PROBABLE VALUES OF e, h, e/m AND α

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ABSTRACT

Calculation of general physical constants.—The calculation of the general physical constants offers a rich and almost untouched field of investigation. In general each equation for the calculation of a given constant contains also other general constants, and therefore does not, strictly speaking, evaluate any one of them. In 1929 the writer solved this problem by a method of successive approximations, i.e., the constant first evaluated was that one least dependent on others. This constant then became an *auxiliary* constant, of known magnitude, in the equations for other constants. The method breaks down when the probable error of any auxiliary constant is not small compared to the experimental error. This is the case in the calculation of h, which always involves e and in which the error in h is usually due *mainly* to the error in e. The present paper gives in detail a logically correct method for the *simultaneous* evaluation of e and h, from the several known functional relations between these two constants. The method was suggested first by W. N. Bond, but was not carried by him to its logical conclusion. The writer also disagrees with portions of Bond's calculations and with the final values of the constants that Bond adopts.

General theoretical relation between e and h.—Every so-called method for the evaluation of h yields an equation of the type $h=A_n e^n$ (1), in which A_n is an experimentally determined magnitude, and $n=3/3, 4/3$, or $5/3$ according to the method employed. For least squares' calculations it is necessary to use an equation *linear* in the unknowns, and in the present case the most convenient form for such an equation is obtained by the introduction of a new parameter h_n, defined by $h_n=A_n e_0^n$, where e_0 is a tentatively adopted value of e. One then gets the linear equation $h_n=h-(h_0 \Delta e/e_0)n$. The intercept at $n=0$ gives h, and from the slope one gets e $(=\Delta e+e_0)$. In the present work we assume $e_0=4.770\times10^{-10}$ *es* units, $h_0=6.547\times10^{-27}$ erg. sec. Plotted h_n:n curves, with the probable error in the *function* h_n indicated by broken lines, are shown in Figs. 1–7 and 10. The value of e so obtained is entirely *independent* of the adopted e_0 and we thus have a new method for the evaluation of e, of greater potential accuracy than any direct measurement. To obtain final most probable values of e and h, our original Eq. (1) is transformed to $e=a_m h^m$ (2), in which $m=1/n$ and $a_m=(A_n)^{-1/n}$. Then $m=0$ corresponds to a direct determination of e and one can thus include in a single equation *all* data on e and h, and can determine final most probable values. This is the essential extension of Bond's method.

Least squares procedure.—The necessary least squares' procedure involves most of the questions discussed in the article preceding this, on the calculation of errors by least squares. In order to make several corrections in Bond's work, and to indicate the proper procedure, a sample calculation is carried through in detail, using Bond's adopted data. The values of h and e so obtained agree with Bond's results, but the probable errors are quite different from those deduced by him. This solution is shown in Fig. 1, curve a. The h_n point for $n=5/3$ depends almost entirely upon the value adopted for e/m. The points resulting from $e/m=1.761\times10^7$ and $e/m=1.769$ are shown on each figure. From $h_{6/3}$ one can calculate the fine structure constant α, and the value of $h_{6/3}$ corresponding to Eddington's theoretical value $1/\alpha=137$ is shown in each figure. Bond uses the results of 36 investigations and these are shown by circles in Fig. 1. The three resulting points (arithmetic averages of observations) for the h_n:n curve are shown by crosses.

Additional solutions of bond's data.—Bond weights each point according to the number of observations composing it. When the reliability of the observations also is included in the weighting, one obtains curve b, Fig. 2. The values of e, h, e/m, and $1/\alpha$, resulting from each solution, are collected in Table II (page 242). The first two solutions differ mainly in respect to the probable errors. Three new investigations on e/m have recently appeared, and Fig. 3 gives a solution corresponding exactly to curve b, except for the inclusion of these three new observations. The new values of e/m are all "low" and the calculated value of $1/\alpha$ (137.115 ± 0.054), for curve c, Fig. 3, deviates appreciably from Eddington's value.

Solutions based on 1929 data, using Eq. (1).—The writer believes that values of constants should be derived mainly, if not wholly, from recent experimental work. The data used by Bond include practically all work, new and old, with each investigation given equal weight. The writer in 1929 calculated a most probable value of h_n for each of the six available methods. Three of these (ionization potentials, photoelectric effect and c_2) correspond to $n=3/3$, two (x-rays and σ) to $n=4/3$, and one (Rydberg constant) to $n=5/3$. The six adopted values of h_n are first given equal weight and combined by the new method. The results are shown in curve d, Fig. 4. Then the six observations are weighted according to their probable errors, and the results appear in curve e, Fig. 5. The point $h_{5/3}$ is based on $e/m=1.761\times 10^7$, since this value now seems well established. The chief difference between curves d and e and the preceding is due to the adoption of this value of e/m, the "spectroscopic" value of the 1929 work.

Solutions based on present data, using Eq. (1).—The data now available for each of the six methods are critically examined, and new most probable values of each h_n obtained. If, as has been customary during the past two years, one assumes that the correct wave-lengths of x-rays are those determined from grating measurements, the x-ray point is moved from $n=4/3$ to $n=3/3$, and an accompanying necessary correction gives a very high resulting value of $h_{3/3}$. At the same time there remains at $n=4/3$ only the very "low" $h_{4/3}$ observation derived from σ. Due mainly to these facts, the resulting solution, curve f, Fig. 6, exhibits a definite incompatibility of the data ($r_c/r_i=2.14$). The results therefore cannot be used, and to get curve g, Fig. 7, the two discordant methods (x-rays and σ) are discarded. The resulting value of e (4.7732 ± 0.0072) represents the best *independent* determination by the new method, on the assumption that the x-ray method belongs at $n=3/3$. Fig. 7 also includes a curve based on $1/\alpha=137$. This curve requires $e/m=1.7679$, and leads to $e=4.7824$, $h=6.5670$.

Solutions based on Eq. (2).—By introducing a new parameter $e_m=a_m h_0{}^m$, one can obtain the linear equation $e_m=e-(e_0\Delta h/h_0)m$, which is to be used in the least squares' solution of Eq. (2). e_m:m curves are shown in Figs. 8, 9, and 11. A simple relation is derived, giving the value of e_m corresponding to any h_n. Curve h, Fig. 8, results from the three h_n values used for curve g (converted to e_m values), *plus* the oil-drop value $e_d=4.768 \pm 0.005$. The resulting constants are practically identical with those adopted in 1929, as given in Table II. Curve i, Fig. 9, is based on the 1929 data, and represents the constants that would have been obtained in 1929, had the data been properly handled. There is a considerable change in some of the probable errors, but not in the constants.

