The Effect of Homogeneous Mechanical Stress on the Electrical Resistance of Crystals

By P. W. Bridgman
Harvard University

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It is shown from general considerations of symmetry that the effect of homogeneous mechanical stress on the electrical resistance of a conducting crystal can be expressed in terms of a set of constants, the number of which is equal to the number of elastic moduli, and which connect the resistance with stress by equations very much like the equations connecting strain with stress, except for the difference of a factor 2 in some of the terms. The results are explicitly applied to the case of bismuth, and formulas developed for the change of resistance of a rod cut from the crystal in any direction when subjected to a longitudinal tension. The formulas are checked against the recent experimental results of Miss Allen for bismuth, and agreement found within the limits of error. It is shown that tension measurements alone do not permit an evaluation of all the constants, but if the tension measurements are supplemented by measurements of the effect of hydrostatic pressure in two independent directions, the six constants are then completely determined. Numerical values of the six constants are given for bismuth. Finally the geometrical meaning of the coefficients is briefly discussed and attention called to an effect produced by stress in crystals which is the analogue of the Hall effect produced by a magnetic field in isotropic materials.

IN SPITE of the great amount of work published on the necessary formal geometrical symmetry of all sorts of physical phenomena in crystals, as, for example, most extensively set forth in W. Voigt's *Lehrbuch der Kristallphysik*, the question of the effect of general mechanical stress on the electrical resistance of crystals has not yet been examined. Doubtless the reason for this is that up till now the only such effects which have been studied experimentally are the effects of hydrostatic pressure, and here the symmetry relations are so simple as to be almost intuitively evident. The first experimental attack on the general question has now been made, however, by Miss Allen,[1] who has measured the effect of mechanical tension on the resistance of single crystal rods of bismuth of different orientations. The time is therefore ripe for an examination of the formal symmetry relations, and in particular the number of physical constants necessary to completely characterize the current flow in a conducting crystal subjected to the most general sort of homogeneous mechanical stress. It is obviously not necessary to complicate the problem by considering non-homogeneous stress, for the solution in any such case may be obtained by an integration of the effects in infinitesimal homogeneous elements.

It is natural to attempt to construct a general geometrical theory along the lines suggested in Voigt's book, but a slavish following of pattern is not quite possible because this problem is more complicated than any treated

[1] Mildred Allen, Phys. Rev. **42**, 848 (1932).

by Voigt. Here we are concerned with the cooperation of four factors; within the crystal the three factors, current vector, potential gradient vector (in general not in the same direction as the current vector), and stress tensor must be connected with the fourth factor, the physical constitution of the crystal, which is to be represented by an array of coefficients, in such a way as to be consistent with the symmetry of vectors, tensor, and crystal.

By splitting the problem into two parts, the methods of Voigt may be applied. Consider first the relation between current vector, q, and potential gradient, E. The most general linear relation (we assume of course Ohm's law), expresses E as a linear vector function of q. This involves nine coefficients. Experimentally, however, these nine coefficients are always found to reduce to six, the so-called rotary terms being absent. Since there is no reason to suppose that mechanical stress will so essentially modify the constitution of the crystal as to call into existence rotary terms, six coefficients will be assumed to suffice for this analysis. Further experimental justification of this assumption will be afforded by the agreement with experiment in the case of bismuth. We shall have then:

$$\left.\begin{aligned} E_x &= r_{11}q_x + r_{12}q_y + r_{13}q_z \\ E_y &= r_{12}q_x + r_{22}q_y + r_{23}q_z \\ E_z &= r_{13}q_x + r_{23}q_y + r_{33}q_z\,. \end{aligned}\right\} \tag{A}$$

A relation of this form holds in general, whether or not there is a stress acting. Now specialize the coefficients above, defining them as those valid in the absence of stress. If a stress is allowed to act, the effect will be to somewhat change the coefficients, so that when the stress is acting we shall have:

$$\left.\begin{aligned} E_x &= (r_{11} + \delta r_{11})q_x + (r_{12} + \delta r_{12})q_y + (r_{13} + \delta r_{13})q_z \\ E_y &= (r_{12} + \delta r_{12})q_x + \text{etc.} \\ E_z &= (r_{13} + \delta r_{13})q_x + \text{etc.} \end{aligned}\right\} \tag{B}$$

The problem is now to determine the most general form allowable for the δr's as a function of the stress (restricting ourselves to the linear terms), which shall be consistent with all the symmetry requirements. It is proved in Voigt that the coefficients above have the geometrical nature of the components of a tensor (understanding by tensor the sort of thing of which an ordinary mechanical stress is the simplest example). It follows that the δr's must also be tensor components. The problem reduces, therefore, to finding the most general tensor a linear function of the applied stress which shall be consistent with the symmetry of the crystal. The strain produced by the stress at once springs to mind. But the actual strain is not quite a tensor, and so does not answer the requirements. It is proved in Voigt, however, that a slightly modified strain is in character a tensor, that is, the aggregate of six quantities obtained by leaving unchanged the three strain components with equal indices, and by dividing by 2 the three shearing components of strain with unlike indices.

