Momentum Relations in Crossed Fields

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It is shown that the mechanical linear or angular momentum gained by ions moving in crossed electric and magnetic fields is at the expense of the electromagnetic linear or angular momentum already present. Hence Ross Gunn's theory of the acquisition of angular momentum by a star as the result of ion motions in its external electric and magnetic fields cannot be upheld.

Three cases are analyzed in detail: (a) Newtonian motion in uniform crossed fields, (b) Newtonian motion in the field of a uniformly magnetized charged sphere, (c) relativity motion in uniform crossed fields.

DURING the last three years Ross Gunn[1] has developed a promising theory of the formation of double stars and of planetary systems based on electromagnetic forces rather than on conventional gravitational attractions. While many of the features of his theory are undoubtedly significant, it will be shown that the mechanism which he invokes to account for the angular momentum of a star would not give rise to the effect which he anticipates. Gunn supposes that a star has combined electric and magnetic fields similar to those of the earth. Ions formed in its atmosphere acquire momentum in the direction of the vector $\boldsymbol{E}\times\boldsymbol{H}$, as shown in an earlier paper of the writer.[2] This momentum, which is in the same sense for ions of opposite signs, is transferred to the star as a consequence of collisions and gives rise to a continuing increase in the angular momentum of that body until finally fission occurs.

In a lecture some thirty years ago J. J. Thomson pointed out that outside a uniformly magnetized charged sphere there exists a Poynting flux everywhere in the direction of the parallels of latitude. This implies the presence of energy and momentum circulating around the axis of the sphere without ever departing from it. While this phenomenon leads to no predictions embarassing to electromagnetic theory, no particular significance appears to have been attached to it. In the course of this paper it will be shown that this flux of momentum plays a very vital role in connection with the motion of ions in the atmosphere of a star.

While certain fundamental theorems of electrodynamics prove that the laws of conservation of both linear and angular momentum hold for an isolated electromagnetic system provided account is taken of both electromagnetic and mechanical momentum, and therefore that Gunn's argument

[1] Ross Gunn, Phys. Rev. **32,** 133 (1928); **33,** 614, 832 (1929); **34,** 335, 1621 (1929); **35,** 107, 635 (1930); **36,** 1251 (1930); **37,** 283, 983, 1129, 1573 (1931); **38,** 1052 (1931); **39,** 130, 311 (1932).

[2] Page, Phys. Rev. **33,** 553 (1929).

cannot be upheld, we shall investigate here the details of the process by which these laws are satisfied in some simple cases, including the case of a uniformly magnetized charged sphere in whose atmosphere ions are present.

A. Uniform Crossed Fields

Consider a uniform electric field E in the direction of the y axis of a set of right-handed rectangular axes x, y, z combined with a uniform magnetic field H in the direction of the z axis. Then, as shown in an earlier paper,[2] ions of both signs progress in the x direction with a drift velocity cE/H which is independent of the charge or mass of the ion. The ion paths are cycloids, being prolate, common or curtate in accord with the magnitude and direction of the initial velocity. The cycloidal paths for the negative ions are obtained from those for the positive ions by rotating the latter through the angle π about the x axis. The integrated equations of motion are[3]

$$x - x_0 = \frac{mc}{e}\left\{\frac{v_{0y}}{H}\left(1 - \cos\frac{eH}{mc}t\right) + \left(\frac{v_{0x}}{H} - \frac{cE}{H^2}\right)\sin\frac{eH}{mc}t\right\} + \frac{cE}{H}t,$$

$$y - y_0 = \frac{mc}{e}\left\{-\left(\frac{v_{0x}}{H} - \frac{cE}{H^2}\right)\left(1 - \cos\frac{eH}{mc}t\right) + \frac{v_{0y}}{H}\sin\frac{eH}{mc}t\right\},$$

$$z - z_0 = v_{0z}t,$$

where x_0, y_0, z_0 are the coordinates of the starting point and v_{0x}, v_{0y}, v_{0z} the components of the initial velocity.

Differentiating the first of these we find for the x component of the momentum

$$p_x = m\left\{v_{0y}\sin\frac{eH}{mc}t + \left(v_{0x} - \frac{cE}{H}\right)\cos\frac{eH}{mc}t + \frac{cE}{H}\right\},$$

and comparing with the second

$$y - y_0 = (c/eH)(p_x - p_{0x}), \tag{1}$$

where $p_{ox} = mv_{ox}$ is the x component of the initial momentum.

