The Analysis of Cosmic-Ray Observations

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A method of rendering non-linear observation equations susceptible to least-square adjustment is adapted to observations on the absorption of cosmic rays in bodies of water, data on which have appeared from time to time; the object being to obtain the most probable values of the constants of the assumed discrete components of the radiation. Applied to the most consistent data so far published, viz., those reported by Millikan and Cameron in 1931, the adjustment shows that the quantitative conclusions deduced from them by the observers are not derivable from the data by the method of least squares.

I. Theory of the Adjustment

The receipt of several requests for details of the least-square adjustment of cosmic-ray observations reported in a recent letter to the Editor,[1] together with the increasing attention which the cosmic-ray problem is attracting, leads the writer to set forth the method at greater length.

When measurements on the absorption of cosmic rays in water were first interpreted as indicating the existence of discrete components of definite intensity and hardness, the question naturally arose as to whether, with the accumulation of precise data, it might not be possible to apply the well-known methods of least-square adjustment to the determination of the several sets of constants.

If, as has apparently been established, the radiation reaches the earth with the same intensity from all directions, if absorption in any one direction proceeds with a constant absorption coefficient, and if the rays are not refracted, it is easy to deduce an exponential law of attenuation with depth in a material substance. To wit: the initial intensity and absorption coefficient of any one component being respectively I_0 and μ, then the intensity at any depth H in a horizontal layer of homogeneous material, which the rays reach only through the upper hemisphere, is

$$I = I_0 \int_1^\infty x^{-2} e^{-H\mu x} dx. \tag{1}$$

x represents the cosecant of the angle of incidence upon the horizontal surface.[2]

The definite integral in this expression is a function only of the argument $H\mu$. We shall call it the "Gold integral" and designate it by $G(H\mu)$; because a table of its values has been published by Gold.[3] If the radiation is composed

[1] Weld, Phys. Rev. **37**, 1368 (1931).

[2] The equation given by Millikan and Cameron, Phys. Rev. **28**, 860 (1926), inadvertently contains a factor 2π. Their computations were, however, apparently based on the correct formula (1) above.

[3] Gold, Proc. Roy. Soc. **A82**, 62 (1909); Millikan and Cameron, Phys. Rev. **31**, 926 (1928). See also part 2 of this paper.

of several distinct monochromatic components with initial intensities I_{01}, I_{02}, $\cdots$ and absorption coefficients μ_1, μ_2, $\cdots$, then at any depth H the sum of their intensities should be

$$I_{01}G(H\mu_1) + I_{02}G(H\mu_2) + \cdots = \Sigma I. \tag{2}$$

This may be taken as a form of observation equation, in which the unknown quantities are the constants I_{01}, μ_1, I_{02}, μ_2, $\cdots$, and in which ΣI is observed by means of the ionization measurements. I believe that the depth H may be justifiably regarded as a known coefficient, since its measurement is presumably subject to a comparatively small percentage of uncertainty.

But (2) is a *non-linear* function of the μ's. Before such observations can be subjected to practicable least-square adjustment, the observation equations must be replaced by equivalent equations of linear form. There is an approximation method of accomplishing this, somewhat analogous to Horner's method of solving non-linear algebraic equations. Its use is contingent upon our having first obtained, by some means, fairly approximate values of the unknown quantities; and it is then the required *small corrections* to these values that are computed in the adjustment.[4]

In this case, let the approximate values assumed for I_{01}, I_{02}, $\cdots$ be designated by α_1, α_2 $\cdots$, and the corresponding corrections by c_1, c_2, $\cdots$; likewise let the approximate values and corrections for μ_1, μ_2, $\cdots$ be β_1, β_2, $\cdots$ and k_1, k_2, $\cdots$, respectively. Then the *true* values of the unknowns are represented by

$$\left.\begin{aligned} I_{01} &= \alpha_1 + c_1 \\ I_{02} &= \alpha_2 + c_2 \\ \cdots &\quad \cdots \\ \mu_1 &= \beta_1 + k_1 \\ \mu_2 &= \beta_2 + k_2 \\ \cdots &\quad \cdots \end{aligned}\right\} . \tag{3}$$

Substitution of these in the typical observation Eq. (2) gives

$$(\alpha_1 + c_1)G[H(\beta_1 + k_1)] + (\alpha_2 + c_2)G[H(\beta_2 + k_2)] + \cdots = \Sigma I; \tag{4}$$

which is still non-linear, but in which the unknowns to be adjusted are now the presumably *small* corrections c_1, c_2, $\cdots$, k_1, k_2, $\cdots$.