New solutions with x-ray method at $n=4/3$.—Very recent observations by Bearden, as well as the discrepancies noted in curve f, give convincing evidence that the crystal wave-lengths of x-rays are correct, rather than the grating values. On this assumption the x-ray method belongs at $n=4/3$ and a recalculation of the data of curve f, with this one change, gives curve j, Fig. 10, which exhibits no inconsistency of data. The resulting value of e (4.7688 ± 0.0059) is the most reliable resulting from the new method, and is in remarkable agreement with the oil-drop value. Curve k, Fig. 11, represents the e_m:m curve corresponding to curve j, with the oil-drop value added.

General conclusions.—(1) The values of e, h and $1/\alpha$ depend in a very direct way upon the value adopted for e/m. In the past ten years all investigations on e/m, with

one exception, have given results in the close vicinity of 1.761, and with this value one obtains (curve *k*) $1/\alpha = 137.307 \pm 0.048$. Conversely, the assumption $1/\alpha = 137$ practically requires $e/m = 1.768$, for which there is now very little experimental evidence. This is the most important result of the present investigation. (2) It would appear that the crystal wave-lengths of x-rays are correct, and the close agreement of the resulting value of e (curve *j*) with the oil-drop value is strong evidence that there is no unsuspected error of any significance in Millikan's work. (3) Curve *k* gives the most probable values of the constants ($h = 6.5443 \pm 0.0091$, $e = 4.7688 \pm 0.0040$, $1/\alpha = 137.307 \pm 0.048$ and $e/m = 1.7611 \pm 0.0009$), but these differ so little from the values adopted in 1929 that *no change is advocated at this time.* However, the probable errors of all constants involving h^x/e^y, as published in 1929, are more or less in error, and a general equation is given for the correct calculation of such probable errors. A full report on all the general constants, including derived constants, will be published by a committee of the National Research Council, whenever this seems necessary.

Introduction

THE calculation of probable values of the various general physical constants offers a rich and hitherto almost untouched field of investigation. Much time and effort have been devoted to the experimental evaluation of each constant, and as a result there is a really extensive amount of experimental material available to the computer. How this material should be handled, mathematically, in order to obtain the most reliable values of the desired constants, constitutes a problem of considerable complexity. This is quite aside from the question of the proper selection and weighting of the data, which involves primarily the individual judgment of the computer. If each general constant could be evaluated by means of an equation that contained no other general constant, the situation would be comparatively simple. Unfortunately this is not the case. Each such equation normally contains one or more additional constants, and in a previous investigation on this subject[1] I have called these "auxiliary constants."

In this previous work I attempted to solve the problem arising from the presence of these auxiliary constants by calculating first the constant that depends least on the other constants, and then consistently adopting this value in all further calculations. In most cases the probable error in the constant being evaluated is due chiefly to the direct experimental errors, and depends only in a minor degree on the errors of the auxiliary constants. In such a situation the procedure adopted in 1929 constitutes essentially a "method of successive approximations" for the simultaneous evaluation of all the constants, and is theoretically quite correct. The method, however, breaks down completely when the error in the constant being investigated is due *chiefly* to the error in one or more auxiliary constants. This is just the situation in the case of the Planck constant h. There is no known method for evaluating h that does not involve the electronic charge e as an auxiliary constant, and the resulting error in h is, in most cases, due mainly to the error in e. The writer pointed out this situation, in the 1929 work, but failed to recognize the procedure necessary to give the best results.

[1] R. T. Birge "Probable Values of the General Physical Constants," Phys. Rev. Supplement (now called Reviews of Modern Physics) **1**, 1, 1929. This article will be denoted as G.C. 1929.

A general statement of the correct procedure is as follows. If there exists a theoretical equation containing, let us say, e, h, and e/m, this equation cannot be said to furnish a value of any one of these constants. If, however, there exist three different theoretical relations involving these three constants, one can use the three relations simultaneously for the evaluation of all three constants. If more than three such relations exist, the *most probable* values of the constants are best determined by least squares. It has been remarked that the mathematically correct way to determine the atomic weights of the elements is to make one simultaneous least squares' evaluation of all atomic weights, using all measured mass ratios. The practical objection is that the necessary calculations might easily take centuries to carry out.

The situation, as it exists in connection with the fundamental physical constants, is far less complex, due to the smaller number of such constants. In fact it appears now as though e, h, and e/m are the only important constants that should be simultaneously evaluated in a straight-forward manner. In the present paper the problem is still further simplified by assuming a value of e/m, so that only e and h remain as unknowns. The chief purpose of the paper is to present what appears to be a theoretically correct method for finding the most probable values of these two constants. It is hoped that this general method will be used in future calculations, and the method is therefore given in detail. The paper also includes a re-examination of available data, and a new calculation of probable values of e, h, and the fine structure constant α. The final conclusion of the paper is that there is at the moment no sufficient cause for advocating a change in the writer's 1929 values of these constants, but there is certainly cause for discarding the very crude method used in these 1929 calculations.

The major principle underlying the new method of calculation has been suggested by Bond,[2,3] but in his two articles Bond fails to carry the theory to its logical conclusion. The writer does not agree at all with Bond's choice of data, nor with his weighting of the data, and it is for that reason that a re-examination of available material has been made. All of Bond's calculations are carried out by least squares' methods, but in several cases he seems to have used formulas not applicable to the situation. In fact there is scarcely an English text on the subject of least squares that even mentions some of the formulas needed for these calculations. It is partly for this reason that the paper just preceding this[4] has been written. The work of the present paper really constitutes an interesting example of the application of least squares, and I open the paper with a discussion of the fundamental equations, and the correct procedure for handling them, using Bond's own data as illustration. This is followed by a recalculation of the data adopted in my 1929 article, and finally by a discussion and calculation of what appears to be the best data available at present.

[2] W. N. Bond, Phil. Mag. **10**, 994 (1930).

[3] W. N. Bond, Phil. Mag. **12**, 632 (1931).

[4] R. T. Birge, "The Calculation of Errors by the Method of Least Squares." To be referred to as L.S. 1932.

