Double Stern-Gerlach Experiment and Related Collision Phenomena

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The theory of the double Stern-Gerlach experiment is developed where the rate of rotation has a maximum at $t=0$ and goes gradually to zero at $t=\pm\infty$. The negative results of the experiments of Phipps and Stern are completely accounted for. The same analysis is applied to those collision phenomena where only two quantum states need be considered, and where their difference of energy, ΔE, is so much smaller than the relative kinetic energy of the two systems that the positions of the centers of gravity of the two systems may be regarded as time parameters. If $f(t)=2\pi/h$ times the matrix element of the perturbation, and if $A=\int_{-\infty}^{\infty}f(t)dt$, then the transition probability is

$$\left|\frac{\sin A}{A}\int_{-\infty}^{\infty}f(t)e^{2\pi i(\Delta E/h)t}dt\right|^2$$

for the cases investigated, and this probably holds for all experimental cases.

I. DOUBLE STERN-GERLACH EXPERIMENT

1. Introduction

A BEAM of alkali atoms all having the same spatial quantization may be obtained by the Stern-Gerlach experiment. The question arises, if such a beam of atoms is sent through a weak magnetic field which is rapidly rotating, will the spatial quantization remain unaltered, or will transitions occur which will separate the beam when it is subjected to a second Stern-Gerlach experiment? This question has been examined both theoretically[1] and experimentally.[2]

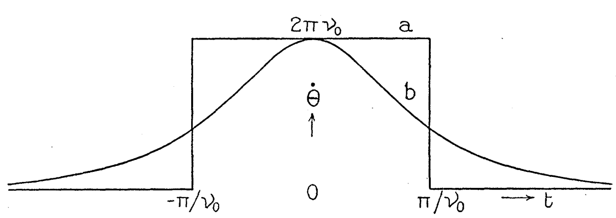

Fig. 1. Rate of rotation of magnetic field.

In view of the lack of agreement between the theoretical calculations of Güttinger and the experiments of Phipps and Stern, it is of interest to reexamine the theory of transitions in spatial quantization. An obvious improvement is to replace the discontinuous rate of rotation of the magnetic field used by Güttinger, curve a in Fig. 1, by a continuous rate of rotation,

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[1] P. Güttinger, Zeits. f. Physik **73**, 169 (1932).
[2] T. E. Phipps and O. Stern, Zeits. f. Physik **73**, 185 (1932).

such as curve b in Fig. 1. In the following section the analysis is carried through for such a rotating magnetic field. The results are in complete agreement with experiment in predicting no transitions in the particular arrangement of Phipps and Stern where the magnetic field rotates through an angle of 2π.[3]

2. Analysis

Since the only dynamical variable of an alkali atom which is appreciably affected by the weak magnetic field, H, is the electron spin, the Hamiltonian may be written simply as $H\cdot\sigma$, where σ is the magnetic moment of the spin. This spin variable can have only two quantum states, so the wave function may be written as the linear combination $\psi = A(t)\psi_\alpha + B(t)\psi_\beta$, where ψ_α, ψ_β are the Pauli spin functions with reference to an arbitrary axis. In particular, if we choose this axis to be parallel to the magnetic field, which makes an angle θ with the fixed z axis, these functions will satisfy the wave equations

$$\begin{aligned}(H\cdot\sigma - h\nu)\psi_\alpha &= 0\\ (H\cdot\sigma + h\nu)\psi_\beta &= 0\end{aligned} \tag{1}$$

where ν is the frequency of the Larmor precession. By means of the transformation relations which relate ψ_α, ψ_β to the spin functions ψ_α', ψ_β' referring to a fixed z axis, namely[4]

$$\begin{aligned}\psi_\alpha &= \cos(\theta/2)\psi_\alpha' - i\sin(\theta/2)\psi_\beta'\\ \psi_\beta &= -\,i\sin(\theta/2)\psi_\alpha' + \cos(\theta/2)\psi_\beta',\end{aligned}$$

we see that ψ_α, ψ_β are related by

$$\begin{aligned}\dot\psi_\alpha &= -\,i(\dot\theta/2)\psi_\beta\\ \dot\psi_\beta &= -\,i(\dot\theta/2)\psi_\alpha.\end{aligned} \tag{2}$$

The exact wave equation is

$$\left(H\cdot\sigma - \frac{h}{2\pi i}\frac{d}{dt}\right)\left\{C_1 e^{2\pi i\nu t}\psi_\alpha + C_2 e^{-2\pi i\nu t}\psi_\beta\right\} = 0. \tag{3}$$

Corresponding to our knowledge that the spin is initially antiparallel, say, to the magnetic field, this equation is given the boundary conditions

$$C_1(-\infty) = 0 \tag{4a}$$

$$|C_2(-\infty)| = 1. \tag{4b}$$

The answer to our problem, namely the fraction, P, of atoms which change their spatial quantization upon passing through the rotating field, will be given by

$$P = |C_1(\infty)|^2.$$

[3] The observed change of quantization caused by impurities may be interpreted as caused by an exchange of electrons having opposite spins.