Each term of (4), when expanded by Taylor's theorem as in the method cited, assumes the form (dropping subscripts):[5]

[4] See Weld, *Theory of Errors and Least Squares*, p. 178; or Merriman, *Method of Least Squares*, p. 175.

[5] In this process we encounter the following:

$$\frac{\partial}{\partial\beta}G(H\beta) = \frac{\partial}{\partial\beta}\int_1^\infty x^{-2}e^{-H\beta X}dx = -H\int_1^\infty x^{-1}e^{-H\beta x}dx \text{ (integrate by parts)}$$

$$= \frac{1}{\beta}\left[\int_1^\infty x^{-2}e^{-H\beta x}dx - e^{-H\beta}\right] = \frac{1}{\beta}[G(H\beta) - e^{-H\beta}];$$

$$\frac{\partial^2}{\partial\beta^2}G(H\beta) = \frac{G(H\beta) - e^{-H\beta} + H\beta e^{-H\beta} - G(H\beta) + e^{-H\beta}}{\beta^2} = \frac{H}{\beta}e^{-H\beta}.$$

$$(\alpha + c)G[H(\beta + k)] = \alpha G(H\beta) + G(H\beta)\cdot c + \frac{\alpha}{\beta}[G(H\beta) - e^{-H\beta}]k$$

$$+ \frac{H\alpha}{2\beta}e^{-H\beta}k^2 + \frac{1}{\beta}[G(H\beta) - e^{-H\beta}]ck + \cdots. \tag{5}$$

This is strictly linear in c. If k is very small, the higher terms can be dropped. Whether k is small enough to justify this must be investigated. The steps necessary to insure the negligibility of the non-linear terms in k will appear in the second part of the paper. Let us assume that they are negligible, and that therefore only the first three terms of (5) need be employed.

Besides the terms included in the first member of (2), one for each cosmic-ray component, there may be added a term R representing such observed effects as are not so included: the *Restgang*, the small residual source of ionization, the leakage of the electroscope, or whatever it is that the ionization appears to approach as an irreducible minimum limit. If R is the true value of this, and ρ its assumed approximate value, then to preserve the symmetry of the method we may let $R = \rho + r$, where r is, again, a small correction.

Using now the linear part of the expansion (5) and adding the residual ionization just mentioned, the observation equation takes the form, linear in the c's, the k's, and r:

$$\Sigma G(H\beta)c + \Sigma \frac{\alpha}{\beta}[G(H\beta) - e^{-H\beta}]k + \Sigma\alpha G(H\beta) + (\rho + r) = \Sigma I; \tag{6}$$

in which the summations carry over whatever number of discrete cosmic-ray components is assumed to exist. Or if we designate the coefficients of the respective c's by A's, those of the k's by B's, and the third terms by P's, and transpose, (6) may be more intelligibly written

$$\begin{aligned} A_1c_1 + A_2c_2 + \cdots + B_1k_1 + B_2k_2 + \cdots + r \\ = \Sigma I - (P_1 + P_2 + \cdots + \rho); \end{aligned} \tag{7}$$

the key to which is

$$\left.\begin{aligned} A_1 &= G(H\beta_1), \text{ etc.} \\ B_1 &= \frac{\alpha_1}{\beta_1}[G(H\beta_1) - e^{-H\beta_1}], \text{ etc.} \\ P_1 &= \alpha_1 G(H\beta_1), \text{ etc.} \end{aligned}\right\} \tag{8}$$

The observation equations actually used in this adjustment were all of the form (7), differing from each other only in the values given to H and in the corresponding observed values of the apparent total ionization ΣI. The rest of the process follows the well-known procedure of the method of least squares for observations assumed to be of equal weight.

II. Application to Actual Cosmic-Ray Data

The first step is a survey of the available experimental data. Millikan and Cameron have taken most elaborate pains to obtain accurate readings on the ionization at various depths in water, and their most recent published results[6] have been used as the basis of this analysis. There are tabulated forty-four observations on the total ionization ΣI at depths varying from about 8 to 80 meters. From their study of the results, Millikan and Cameron conclude that there are four principal components, whose constants they estimate as follows (using the above notation):

$$\left.\begin{array}{rr} \text{Initial intensity} & \text{Absorption coefficient} \\ 33 = \alpha_1 & 0.03 = \beta_1 \\ 80 = \alpha_2 & 0.10 = \beta_2 \\ 130 = \alpha_3 & 0.20 = \beta_3 \\ 141{,}000 = \alpha_4 & 0.80 = \beta_4 \end{array}\right\} \tag{9}$$

Also their preliminary estimate of R, the residual minimum ionization, is 1.2, which is therefore the value of our ρ. This makes *nine* unknowns, viz., the corrections c_1, c_2, c_3, c_4, k_1, k_2, k_3, k_4, and r.