THE GENERAL THEORETICAL RELATION BETWEEN e AND h

Each so-called method for the evaluation of h yields an equation of the type

$$h = A_n e^n \tag{1}$$

where e is the electronic charge and A_n is an experimentally determined magnitude. The value of n, for each method, is listed in the last column of the table on page 57 of *G. C.* 1929. One observes that three methods (ionization potentials, photoelectric effect and c_2) involve $n=3/3$, two methods (x-rays and σ) involve $n=4/3$, and one method (Rydberg constant) involves $n=5/3$. Now, as noted in the Introduction, Eq. (1) is, strictly speaking, not an equation for the evaluation of h. It does, however, permit the simultaneous evaluation of e and h, since at least two values of n, together with the corresponding values of A_n, are available. This is the major point brought out in Bond's papers. His procedure, however, does not lead to the final most probable values of e and h because he omits all directly measured values of e, such as for example that obtained by the oil-drop method. What Bond does obtain is a new value of e, entirely *independent* of the oil-drop value.

To get the final most probable values of e and h, Eq. (1) should be rewritten as follows[5]

$$e = a_m h^m \tag{2}$$

where

$$m = 1/n \text{ and } a_m = (A_n)^{-1/n}. \tag{3}$$

Then $m=0$ corresponds to any *directly* determined value of e, while $m=3/5$, $3/4$ and $3/3$ correspond to the above-mentioned values $n=5/3$, $4/3$ and $3/3$. It is thus possible to include simultaneously *all* work on e and h, and so to obtain final most probable values of both constants. There is, however, an important advantage in the preliminary use of Eq. (1). By means of this equation one can determine a value of e that should be consistent with the directly observed value. Only when such consistency exists is it permissible to use Eq. (2) to obtain a final most probable value of e. For that reason a number of calculations will be made, using Eq. (1), and various deductions will be drawn from the results. Then a selected number of these calculations will be repeated, using Eq. (2).

Eqs (1) and (2) are non-linear in the unknowns, e and h, and the proper method of procedure, in order to obtain least squares' values of such unknowns, is to be found in every good text. The probable errors in the resulting values of e and h are given by formulas that appear in many, but not all, texts on least squares. Finally, in order to evaluate the fine structure constant α, it is necessary to know the value of a certain $f(n)$ at the point $n=2$, and the probable error in α follows directly from the probable error of the *function* at this point. It is therefore necessary to determine the probable

[5] The writer is indebted to Professor R. B. Brode for this important suggestion.

error of a function whose coefficients (e and h) have been simultaneously evaluated by least squares. This type of error, as noted in L. S. 1932, is scarcely mentioned in the literature, and has accordingly been discussed rather fully in that article.

The Least Squares Procedure

As already noted, Eq. (1) involves h and e as unknown quantites, and n and A_n as known quantities. When an equation is non-linear in the unknowns, the standard least squares procedure[6] is to adopt approximately correct values of the unknowns, and then to calculate the proper corrections to be applied to these tentative values. Using a Taylor's expansion one obtains the necessary linear observational equations. In this case let e_0 and h_0 be the tentatively adopted values. Also let

$$\left.\begin{aligned} e &= e_0 + \Delta e \\ h &= h_0 + \Delta h \end{aligned}\right\} \tag{4}$$

Eq. (1) is now to be written as

$$f(h,\, e) = he^{-n} = A_n. \tag{5}$$

The general form of the resulting observational equation is

$$\left(\frac{\partial f}{\partial h}\right)_0 \Delta h + \left(\frac{\partial f}{\partial e}\right)_0 \Delta e = l \tag{6}$$

where

$$l = f(h,\, e) - f(h_0,\, e_0) = A_n - h_0 e_0^{-n}. \tag{7}$$

Eqs. (5), (6) and (7) give

$$e_0^{-n}\Delta h - nh_0 e_0^{-(n+1)}\Delta e = A_n - h_0 e_0^{-n}. \tag{8}$$

Eq. (8) is not in a convenient form for use as an observational equation, for the following reason. The various precision methods for determining h involve *proportional* errors in A_n (and in h) of the same order of magnitude. The absolute value of A_n (and accordingly its absolute error) varies, however, as e^{-n}, as shown by Eq. (5). Now for each increase of 1/3 in the value of n, e^{-n} and A_n increase about 1280 fold. Hence the *weight* of each observational equation (which is to be taken inversely proportional to the square of the absolute probable error) varies enormously with n, and this introduces a troublesome feature in the numerical calculations.

The whole situation is greatly simplified by the introduction of a new parameter

$$h_n \equiv A_n e_0^{\,n}. \tag{9}$$

Substituting in Eq. (8) the value of A_n from Eq. (9) and using Eq. (4) we obtain

$$h_n = h - \left(h_0 \frac{\Delta e}{e_0}\right) n. \tag{10}$$

[6] G. C. Comstock "Method of Least Squares" pp. 21–23. Merriman "Method of Least Squares" pp. 200–204. W. W. Johnson "Theory of Errors etc.," pp. 93–94.

Eq. (10) is of the form

$$y = a + bx \tag{11}$$

where

$$\left.\begin{aligned} a &= h \qquad b = -\left(h_0\frac{\Delta e}{e_0}\right) \\ y &= h_n \quad x = n \end{aligned}\right\} \tag{12}$$

This new observational equation is linear in the unknowns, a and b, and the absolute value of y, or h_n, changes only very slightly with n. Hence the absolute error in h_n is always of the same order of magnitude, and the various observational equations have comparable weights. Moreover, as discussed in footnote 20 of L. S. 1932, the least squares' formulas are strictly applicable, since the errors of observation are confined to the ordinate y.

In terms of a graph, the various observations h_n obtained from Eq. (9) are to be plotted as ordinates against n as abscissa. The best straight line through the data is then to be calculated by least squares. The intercept of this line on the h_n axis, at $n=0$, gives the desired value of h. From the slope of the line we can obtain Δe and so, by Eq. (4), the desired value of e. This is the method used by Bond.