[4] W. Pauli, Zeits. f. Physik **43**, 601 (1927).

Upon applying the relations (1) and (2) to the wave Eq. (3), we obtain[5]

$$\dot{C}_1 = i\frac{\dot{\theta}}{2}e^{-2\pi i\Delta\nu t}C_2 \tag{5a}$$

$$\dot{C}_2 = i\frac{\dot{\theta}}{2}e^{2\pi i\Delta\nu t}C_1 \tag{5b}$$

where $\Delta\nu = 2\nu$. Elimination of C_2 leads to the second order equation

$$\ddot{C}_1 + (2\pi i\Delta\nu - \ddot{\theta}/\dot{\theta})\dot{C}_1 + \left(\frac{\dot{\theta}}{2}\right)^2 C_1 = 0. \tag{6}$$

In order that the results of this analysis may be legitimately compared with an actual experiment, $\dot{\theta}$ and $\ddot{\theta}$ must be continuous functions of time. Our choice of $\dot{\theta}$ is limited, however, to such functions that render (6) reducible to a known type of differential equation. Such a function is

$$\dot{\theta} = 2\pi\nu_0 \operatorname{sech} \pi t/\tau, \tag{7}$$

the plot of which is b of Fig. 1. The constants ν_0, τ have been so chosen that the total change in θ,

$$\Delta\theta = \int_{-\infty}^{\infty}\dot{\theta}dt = 2\pi\nu_0\tau,$$

is that produced by a constant rotation of frequency ν_0 and of duration τ.

This function, together with the transformation

$$z = [\tanh(\pi t/\tau) + 1/2],$$

reduces (6) to the hypergeometric equation

$$z(1-z)\frac{d^2C_1}{dz^2} + \{c - (a+b+1)z\}\frac{dC_1}{dz} - abC_1 = 0 \tag{8}$$

where $a = -b = \Delta\theta/2\pi$ and $c = \frac{1}{2} + i\tau\Delta\nu$. As t goes from $-\infty$ to ∞, z goes from 0 to 1. The general solution of (8) which is defined in this range is

$$C_1 = AF(a, b, c, z) + Bz^{1-c}F(a+1-c, b+1-c, 2-c, z).$$

In order that the boundary condition (4a) be satisfied, we must set $A = 0$. Using Eq. (5a) we find that the boundary condition (4b) is satisfied when

$$B = \frac{(\Delta\theta/2\pi)}{|c|}\cdot$$

The desired transition probability is thus given by

$$\begin{aligned} P &= \left|\frac{\Delta\theta/2\pi}{c}F(a+1-c, b+1-c, 2-c, 1)\right|^2. \\ &= \sin^2\frac{\Delta\theta}{2}\operatorname{sech}^2(\pi\tau\Delta\nu). \end{aligned} \tag{9}$$

[5] This direct method of obtaining this simultaneous set of equations was suggested to the authors by Professor H. P. Robertson.

3. Discussion

We have found the fraction (9) of electrons which change their spatial quantization when subjected to a rotating magnetic field of the type (7). It is reasonable to expect that any experimentally realizable rotating magnetic field of the same general type as (7), Fig. 1b, will lead to the same general results.

The fraction, P, is zero whenever the total rotation is a multiple of 2π. In the experiment of Phipps and Stern, where the total rotation was 2π, no transition was observed. Even if their rotation had not been quite 2π the "sech" in (9) would probable have given a result with their apparatus too small to be observed.

Another equally surprising result is, that if we multiply $\dot{\theta}$ by an arbitrarily large number, we do not change P, apart from the sine factor.

The dependence of P upon the duration of the rotation τ and upon the difference in energy of the two states, $\Delta E = 2h\nu$, is of a type to be expected from previous studies on transitions.[6] The decrease of P with increasing values of the product $\tau\Delta\nu$ is much more marked than as found by Güttinger. A comparison is given in Fig. 2.

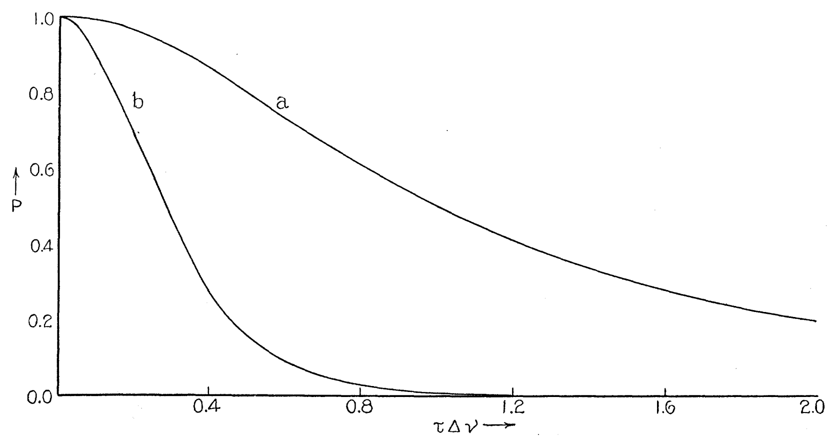

Fig. 2. Transition probabilities. Curve a, $P = 1/[1+(\tau\Delta\nu)^2]$; curve b, $P = \text{sech}^2\, \pi\tau\Delta\nu$.