This equation states that the displacement of the ion in the direction of the electric field is proportional to the gain of momentum in the direction at right angles to the two fields, that is, in the direction of the Poynting flux. Increase in momentum involves a displacement of positive ions in the same direction as the electric field, and of negative ions in the opposite direction.

Now suppose that at a certain instant n ion pairs per unit volume are formed. Under the action of the fields they separate, acquiring at the same time momentum in the direction at right angles to the two fields. Whether they make collisions with neutral particles and thereby transfer part of their momentum to the latter or not, the total mechanical momentum generated is proportional to the separation of the ions. If Δy denotes the mean separation of the positive from the negative ions, an effective electric moment

[3] Page and Adams, *Principles of Electricity*, p. 289. In the present paper we are using Heaviside-Lorentz units.

$ne\Delta y$ is set up in the region under consideration. This gives rise to an electric field of the same magnitude in the direction opposite to that originally existing. Consequently the electromagnetic momentum g per unit volume increases in the amount $\Delta g = -(1/c)ne\Delta yH$.

However, if Δp is the increase in mechanical momentum per unit volume, it follows from (1) that

$$\Delta p = neH\Delta y/c$$

and consequently

$$\Delta p + \Delta g = 0. \tag{2}$$

So the increase in mechanical momentum of the ions is at the expense of the electromagnetic momentum already present. Production of ions acts as a mechanism for converting the latter into the former. When all the electromagnetic momentum has been transformed, the electric field is reduced to zero, and no more transverse momentum is acquired by the ions. Since electromagnetic momentum is consumed in the same region of space as that in which mechanical momentum is generated, Eq. (2) applies to angular momentum about any arbitrary axis as well as to the case of linear momentum for which it was derived. However, we shall discuss in the next section the specific angular momentum relations existing in the case of a uniformly magnetized sphere which has an electric charge.

B. Uniformly Magnetized Charged Sphere

Take origin at the center of the sphere with polar axis in the direction of magnetization. The spherical coordinates consisting of the radius vector r, polar angle θ and azimuth ϕ are chosen so as to constitute a right-handed set in the order named. If Q is the charge on the sphere and M its magnetic moment, the nonvanishing field components are

$$E_r = \frac{Q}{4\pi r^2}, \quad H_r = \frac{2M}{4\pi r^3}\cos\theta, \quad H_\theta = \frac{M}{4\pi r^3}\sin\theta,$$

and the equation of motion of an ion with charge e and mass m is

$$m\boldsymbol{f} = e\{\boldsymbol{E} + (1/c)\boldsymbol{v} \times \boldsymbol{H}\},$$

which is equivalent to the three component equations

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 = \frac{e}{4\pi m}\left\{\frac{Q}{r^2} - \frac{M}{cr^2}\sin^2\theta\dot{\phi}\right\},$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2 = \frac{e}{4\pi m}\left\{\frac{2M}{cr^2}\sin\theta\cos\theta\dot{\phi}\right\},$$

$$2\sin\theta\dot{r}\dot{\phi} + 2r\cos\theta\dot{\theta}\dot{\phi} + r\sin\theta\ddot{\phi} = \frac{e}{4\pi m}\left\{\frac{M}{cr^3}\sin\theta\dot{r} - \frac{2M}{cr^2}\cos\theta\dot{\theta}\right\}.$$

The third equation is the significant one for our purposes. It can be written

$$\frac{d}{dt}(r^2 \sin^2 \theta\dot{\phi}) = -\frac{eM}{4\pi mc}\frac{d}{dt}\left(\frac{\sin^2 \theta}{r}\right),$$

or, if we put $p \equiv mr^2 \sin^2 \theta\dot{\phi}$ for the angular momentum of the ion about the axis of the sphere,

$$dp = -(eM/4\pi c)d(\sin^2 \theta/r). \qquad (3)$$

Next we must calculate the change in electromagnetic angular momentum due to motion of the ion. To do this we shall first compute the electromagnetic angular momentum of an isolated system (Fig. 1) consisting of a point pole m and a point charge e. It is evident from symmetry that the resultant angular momentum is entirely about the line connecting m to e. As