In my 1929 paper, h was calculated by the use of Eq. (9). That is, a most probable value of e (here called e_0) was assumed, and then from each observed A_n one calculated an h_n. The weighted average of the various h_n values was taken as the most probable value of h. Such a procedure is equivalent to fitting the h_n values to the best *horizontal* straight line ($h_n=$constant). It is legitimate only when h_n, plotted against n, shows entirely irregular variations (due solely to the experimental errors in A_n). If now the assumed value of e_0 is *not* correct, the values of h_n will show a regular trend with n. The least squares' calculation of the best straight line through the various points is essentially the calculation of a new value of e_0 (to be called e) which, used in Eq. (9), will give a new set of values of h_n showing the least possible trend with n, and is, simultaneously, the calculation of the best average value (to be called h) of this new set of values of h_n. Even if the trend of h_n with n is so small as to be undetected on an $h_n{:}n$ plot, the least squares' solution should be carried out, since only in this way can the most reliable values of e and h be obtained.[7]

In G. C. 1929 six methods for calculating h were discussed. These methods, as already noted, correspond to values of $n=3/3$, $4/3$ and $5/3$. There is a seventh method, corresponding to $n=6/3$, which was not mentioned because its precision was not yet comparable with the others. This seventh method concerns the fine structure constant α, which is given by the equation

[7] The reader is reminded that in such a method the value e_0 first assumed is naturally some directly determined value of e, but the final calculated value of e quite ignores this assumed value. Hence, as emphasized by Bond, the procedure now under discussion constitutes an entirely *new and independent* method for evaluating e, but as noted earlier, a really proper procedure for determining a final most probable value of e must use *both* the new value and the directly determined value. This is accomplished by the use of Eq. (2), in place of Eq. (1).

$$\alpha = 2\pi e^2/hc \tag{13}$$

Hence

$$h = \left(\frac{2\pi}{\alpha c}\right)e^2. \tag{14}$$

The most accurate direct measurement of α is that by Paschen,[8] who found $\Delta\nu_{He} = R_{He}\cdot\alpha^2/16 = 0.3645 \pm 0.0045$ cm^{-1}. Using Paschen's value[9] $R_{He} = 109$-722.14 ± 0.04 cm^{-1}, one obtains $\alpha = (7.291 \pm 0.045) \times 10^{-3}$. It is more convenient to state α in terms of its reciprocal, and in this case $1/\alpha = 137.16 \pm 0.85$. With $e = 4.770 \times 10^{-10}$, this yields $h = (6.54 \pm 0.04) \times 10^{-27}$ erg·sec., exclusive of the error in e. This is a much less accurate value of h than that obtained by the other six methods, as will appear in the discussion to follow. Accordingly this seventh method is omitted, and we shall use our final calculated values of h and e to determine a most probable value of $1/\alpha$. The most important conclusion of Bond's two articles is that the available data are in agreement with Eddington's theory[10] that $1/\alpha = 137$. The value of $1/\alpha$ will accordingly be discussed with special reference to this point.

In terms of Eq. (9) we may write Eq. (14) as

$$h_2 = \left(\frac{2\pi}{\alpha c}\right)e_0^2. \tag{15}$$

Hence, after evaluating the constants of Eq. (10), we use that equation to calculate h_2 and then Eq. (15) to calculate $1/\alpha$. The probable error in h_2 is the error in the function

$$f(n) \equiv h - \left(h_0\frac{\Delta e}{e_0}\right)n \tag{16}$$

at the point $n = 2$, and from Eq. (15) the proportional error in $1/\alpha$, aside from the negligibly small error in c, equals the proportional error in h_2. The results published by Bond indicate that he used an incorrect formula for the error in $1/\alpha$, as well as for the errors in other quantities. It therefore seems advisable to go through all steps of a typical solution. For this purpose I shall use the 36 values of h_n quoted on page 634 of Bond,[3] without any comment as to the scientific value of the data.[11] Such comment will appear later.

The data listed by Bond consist of 14 observations of h_n corresponding to $n = 3/3$, 9 corresponding to $n = 4/3$, and 13 corresponding to $n = 5/3$. These observations are plotted as circles in Fig. 1 of the present article. Double and triple circles indicate two and three coincident observations. Now we desire a solution of Eq. (10), and in this case the final data consist of *three* values of h_n (to be called *points*), one for each of the three values of n (3/3, 4/3 and

[8] F. Paschen, Ann. d. Physik **50**, 901 (1916). See also **82**, 689 (1927).

[9] See G.C. 1929, pp. 45 and 46.

[10] A. S. Eddington, Proc. Roy. Soc. **A126**, 696 (1930).

[11] In calculating values of h_n by means of Eq. (9), Bond uses $e_0 = 4.770 \times 10^{-10}$ *es* units, and $h_0 = 6.547 \times 10^{-27}$ erg·sec. These are the values given in G.C. 1929, and Bond's paper, like the present one, is concerned with possible changes in these values.

5/3). The function obviously cannot have two or more values for a single value of n, and any observed scattering of observations at one value of n must be due solely to experimental errors. On the other hand, a *variation* of h_n with n may well exist, due to the fact that our adopted e_0 is not the best value of e, and it is this variation that we are trying to evaluate. The first step is then to calculate a weighted average value of h_n for each different value of n. Bond gives each original observation[12] unit weight, so that each resulting point is to be given a weight (p) equal merely to the number of observations of which it is the arithmetic average. These points[13] are denoted by crosses in Fig. 1, and are listed in Table I, col. 2.

TABLE I.

1 n	2 h_n(obs) curve a	3 p	4 h_n(calc)	5 r_a'	6 r curve b	7 p	8 h_n(calc)	9 r_a'
0			6.55751	0.00960			6.55681	0.00635
3/3	6.5473	14	6.54561	0.00307	0.0019	2.77	6.54630	0.00243
4/3	6.5364	9	6.54165	0.00204	0.0034	0.86	6.54280	0.00132
5/3	6.5395	13	6.53768	0.00317	0.00085	13.84	6.53930	0.00112
6/3			6.53373	0.00521			6.53579	0.00210

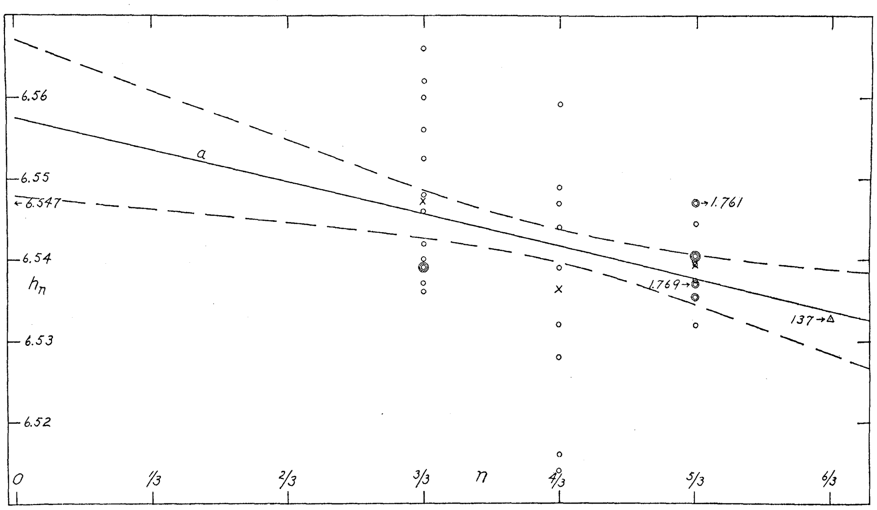

Fig. 1. Solution of Eq. (17). 36 observations (circles) used by Bond, with each resulting point (cross) weighted according to the number of observations composing it. $h = 6.5575$, $e = 4.7787$. Probable error of the calculated *function* h_n shown by broken lines.