II. Related Collision Phenomena

The Born-Dirac collision method has been applied with success to numerous high velocity collision problems. No such general method has been found for collisions which involve low velocities. The best that can be hoped for at present is to have a variety of techniques each restricted to a limited class of collisions. Such special techniques are being rapidly developed.

The most obvious procedure is to use the Born-Dirac method with modifications which reduce the perturbations by keeping the colliding systems apart. One method is to consider only grazing incidences.[7] Another is to use

[6] C. Zener, Phys. Rev. **38**, 277 (1931).

[7] O. K. Rice, Proc. Nat. Acad. Sci. **17**, 34 (1931).

as the initial solutions those corresponding to elastic collisions.[8] Even then the perturbation method is not applicable if the eigenwert of the initial state is very close to an eigenwert of another state.

Fortunately in this case a simplification is introduced by the legitimacy of considering only the two states which are in close resonance. The problem may then be reduced to the solution of two simultaneous equations of the second order in the coordinates of the centers of gravity of the two systems.[9] These can at present be handled only by a perturbation method starting from the adiabatic solutions. This is, however, exceedingly cumbersome.

These complications may be avoided in those cases in which the difference in energy between the two quantum states is small in comparison with the relative energy of translation. We are then justified in treating the positions of the centers of gravity not as dynamical variables but as parameters dependent upon time. This is in the spirit of the pioneer work of Kallmann and London.[10] The problem is then reduced to the solution of two simultaneous equations of the first order.

The initial wave equation is of the form

$$\left\{H_0(x) + V(t, x) - \frac{h}{2\pi i}\frac{d}{dt}\right\}\left\{C_1(t)e^{2\pi i(E_1/h)t}\psi_1(x) + C_2(t)e^{2\pi i(E_2/h)t}\psi_2(x)\right\} = 0. \tag{10}$$

H_0 is the unperturbed Hamiltonian for the internal coordinates x. V is the perturbation energy. Since the relative coordinates are here being regarded as definite functions of time, V is a function of t and x. The wave functions ψ_1, ψ_2 satisfy

$$(H_0 - E_n)\psi_n = 0.$$

The two simultaneous equations are obtained by multiplying (10) by ψ_1^* and by ψ_2^* and integrating.

$$\dot{C}_1 = ife^{-2\pi i\Delta\nu t}C_2 \tag{11a}$$

$$\dot{C}_2 = ife^{2\pi i\Delta\nu t}C_1. \tag{11b}$$

Here

$$f(t) = \frac{2\pi}{h}\int \psi_1 V(t/x)\psi_2^* dx$$

and $\Delta\nu = (E_1 - E_2)/h$. We must specify that V be such that $(n|V|n) = 0$.

The analysis of the preceding section has shown that this set of equations can be solved exactly if we give f the form

$$f(t) = f_0 \operatorname{sech}(\pi t/\tau). \tag{12}$$

[8] C. Zener, Phys. Rev. **37**, 556; **38**, 277 (1931); P. M. Morse and E. Stueckelberg, Ann. d. Physik **9**, 579 (1931).

[9] O. K. Rice, Phys. Rev. **37**, 1187; 1551; **38**, 1943 (1931); F. London, Zeits. f. Physik **74**, 143 (1932).

[10] H. Kallmann and F. London, Zeits. f. physik. Chem. **2B**, 207 (1929).

In particular, we have seen that the solution which satisfies the boundary conditions

$$C_1(-\infty) = 0$$
$$|C_2(-\infty)| = 1$$

gives for the asymptotic value of the transition probability

$$P = |C_1(\infty)|^2 = \sin^2 A \operatorname{sech}^2 (\pi\tau\Delta\nu),$$

where

$$A = \int_{-\infty}^{\infty} f(t)dt.$$

This may be written in the more general form

$$P = \left| \frac{\sin A}{A} \int_{-\infty}^{\infty} f(t) e^{2\pi i \Delta\nu t} dt \right|^2. \tag{13}$$

If we retain only the first term in the expansion of sin A, we obtain the P given by the perturbation method of setting $C_2=1$ in (11a). This simple relationship between the exact solution and that obtained by the perturbation method suggests that (13) is not confined to the particular $f(t)$ given by (12), but is the general expression for all non-singular f's, i.e., all functions which are continuous and whose first derivatives are continuous. This is borne out by the fact that for $\Delta\nu=0$, it is found that for *any* f, P is given by $\sin^2 A$. This generalization, applied to (9), says that the factor $\sin^2(\Delta\theta/2)$ is present for all experimentally obtainable $\dot{\theta}$'s.

As an example, let

$$f(t) = f_0/(1 + \pi^2 t^2/\tau^2).$$

Then (13) gives

$$P = \sin^2(\tau f_0) e^{-2\tau\Delta\nu/\pi}.$$