$$E = e/4\pi\rho^2, \quad H = m/4\pi r^2,$$

the component of angular momentum parallel to this line is

$$g = -(me/16\pi^2 c r \rho^2) \sin \theta \sin \alpha$$

per unit volume. Therefore the total angular momentum can be written

$$G = -\frac{mea}{8\pi c}\int_0^\infty \int_0^{2\pi} \frac{r \sin^3 \theta}{\rho^3} d\theta dr$$

if we eliminate α by the relation $\rho \sin \alpha = a \sin \theta$. First we shall integrate with respect to θ, keeping r constant. In doing this it will be convenient to change

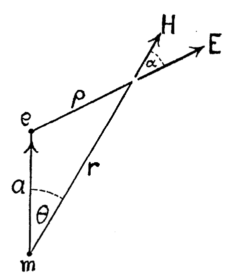

Fig. 1.

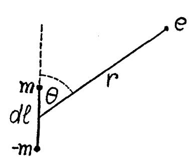

Fig. 2.

the variable of integration to ρ by means of the relation $\rho^2 = r^2 - 2ar \cos \theta + a^2$. We have then

$$ar\int \frac{\sin^3 \theta}{\rho^3} d\theta = \int \left\{1 - \left(\frac{r^2 + a^2 - \rho^2}{2ar}\right)^2\right\}\frac{d\rho}{\rho^2}$$

$$= \left\{\frac{(r^2 - a^2)^2}{4a^2r^2}\frac{1}{\rho} + \frac{r^2 + a^2}{2a^2r^2}\rho - \frac{1}{12a^2r^2}\rho^3\right\}.$$

When $r < a$ the limits of ρ are $a - r$ and $a + r$, whereas when $r > a$ the limits are $r - a$ and $r + a$. For the first region

$$G_1 = -\frac{me}{6\pi a^2 c}\int_0^a r dr = -\frac{me}{12\pi c},$$

and for the second

$$G_2 = -\frac{mea}{6\pi c}\int_a^\infty \frac{dr}{r^2} = -\frac{me}{6\pi c}\cdot$$

Adding, the total electromagnetic angular momentum is found to be

$$G = G_1 + G_2 = -me/4\pi c. \tag{4}$$

At first sight it seems strange that the angular momentum should be independent of the distance a between the charge and the pole. A little consideration, however, shows that this is to be expected. For if the charge is projected directly toward the pole the magnetic field exerts no force on it and its mechanical angular momentum along the line joining the two remains constant. So the electromagnetic angular momentum should be independent of a.

The magnetic field in which we are interested is that of a dipole. If the line joining the center of the dipole to the charge e (Fig. 2) has a length r and makes an angle θ with the axis of the dipole, the component of the electromagnetic angular momentum of the system along the axis of the dipole is

$$G = -\frac{me}{4\pi c}d(\cos\theta) = \frac{me}{4\pi c}\sin\theta d\theta.$$

Now if dl is the length of the dipole, $d\theta = dl \sin\theta/r$, and

$$G = (eM/4\pi c)(\sin^2\theta/r), \tag{5}$$

where M is the magnetic moment mdl of the dipole. Comparing (5) with (3) we have

$$dp + dG = 0. \tag{6}$$

This equation shows that any gain in mechanical angular momentum by an ion is accompanied by an equal loss in electromagnetic angular momentum. Suppose that the uniformly magnetized sphere is initially uncharged electrically and that ions of one sign (electrons, for instance) are projected toward it from a great distance, the impinging ions having on the average no initial angular momentum about the axis of the sphere. When these ions strike the sphere they impart to it mechanical angular momentum in one sense while giving rise to an equal amount of electromagnetic angular momentum in the opposite sense. If, after the sphere has been charged by the impacting ions, ion pairs are formed in its atmosphere, the action of these ion pairs is to transform the electromagnetic angular momentum into mechanical angular momentum until finally the mechanical angular momentum imparted by the ions impinging from outside is just annulled. Therefore the electromagnetic process postulated by Gunn cannot lead to a continuing increase in mechanical angular momentum of the sphere.