In discussing least squares' formulas, it is more convenient to use Eq. (10) in the general form of Eq. (11), to which it is connected by the relations given

[12] Just as in L.S. 1932, I shall attempt to avoid confusion by using the word *observation* for each h_n, as determined directly from one investigation, and the word *point* for the average value of h_n from all observations at any one value of n.

[13] In discussing h and e, the factors 10^{-27} and 10^{-10} respectively are omitted whenever this can cause no misunderstanding.

in Eq. (12). The least squares' formulas for a and b, and for their probable errors r_a and r_b, are given as Eqs. (33) to (38) of L.S. 1932. With each point given the *arbitrary* weight shown in Table I, col. 3, it is possible to calculate all probable errors only on the basis of external consistency[14] i.e., by the use of r_e, as defined by Eq. (38), L.S. 1932. Eq. (10), thus evaluated, is plotted as curve a in Fig. 1, and is given by

$$h_n = 6.5575 \pm 0.0096 - (0.0119 \pm 0.0071)n. \tag{17}$$

Hence $h_0(\Delta e/e_0) = +0.0119 \pm 0.0071$, and since $h_0 = 6.547$ and $e_0 = 4.770$, we find $\Delta e = +0.0087 \pm 0.0052$ or $e = 4.7787 \pm 0.0052$. Also, from Eq. (17), $h = 6.5575 \pm 0.0096$.

In place of these results, Bond gives $h = 6.5575 \pm 0.0053$, and $e = 4.7787 \pm 0.0029$. His published errors seem to have been obtained in the following way. Instead of getting the three average values of h_n (the points), he uses Eqs. (33) and (34), L.S. 1932, as though there were 36 different points. It is easily shown that this procedure will give values of a and b identical with those obtained by the method of weighting used in Table I, col. 3. The calculated errors will however *not* be the same, as the results show. This is due to the fact that his value of r_e is not correct. The rest of Eqs. (36) and (37), L.S. 1932, is unchanged. Bond calculates r_e from an equation of the correct form, i.e.,

$$r_e = 0.6745\left(\frac{\sum pv^2}{(s-2)}\right)^{1/2} \tag{18}$$

but he uses $p = 1$, $s = 36$,[15] and $v =$ the deviation of *each* original observation (36 in all) from its value calculated by Eq. (17). In the correct interpretation of Eq. (18), (or of Eq. (38), L.S. 1932), v is the deviation of each of the three *points* (average values of h_n) from its value calculated by Eq. (17), p is the weight of the point, and $s = 3$. By using the former incorrect interpretation of Eq. (18), I have been able to reproduce all Bond's published probable errors, and therefore deduce that that was the procedure he used. That his procedure cannot be correct is immediately evident from the fact that if all observed h_n values correspond to the same value of n (giving only one point), no solution at all would be possible. An increase in the number of observations at any one value of n is useful only in giving a more reliable point. The accuracy of the resulting coefficients in Eq. (10) depends solely on the number of *different* values of n represented (i.e., on the number of points), and the accuracy with which these points fit a linear relation. These facts are brought out more clearly in the actual calculations given later.

The values of h_n, as calculated from Eq. (17), are given in col. 4 of Table I, and from the calculated h_2 we obtain $1/\alpha$ by writing Eq. (15) as

$$1/\alpha = \left(\frac{c}{2\pi e_0^2}\right)h_2 = \frac{h_2}{47.68575 \times 10^{-30}}\,. \tag{19}$$

[14] See section on "Internal versus External Consistency," L.S. 1932.

[15] Throughout this article s is used for the number of observations, (or the number of points), in place of the usual n, in order to avoid confusion with n in Eqs. (1) and (10).

The result is $1/\alpha = 137.016$. The proportional probable error in $1/\alpha$, as has been pointed out in connection with Eq. (15), equals the proportional error in the function h_n at $n=2$. The formula for obtaining such an error of a function is given by Eq. (40) L.S. 1932. It is interesting to calculate the probable error in the function h_n, not only at $n=0$ and 2, but also at each value of n considered. This error is given in col. 5, Table I, and is indicated by the broken lines of Fig. 1. The error has been calculated and is similarly shown in each of the succeeding figures. In the case of Eq. (17), $h_2 = (6.5337 \pm 0.0052) \times 10^{-27}$. Hence, by Eq. (19), $1/\alpha = 137.01_6 \pm 0.10_9$ (Bond gives $137.01_7 \pm 0.05_9$).

It is of interest to calculate also the value of e/m that corresponds to $h_{5/3}$, as given by Eq. (10), From the formula for the Rydberg constant[16] and from Eq. (9) we obtain

$$e/m = \frac{2\pi^2 \cdot e_0{}^5}{R_\infty c^2 h_{5/3}{}^3} = \frac{4.94214 \times 10^{-72}}{h_{5/3}{}^3}. \tag{20}$$

Conversely, in obtaining *observed* values of $h_{5/3}$ from any observed value of e/m, we have the relation

$$h_{5/3} = \frac{7.90625 \times 10^{-27}}{(e/m \times 10^{-7})^{1/3}}. \tag{21}$$

In taking the cube root, it is convenient to use the form $e/m \times 10^{-7} = 1.761$ or thereabouts, and the factor in the numerator of Eq. (21) corresponds to this.

The proportional error in the calculated value of e/m is, from Eq. (20), three times the proportional error in the calculated $h_{5/3}$. Using this value of $h_{5/3}$ and its error, as given in Table I, we obtain from Eq. (20), $e/m = (1.7686 \pm 0.0026) \times 10^7$. This is merely the value of e/m that it is necessary to assume, in Bohr's formula for the Rydberg constant, in order to be consistent with the values of h and e calculated from Eq. (17). As in the case also of $1/\alpha$, the writer made such a calculation of e/m in his 1929 work, but it was not evident at that time how the corresponding error in $1/\alpha$ and in e/m should be calculated.