The solution, therefore, is now in our hands. Build up from the stress a set of quantities involving coefficients entering in the same way as the coefficients which determine the ordinary elastic strains as a function of stress, except that the constants in the terms analogous to the shearing strains must be divided by 2. This completes the formal solution, since, given the stress, we can now compute the δr's, and then the equations determine E completely as a function of q, so that the particular connections between E and q which may be expressed in terms of resistance may also be computed under any desired conditions. In particular, we have found that the number of constants necessary to completely define resistance is equal to the number of ordinary elastic constants, 21 at a maximum.

As an illustration of this general analysis I now apply it to the case of bismuth, eventually coming out with the numerical values of the coefficients. The starting point is the relation between strain and stress. The necessary information is on page 585 of Voigt's book, 1928 printing, noticing, however, that our scheme calls for the use of the elastic moduli as distinguished from the elastic constants, which Voigt tabulates, and that this change of itself introduces a factor 2 in certain places. Following the instructions above we now obtain:

$$\left.\begin{array}{llllllll}
\delta r_{11} = & \rho_{11}X_x + & \rho_{12}Y_y + & \rho_{13}Z_z + & \rho_{14}Y_z & 0 & 0 \\
\delta r_{12} = & \rho_{12}X_x + & \rho_{11}Y_y + & \rho_{13}Z_z - & \rho_{14}Y_z & 0 & 0 \\
\delta r_{33} = & \rho_{13}X_x + & \rho_{13}Y_y + & \rho_{33}Z_z & 0 & 0 & 0 \\
\delta r_{23} = & \frac{1}{2}\rho_{14}X_x + & \frac{1}{2}\rho_{14}Y_y & 0 & + \frac{1}{2}\rho_{44}Y_z & 0 & 0 \\
\delta r_{31} = & 0 & 0 & 0 & 0 & \frac{1}{2}\rho_{44}Z_x + \rho_{14}X_y & \\
\delta r_{12} = & 0 & 0 & 0 & 0 & \rho_{14}Z_x + (\rho_{11} - \rho_{12})X_y &
\end{array}\right\} \quad \text{(C)}$$

in which 6 stress-resistance coefficients appear, which, of course, have no numerical relation to the elastic coefficients, but only a formal relation. In this scheme the Z axis is the axis of trigonal symmetry, and the X axis is the axis of two-fold rotational symmetry in the basal plane.

Furthermore it is known that with this choice of axes the resistance coefficients of equations (A) reduce to 2 only, r_{11} and r_{33}. The connection between E and q therefore becomes:

$$\left.\begin{array}{llll}
E_x = & (r_{11} + \delta r_{11})q_x + & \delta r_{12}q_y + & \delta r_{13}q_z \\
E_y = & \delta r_{12}q_x & + (r_{11} + \delta r_{22})q_y + & \delta r_{23}q_z \\
E_z = & \delta r_{13}q_x + & \delta r_{23}q_y & + (r_{33} + \delta r_{33})q_z.
\end{array}\right\} \quad \text{(D)}$$

This solution is now to be applied to the case of a slender cylindrical rod cut from the crystal in any direction, making angles α, β, and γ with the X, Y, Z axes. The cross section of the rod may be of any shape. The current q has access to the rod only through electrodes at the two ends, so that within the rod the current flow is entirely along the rod, with no transverse components. This gives $q_x = q \cos \alpha$ etc., and hence:

$$\left.\begin{aligned} E_x &= q[(r_{11} + \delta r_{11}) \cos \alpha + \delta r_{12} \cos \beta + \delta r_{13} \cos \gamma] \\ E_y &= q[\delta r_{12} \cos \alpha \qquad \text{etc.} \qquad] \\ E_z &= q[\delta r_{13} \cos \alpha \qquad \text{etc.} \qquad]. \end{aligned}\right\} \tag{E}$$

In general, E has components transverse to the rod, but it is only the component along the rod which determines the measured resistance. The resistance is obviously $R = (E_x \cos \alpha + E_y \cos \beta + E_y \cos \gamma)/q$, or:

$$R = (r_{11} + \delta r_{11}) \cos^2 \alpha + (r_{11} + \delta r_{22}) \cos^2 \beta + (r_{33} + \delta r_{33}) \cos^2 \gamma$$
$$+ 2\delta r_{23} \cos \beta \cos \gamma + 2\delta r_{31} \cos \gamma \cos \alpha + 2\delta r_{12} \cos \alpha \cos \beta. \tag{F}$$