If we replace the point pole by a finite sphere of magnetic charge in deducing (4) we are led to a different result, obtaining in addition to the term appearing on the right-hand side of that equation a second term containing in the denominator the square of the distance a between the center of the sphere and the point charge. This additional term remains when we superpose two slightly displaced spheres of opposite sign so as to construct a uniformly magnetized sphere. Its presence is due to the fact that we erroneously employ H inside the magnetized sphere in calculating the electromagnetic momentum when we should in fact make use of B. The added term disappears when this error is corrected, and (5) is found to hold rigorously for a uniformly magnetized sphere as well as for a magnetic dipole. The analysis, which is somewhat laborious, will not be reproduced here.

C. Relativity Dynamics

The preceding analysis is inapplicable to the case where the ions have velocities comparable with the velocity of light. In such an event the variation of mass with velocity must be taken into account. We shall develop the theory for the case of uniform crossed fields on the relativity dynamics.

As before we shall take E in the direction of the y axis and H in that of the z axis of a right-handed set of rectangular axes fixed in the observers' inertial system S. In addition we shall have occasion to refer to a set of parallel axes x', y', z' located in an inertial system S' moving in the x direction relative to S with the constant velocity $u=cE/H$. As shown in a previous paper[2] the electric field vanishes in S' and the ions describe helices around the lines of magnetic force with angular velocity

$$\omega' = -eH'/m_t c,$$

where m_t is the transverse mass. If v_0' is the initial velocity of an ion relative to S', the components of velocity at any time t' are

$$\begin{aligned} v_x' &= v_{0x}' \cos \omega' t' - v_{0y}' \sin \omega' t', \\ v_y' &= v_{0x}' \sin \omega' t' + v_{0y}' \cos \omega' t', \\ v_z' &= v_{0z}'. \end{aligned}$$

Now the displacement of an ion along the electric field is

$$\begin{aligned} y - y_0 = y' - y_0' &= (1/\omega')\{v_{0x}'(1 - \cos \omega' t') + v_{0y}' \sin \omega' t'\} \\ &= -(1/\omega')(v_x' - v_{0x}'). \end{aligned}$$

Making use of the relativity transformations for the components of velocity,

$$y - y_0 = -\frac{1}{\omega'}\left\{\frac{v_x - u}{1 - \beta\dfrac{v_x}{c}} - \frac{v_{0x} - u}{1 - \beta\dfrac{v_{0x}}{c}}\right\},$$

where $\beta \equiv u/c$ is the ratio of the relative velocity of the two inertial systems to the velocity of light. Now

$$v_x'^2 + v_y'^2 + v_z'^2 = v_{0x}'^2 + v_{0y}'^2 + v_{0z}'^2 \equiv v'^2$$

since the speed relative to S' does not change with the time. Transforming to system S we find

$$\frac{(1 - v^2/c^2)^{1/2}}{1 - \beta v_x/c} = \left(\frac{1 - v'^2/c^2}{1 - \beta^2}\right)^{1/2} \equiv \text{constant}.$$

Also

$$-\frac{1}{\omega'} = \frac{m_t c}{eH'} = \frac{m_0 c}{eH} \frac{1}{[(1 - \beta^2)(1 - v'^2/c^2)]^{1/2}},$$

where m_0 is the rest mass of the ion.

Therefore

$$y - y_0 = \frac{m_0 c}{eH} \frac{1}{1 - \beta^2}\left\{\frac{v_x - u}{(1 - v^2/c^2)^{1/2}} - \frac{v_{0x} - u}{(1 - v_0{}^2/c^2)^{1/2}}\right\}$$

$$= \frac{c}{eH} \frac{1}{1 - \beta^2}\left\{p_x - p_{0x} - \frac{u}{c^2}(T - T_0)\right\}$$

where p_x is the momentum and T the kinetic energy of the ion. Now

$$(u/c^2)(T - T_0) = (\beta/c)eE(y - y_0) = (eH/c)\beta^2(y - y_0).$$

Solving for $y-y_0$, we find identically the same relation

$$y - y_0 = (c/eH)(p_x - p_{0x}) \tag{7}$$

between the displacement along the electric lines of force and the gain in momentum at right angles to the two fields as was obtained on the Newtonian dynamics. Therefore electromagnetic momentum is converted into mechanical momentum by ions produced in the atmosphere of a star or planet in precisely the same manner as in the cases previously discussed.