It will appear from the discussion to follow that the most probable values of h and e depend chiefly on the value adopted for e/m. In 1929 the writer gave *two* values, the so-called "spectroscopic" value 1.761×10^7 *em* units, and the "deflection" value 1.769×10^7. These, by Eq. (21), correspond to values $h_{5/3} = 6.54714 \times 10^{-27}$ and 6.53724×10^{-27} respectively. These two points are located by arrows on each figure of this article. The 1929 value, $h = 6.547$ (our present h_0), is also shown on each figure. Finally, by Eq. (19), Eddington's value $1/\alpha = 137$ corresponds to $h_2 = 6.53295 \times 10^{-27}$, and this point is indicated by a triangle, on each figure. The discrepancy between each new value of e, and the previously adopted $e_0 = 4.770$ can be judged from the slope of the h_n curve. A horizontal line would indicate $e = e_0$.

This completes the discussion of the various formulas needed for the solution of Eq. (1), and we pass now to a consideration of the scientific aspects of the problem.

[16] G.C. 1929, page 49.

Other Possible Values of e and h Based on Bond's Observational Data

Bond[3] has chosen, from all available data, the 36 values listed in his article and used as illustrative material in the preceding section. Even assuming that these 36 observations should be chosen, it does not follow by any means that the values of e and h corresponding to Fig. 1 are the most probable. In the first place, Bond gives each observation unit weight and, as has been discussed, this is equivalent to weighting each of the three resulting points merely according to the number of observations that it represents. It is however quite obvious that the reliability of any point h_n, for a given value of n, should be judged by the *consistency* of the observations as well as by their number. Accordingly we shall first consider other possible ways in which these 36 observations, and the resulting 3 points, may be weighted, and shall draw certain conclusions as to the most desirable method of weighting. These conclusions will then be applied to the newer data now available.

The most rigorous method of weighting should start with the 36 observations. Each should be weighted according to what is judged to be its probable error. Such a probable error should in turn be based not only on the purely accidental errors of observation, but also on all other possible errors, constant or systematic, so far as these can be estimated. Since the 36 observations listed by Bond include practically all work in the field, old as well as new, such an evaluation would be very difficult to make in an intelligent manner, and does not seem worth the effort. The method will however be applied to the solutions given later.

(1) Curve *b*. A simpler method of weighting the observations is that adopted by Bond. Each observation is given equal weight and the arithmetic average of all observations at a given value of n is calculated. We thus obtain again the three points listed in col. 2 of Table I. We shall, however, weight each of these *points* according to its probable error r, and not merely according to the number of observations included by it.[17] The values of r calculated from Eq. (13) L.S. 1932, and the resulting values of $p(=c/r^2)$, are listed in cols. 6 and 7 of Table I. In this, as in succeeding calculations, the arbitrary constant c is taken as 10^{-5}. A comparison of cols. 3 and 7 of Table I shows how greatly the weighting is modified when the consistency of the observations is considered.

The least squares' solution of Eq. (10) is now

$$h_n = 6.5568 \pm 0.0063 - (0.0105 \pm 0.0041)n. \qquad (22)$$

This equation is plotted as curve b, Fig. 2, together with the probable error of the function, at each value of n. The arrow attached to each observed point measures its assumed error as listed in Table I, col. 6. The calculated values of h_n and the probable errors of these calculated values are listed in cols. 8 and 9 of Table I. All of these probable errors associated with Eq. (22) have been calculated, just as in the case of Eq. (17), on the basis of external

[17] See Eqs. (13), (14), (15), L.S. 1932 and accompanying discussion.

consistency. In the present case, however, the three individual points have been given weights based on their probable errors and it is accordingly possible to calculate errors also on the basis of internal consistency.

The subject of internal versus external consistency has been discussed fully in L.S. 1932, and the reader is reminded that there is a constant ratio between all errors, as calculated by the two systems. This ratio is computed most simply by using a hypothetical point of unit weight. For this point the probable error on the basis of external consistency is given by r_e, Eq. (18), and the corresponding error on the basis of internal consistency is, in this article, always given by

$$r_i = (c)^{1/2} = (10^{-5})^{1/2} = 3.162 \times 10^{-3}. \tag{23}$$

In the case of the solution now under discussion, the data needed for Eq. (18) are given in cols. 2, 7 and 8 of Table I, with $s=3$. The result is $r_e=4.188 \times 10^{-3}$. Hence $r_e/r_i=1.32$.

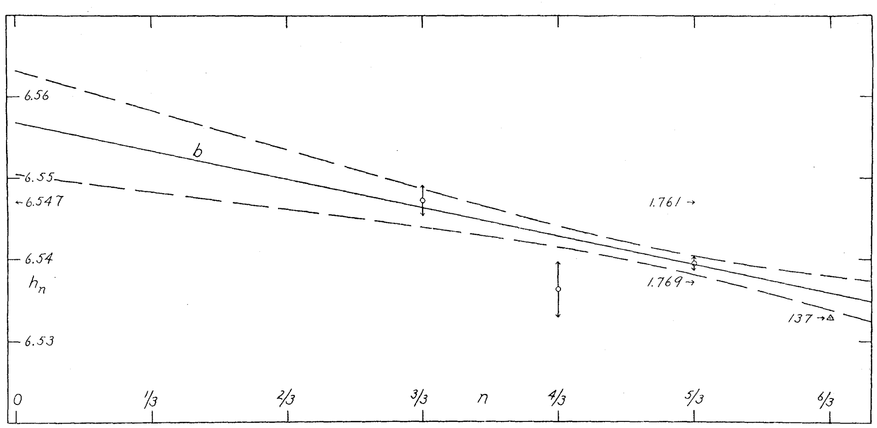

Fig. 2. Solution of Eq. (22). Same data as in Fig. 1, but points weighted according to the consistency of the observations, as well as their number. Probable error of each point shown by attached arrows. $h=6.5568$, $e=4.7777$.