Next consider the effect of a mechanical tension T applied along the rod. The stress system thereby produced within the rod must satisfy the following conditions:

$$\left.\begin{aligned} X_x \cos \alpha + X_y \cos \beta + X_z \cos \gamma &= T \cos \alpha \\ X_y \cos \alpha + Y_y \cos \beta + Y_z \cos \gamma &= T \cos \beta \\ X_z \cos \alpha + Y_z \cos \beta + Z_z \cos \gamma &= T \cos \gamma \end{aligned}\right\} \tag{G}$$

and

$$\left.\begin{aligned} X_x \cos \alpha' + X_y \cos \beta' + X_z \cos \gamma' &= 0 \\ X_y \cos \alpha' \qquad + \text{etc.} \qquad &= 0 \\ X_z \cos \alpha' \qquad + \text{etc.} \qquad &= 0 \end{aligned}\right\} \tag{H}$$

where α', β', and γ' are any direction angles satisfying the condition

$$\cos \alpha' \cos \alpha + \cos \beta' \cos \beta + \cos \gamma' \cos \gamma = 0. \tag{I}$$

The conditions (G) come from the requirement that the force across any plane perpendicular to the length of the rod must be T, perpendicular to this plane, and the conditions (H) from the requirement that there is no external force acting across any lateral surface of the rod. By reflecting that the stress quadric in this case reduces to a couple of planes, the solution may be found almost by inspection, and is:

$$\begin{aligned} X_x &= T \cos^2 \alpha, & Y_y &= T \cos^2 \beta, & Z_z &= T \cos^2 \gamma, \\ Y_z &= T \cos \beta \cos \gamma, & Z_x &= T \cos \gamma \cos \alpha, & X_y &= T \cos \alpha \cos \beta. \end{aligned} \tag{J}$$

The δr's now assume the values:

$$\left.\begin{aligned} \delta r_{11} &= T[\rho_{11} \cos^2 \alpha + \rho_{12} \cos^2 \beta + \rho_{13} \cos^2 \gamma + \rho_{14} \cos \beta \cos \gamma] \\ \delta r_{22} &= T[\rho_{12} \cos^2 \alpha + \rho_{11} \cos^2 \beta + \rho_{13} \cos^2 \gamma - \rho_{14} \cos \beta \cos \gamma] \\ \delta r_{33} &= T[\rho_{13} \cos^2 \alpha + \rho_{13} \cos^2 \beta + \rho_{33} \cos^2 \gamma] \\ \delta r_{23} &= T[\tfrac{1}{2}\rho_{14}(\cos^2 \alpha - \cos^2 \beta) + \tfrac{1}{2}\rho_{44} \cos \beta \cos \gamma] \\ \delta r_{31} &= T[\tfrac{1}{2}\rho_{44} \cos \alpha \cos \gamma + \rho_{14} \cos \alpha \cos \beta] \\ \delta r_{12} &= T[\rho_{14} \cos \alpha \cos \gamma + (\rho_{11} - \rho_{12}) \cos \alpha \cos \beta]. \end{aligned}\right\} \tag{K}$$

The material is now at hand for substituting in the expression (F) for R. Comparison with experiment will be simplified by introducing two new angles. Project the length of the rod on the basal plane and denote the angles between this projection and the X and Y axes by α'' and β'', where $\cos \alpha'' = \cos \alpha/(\cos^2\alpha + \cos^2\beta)^{1/2}$, and $\cos \beta'' = \cos \beta/(\cos^2\alpha + \cos^2\beta)^{1/2}$. Substitution gives, after some simple reductions, for the tension coefficient of resistance:

$$K_T \equiv \frac{1}{T}\frac{\Delta R}{R_0}$$
$$= \frac{\rho_{11}\sin^4\gamma + (\rho_{44}+2\rho_{13})\cos^2\gamma\sin^2\gamma + \rho_{33}\cos^4\gamma - 2\rho_{14}\cos\gamma\sin^3\gamma\cos 3\beta''}{r_{11}\sin^2\gamma + r_{33}\cos^2\gamma}. \quad \text{(L)}$$

Notice that the constant ρ_{12} has cancelled, and ρ_{44} and ρ_{13} enter only through the combination $\rho_{44}+2\rho_{13}$. Tension measurements are, therefore, not sufficient to exhaustively determine the coefficients, but at most only four relations between the six coefficients can be fixed by such measurements. Explicitly, by appropriately varying the orientation, the constants ρ_{11}, ρ_{33}, and ρ_{14} may be determined, and the combination $\rho_{44}+2\rho_{13}$. Furthermore, K_T is seen to have three-fold symmetry about the Z axis, as insured by the term in $\cos 3\beta''$. This, of course, is necessary, and constitutes one check on the correctness of the analysis. Another important feature is that K_T has complete rotational symmetry (that is, the term in β'' vanishes) both when the rod is parallel to the trigonal axis and when it is in the basal plane. That this must be the case when the length is along the trigonal axis is evident from most elementary symmetry considerations, but it is not so easily obvious that the coefficient should be independent of orientation in the basal plane. This latter fact was found experimentally by Miss Allen, and was looked on as one of the important results of the paper, although at the time it did not appear whether this was general, or only a fortuitous result for bismuth. The relation now appears necessary for any crystal of the same symmetry as bismuth.