Now α_4 is so large that there is grave question whether the k^2 term in (5), viz.,

$$\frac{H\alpha_4}{2\beta_4}e^{-H\beta_4}k_4{}^2, \tag{10}$$

can be ignored; even if k_4 were small, which is doubtful. At the greater depths, however, the exponential factor of (10) becomes sensibly zero, as does the coefficient of ck in (5). Moreover, Millikan and Cameron express their belief that the presence of secondary radiation due to the Compton effect may render the absorption coefficients actually variable during the earlier stages of the absorption, that is, at the lesser depths. For both reasons it has been thought best to confine the analysis at first to the greater three-fourths of the depth range. Beyond $H = 16$ meters, the $k_4{}^2$ term (10), and at the same time the coefficients of c_4, k_4, and c_4k_4, become vanishingly small, so that these two unknowns are eliminated entirely, leaving only to the *seven* unknown corrections c_1, c_2, c_3, k_1, k_2, k_3, r to be adjusted.

To test the validity of the linear approximation (6), an arbitrary correction of one percent was assumed for each unknown ($C_1 = 0.01\ \alpha_1$, etc.), and (6) was applied at the most unfavorable point of the range (16 meters). The discrepancy due to the approximation was found to be only one part in 10,000.

Proceeding therefore along this line, the twenty-two observation equations corresponding to the last twenty-two depth-ionization measurements were set up in accordance with (7) and adjusted in the usual manner, giving seven normal equations corresponding to the seven required corrections.

Before this adjustment could be undertaken, it was found necessary to compute values of the Gold integral intermediate between those given in Gold's published table[3] and to extend the table to include somewhat larger values of the argument $H\mu$;—a laborious computation in itself.[7] A calculating

[6] Millikan and Cameron, Phys. Rev. **37**, 235 (1931).

[7] Copies of the enlarged table are available to anyone interested.

machine was used throughout, all work was carefully checked, and the final results were found to satisfy the normal equations exactly.

The adjusted values of the seven unknowns are as follows:

$$\left.\begin{array}{ll} c_1 = -\ \ 0.7078 & k_1 = -\ 0.001269 \\ c_2 = -\ \ 513.63 & k_2 = -\ 0.09225 \\ c_3 = -\ 3958.4 & k_3 = -\ 2.5138 \\ \multicolumn{2}{c}{r = +\ 0.0008} \end{array}\right\} \tag{11}$$

If these corrections were applied to the tentative values given by Millikan and Cameron the results would be:

$$\left.\begin{array}{ll} I_{01} = \ \ 32.2922 & \mu_1 = \ \ 0.028731 \\ I_{02} = -\ \ 433.63 & \mu_2 = \ \ 0.00775 \\ I_{03} = -\ 3828.4 & \mu_3 = -\ 2.3138 \\ \multicolumn{2}{c}{R = 1.2008} \end{array}\right\} \tag{12}$$

The values of I_{01}, μ_1, and R are reasonable enough; but in the other cases the results are entirely devoid of meaning. Most of the corrections are larger than the tenative values. We computed the residuals from the adjusted values (11) by substitution in (7) and found the sum of the squares to be 0.3572, while the algebraic sum of the residuals is exactly zero. If the tenative values were correct, that is, if the seven corrections were all zero, the same substitution would give the sum of the squared residuals as 0.5573, and that of the residuals themselves as -0.06. All this indicates that the utterly impossible results (11) are, nevertheless, more probable than the consequences of assuming the tentative values to be correct.

This is the disconcerting report that has to be made on the matter. We were looking for a refinement of Millikan and Cameron's estimates, and there is no question that, if their assumptions as to the makeup of the radiation were true and if (9) were fair approximations of the involved constants, this adjustment would have yielded a reasonable set of small corrections for the three more penetrating components. The writer can arrive at only one alternative conclusion, namely, that however closely the values fixed upon by Millikan and Cameron may fit their observational curve, those values are not, reciprocally, deducible from the observations by the method of least squares.

It should be remarked, in passing, that half of the observations are crowded into the upper one-fourth of the depth range; and that there is a tendency to "bunch" the observations, that is, to take several at nearly the same depth. While the reasons for this are obvious, it would better serve the purposes of analysis if the same number of readings could have been taken at approximately uniform intervals of, say, two meters.

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