Now, as shown in L.S. 1932, these two methods of calculating errors should agree, except for statistical fluctuations, i.e., the ratio r_e/r_i should be unity. Any considerable deviation from unity is an almost sure indication of the presence of constant or systematic errors in the data. In the present case the 32 percent deviation from unity can well be attributed to chance, since the proportional probable error in the ratio is given by $0.4769/s^{1/2}=27.5$ percent, with $s=3$.[18] We therefore conclude that there is here no definite evidence of systematic errors. In such a situation I adopt the policy suggested in

[18] This formula for the probable error of the ratio appears as Eq. (6), L.S. 1932. As discussed in footnote 26 of that article, the probable error in any one of the actual constants, such as h or e, is more uncertain than this, since the uncertainty in the weight of the constant must also be included. Bond[3] does not include this second source of uncertainty and uses the equation just quoted with $s=36$. He thus obtains an apparent reliability for his stated probable errors greatly in excess of the true value.

L.S. 1932, and use r_e, since it is larger than r_i. The errors given in Eq. (22) are therefore retained, and the various results that may be derived from this equation are as follows,

$$h = 6.5568 \pm 0.0063 \quad 1/\alpha = 137.060 \pm 0.044$$
$$e = 4.7777 \pm 0.0030 \quad e/m = 1.7673 \pm 0.0009.$$

These two solutions differ mainly in the values of the probable errors and illustrate in a striking way how dependent such results are upon the adopted method of weighting.

For convenience of comparison, the values of the four constants, for all the different solutions discussed in this paper, are listed in Table II. Each of these solutions is plotted, and the curve designation appears in the first column of the table. The second column gives the number of the equation stating each solution.

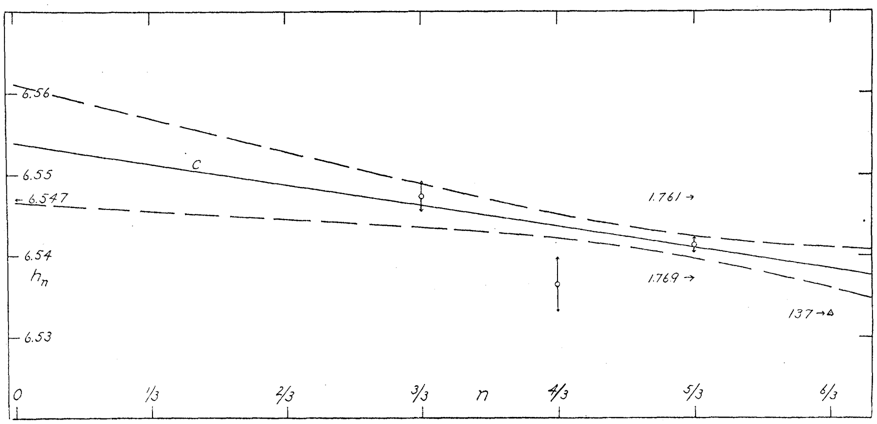

Fig. 3. Solution of Eq. (24). Same data and method of weighting as in Fig. 2. but with the addition of three new observations of $h_{5/3}$ (from values of e/m). $h=6.5539$, $e=4.7756$.

(2) Curve c. In listing the 36 observations used in the preceding analysis, Bond apparently attempted to include all available data. Even since the writing of his paper, however, three new investigations on e/m have been completed. Campbell and Houston[19] from the Zeeman effect, get 1.7577; Perry and Chaffee,[20] from electrostatic acceleration of electrons get 1.761 ± 0.001; and Kirchner,[21] by the same method, gets 1.7598 ± 0.0025. It is of interest to add these three new observations, and to obtain a new solution by the same procedure (including method of weighting) as used for curve b. The points at $n=3/3$ and $4/3$ and their weighting are unchanged. For $n=5/3$ we have $h_n=6.5413 \pm 0.00096$, giving $p=10.85$. The solution is

$$h_n = 6.5539 \pm 0.0072 - (0.0077 \pm 0.0047)n. \qquad (24)$$

[19] J. S. Campbell and W. V. Houston, Phys. Rev. **38,** 581 (1931). Their final value (Phys. Rev. **39,** 601 (1932)) is 1.7579 ± 0.0025, but this small change does not affect the final average value of $h_{5/3}$.

[20] C. T. Perry and E. L. Chaffee, Phys. Rev. **36,** 904 (1930).

[21] F. Kirchner, Ann. d. Physik (5) **8,** 975 (1931).

This is shown by curve *c*, Fig. 3, and the various resulting constants are tabulated in Table II. The errors are again calculated on the basis of external consistency, since the ratio r_e/r_i is now 1.49.

TABLE II.

Curve	Eq.	h	e	$1/\alpha$	e/m
a	17	6.5575±0.0096	4.7787±0.0052	137.016±0.109	1.7686±0.0026
b	22	6.5568±0.0063	4.7777±0.0030	137.060±0.044	1.7673±0.0009
c	24	6.5539±0.0072	4.7756±0.0034	137.115±0.054	1.7659±0.0011
d	25	6.5568±0.0064	4.7753±0.0037	137.193±0.087	1.7631±0.0021
e	26	6.5360±0.0117	4.7652±0.0053	137.338±0.064	1.7612±0.0009
f	36	6.563 ±0.022	4.777 ±0.010	137.22 ±0.12	1.7611±0.0021
g	44	6.5543±0.0166	4.7732±0.0072	137.266±0.079	1.761 ±0.001
h	55	6.5464±0.0095	4.7696±0.0041	137.302±0.048	1.7610±0.0009
1929		6.547 ±0.008	4.770 ±0.005	137.29 ±0.11	1.761 ±0.001
i	56	6.5441±0.0079	4.7688±0.0035	137.303±0.046	1.7612±0.0009
j	57	6.5443±0.0133	4.7688±0.0059	137.305±0.069	1.7611±0.0010
k	58	6.5443±0.0091	4.7688±0.0040	137.307±0.048	1.7611±0.0009

Certain conclusions may be drawn from the constants for this solution. In the first place, the value of $1/\alpha$ differs from 137 by more than twice the probable error, so that the data can hardly be said to support Eddington's theory. The new calculated value of *e* agrees with the directly determined value (4.770±0.005) as well as is to be expected, considering the probable errors. This is very important, for it indicates that there are no serious systematic errors in the direct measurements. Points like these will, however, be discussed more critically after various other solutions have been obtained.

SOLUTIONS BASED ON 1929 DATA, USING EQ. (1)

The results given by curve *c* might be considered the best that can be obtained, if it is wise to include all possible investigations and to give to each investigation the same weight. The writer, however, emphatically disagrees with such a choice of data. At the risk of repeating remarks made in previous articles, the following statement of policy is presented.