The two remaining relations necessary to completely determine the six constants must be determined by the imposition of other kinds of stress. The simplest is a hydrostatic pressure, and the calculations can be made at once for this case. The stress system is $X_x = Y_y = Z_z = -P$, $Y_z = Z_x = X_y = 0$.

If the rod is cut parallel to the Z axis:

$$\left(\frac{1}{P}\frac{\Delta R}{R_0}\right)_{\parallel} = -\frac{2\rho_{13}+\rho_{33}}{r_{33}}, \quad \text{(M)}$$

and when the rod is perpendicular to the Z axis, parallel to the basal plane:

$$\left(\frac{1}{P}\frac{\Delta R}{R_0}\right)_{\perp} = -\frac{\rho_{11}+\rho_{12}+\rho_{13}}{r_{11}}. \quad \text{(N)}$$

Examination shows at once that these two additional relations permit explicit solution for the remaining coefficients, so that the six coefficients may be completely determined in terms of tension measurements on four orientations and hydrostatic pressure measurements on two orientations.

The detailed data of Miss Allen permit further check of the above expression for K_T. Such a check may be made in various ways. For example, at constant γ the variation of K_T with β'' may be studied. The formula demands that the variable part of K_T be proportional to cos $3\beta''$. Miss Allen's Fig. 5 exhibits the coefficients in this way, and inspection will show that within the limits of experimental error each of the curves of Fig. 5 has the shape of a cosine curve. (Her ϕ and θ are the β'' and γ of this paper respectively.) It may be taken, therefore, that the geometrical theory checks sufficiently well against experiment.

The numerical coefficients may now be computed. The tension coefficients are given in Miss Allen's paper. The pressure coefficients I have found[2] to be 1.05×10^{-5} for $\gamma=90°$, and 2.03×10^{-5} for $\gamma=0°$. The specific resistance I have also found to be $r_{11}=114.0\times 10^{-6}$ and $r_{33}=144.2\times 10^{-6}$. All these values are at 30°C. The numerical coefficients are now found:

$$\rho_{11} = -7.7 \times 10^{-9},\ \rho_{33} = -6.6 \times 10^{-9},\quad \rho_{12} = +5.6 \times 10^{-9}$$

$$\rho_{13} = +1.8 \times 10^{-9},\ \rho_{14} = +31.3 \times 10^{-9},\ \rho_{44} = -12.3 \times 10^{-9}.$$

The stress unit is 1 kg/cm². In this computation the corrections for change of dimensions and of angle with stress are neglected. These corrections are just about on the margin of experimental error.

Finally, it is interesting to go back and examine the geometrical significance of the various coefficients by determining what sort of simple measurement would give the isolated coefficient. ρ_{33} and ρ_{11} have already been dealt with, and are directly determined in terms of the tension coefficient of rods parallel and perpendicular to the trigonal axis. The coefficients ρ_{12} and ρ_{13} determine transverse components of e.m.f. when current flows lengthwise in a rod subjected to tension acting lengthwise. For example, if a rod is cut parallel to the Y (or X) axis, and a current passed lengthwise of the rod, then when a tension is applied along the rod, a transverse component of e.m.f. will appear along the X (or Y) axis which determines ρ_{12}. The other cross coefficient ρ_{13} has similar significance with a proper change of letters. The term ρ_{14} points to a formal analogy in crystals to the Hall effect in isotropic metals, the magnetic field being replaced by a compressional force. If a rod of rectangular section is cut with its length along the Z axis and with the X and Y axes along the sides of the rectangular section, and if a current is passed lengthwise of the rod, then a transverse e.m.f. along the Y axis will appear if a compression along the X axis is applied between the opposite faces of the section. Finally, if a bar of rectangular section is cut along the X axis and a shearing stress Y_z is applied to the sides of the bar distorting the cross section, and if a transverse current is led between opposite faces along the Z axis, an e.m.f. between the other two faces is produced by the shearing stress, the magnitude of the effect being determined solely by ρ_{44}.

[2] P. W. Bridgman, Proc. Amer. Acad. **63**, 351 (1929).