of flexibility. If it is true that the theory of general relativity eliminates the observer (through the principle of covariance), and if it is true that the quantum theory retains the observer (through the indeterminacy principle, or whatever physical imagery is supposed to justify this principle), then a unification of relativity and quantum mechanics transcends a two-valued logic because it controverts the law of the excluded middle. To effect any unification which shall be more than superficial algebra, one or other of the theories to be unified must be radically changed, or resort must be made to a more than two-valued logic.

From the considerations adduced in Professor Zwicky's note, it appears that the time is now ripe for the adoption of the theory of many-valued truths as a working hypothesis.

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