Most important constants have been measured many times, and in some cases by a number of different methods. The only object of repeating previous work is to obtain a greater precision in the result. Very often this precision is attained by the elimination of various sources of constant and systematic error that there is reason to believe existed in the earlier work. Now the assignment of equal weights to the various results is merely the assumption of equal reliability. If, however, the newer results are no more reliable than the older, it would appear that these newer investigations represent more or less wasted effort. The computer may remark that he does not wish to pass judgment on the various investigations, and therefore gives them all equal weight. I feel, however, that the computer is practically compelled to pass such judgment; otherwise the computation should not be made.

When one thus proceeds to consider the available data, it becomes immediately evident that the newer investigations are in general entitled to far greater weight than the older. The original experimental work on a given constant is often very brilliant, but the numerical results are likely to be marred by the presence of constant errors that is revealed only by later and more detailed investigation. For this reason it is, I believe, generally agreed at the present time that values of constants should be based almost wholly on the most recent work in the field. This is the theory adopted for my 1929 work, and it is the theory on which the rest of this article is based.

(1) Curve *d*. In the case of curves a, b and c, all observations corresponding to a given value of n have been averaged together. It has, however, been noted that three different methods for evaluating h correspond to $n=3/3$, two to $n=4/3$, and one to $n=5/3$. It seems best to consider separately the value of h_n resulting from each of these six methods. That procedure was adopted in 1929 and the resulting values are listed in the table on page 57 of G.C. 1929. The weighted average obtained then was based on the assumption that there is

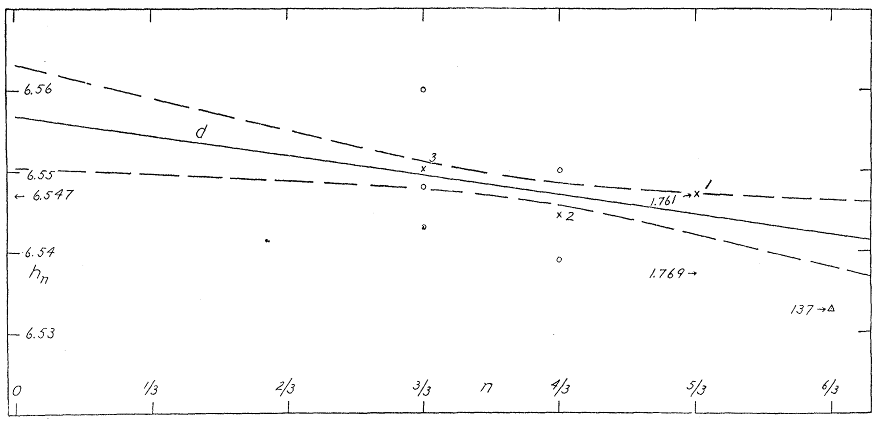

Fig. 4. Solution of Eq. (25). Most probable value of h_n, from each of the six methods, as adopted in 1929. Each point weighted according to the number of observations composing it. $h=6.5568$, $e=4.7753$.

no variation of h_n with n. It is therefore of interest to consider what values of h and e would have been obtained in 1929, had the data been handled by the new method. In making this calculation, all six values of h_n are first given the same weight, and are later weighted according to their probable errors. This is done deliberately, to show the change in the resulting values of e, h etc., brought about by the change of weighting.

Proceeding to the first calculation, we obtain the final value of $h_{3/3}$ from the arithmetic average of the three observations (methods) for which $n=3/3$. This point is given a weight of three. Similarly the final value of $h_{4/3}$ is given a weight of two, and $h_{5/3}$ a weight of unity. It is impossible here to base the relative weights of these three points on the consistency of the observations, since in the case of $h_{5/3}$ there is only one observation. The three points and their weights are listed in Table III, cols. 2 and 3. The solution of Eq. (10) is now given by

$$h_n = 6.5568 \pm 0.0064 - (0.0073 \pm 0.0051)n. \quad (25)$$

This is plotted as curve d, Fig. 4, and the resulting constants are listed in Table II. In Fig. 4 the six observations are indicated by circles, and the three resulting points by crosses. The divergence of this solution from the preceding is due chiefly to the fact that the value of $h_{5/3}$ is now based solely on the "spectroscopic" value of e/m (=1.761). As is evident from Fig. 4, this raises the value of $1/\alpha$ considerably above the 137 figure.

TABLE III.

1 n	2 h_n(obs) curve d	3 p	4 h_n(obs) curve e	5 p	6 h_n (obs) curve f	7 p
3/3	6.5503	3	6.5472 ± 0.0058	0.30	6.5605 ± 0.0046	0.47
4/3	6.5445	2	6.5409 ± 0.0037	0.73	6.539 ± 0.0040	0.625
5/3	6.5471	1	6.5471 ± 0.0012	6.94	6.5471 ± 0.0012	6.94

(2) Curve *e*. We now proceed to consider the probable error in each of the six observations, in order to obtain a more reliable basis for weighting. The error we desire is that in h_n. As shown by Eq. (9) this is merely the error in h due to all sources except the electronic charge. It is *not* the error listed in the table, page 57 of *G. C.* 1929, since there the error in e is included. From the discussion on pp. 48–57 of G.C. 1929, the desired errors are as follows.

(a) In the case of the Rydberg constant, the errors in c and R_∞ are negligible. The proportional error in $h_{5/3}$ is one-third the proportional error in e/m. Assuming $e/m = 1.761 \pm 0.001$, we obtain the value of $h_{5/3}$ listed in Table III, col. 4. The corresponding weight p is listed in col. 5.

(b) At $n = 4/3$ there are two methods. The first involves the Stefan-Boltzmann constant σ, for which 5.735 ± 0.011 was adopted in 1929. The only significant proportional error in $h_{4/3}$ is one-third the proportional error in σ. Hence[22] from σ, we obtain $h_{4/3} = 6.539 \pm 0.00405$ ($p = 0.610$). The second method involves measurements on the continuous spectrum of x-rays. The value adopted in 1929 is 6.550 ± 0.009 ($p = 0.124$). This result was obtained by giving an arbitrary weight of two to the value 6.559, as calculated by the writer from the experimental results of Duane, Palmer and Yeh, and a weight of unity to the similarly calculated value 6.532, based on Wagner's work. The error ± 0.009 just quoted is, as noted on page 53 of G.C. 1929, the regular least squares' probable error, and is therefore the error desired here.[23] The weighted average result of these two methods is 6.5409. The ratio r_e/r_i is 0.75 and an error based on internal consistency (r_i) is accordingly used. The final result for $h_{4/3}$ is given in Table III, col. 4. The weight of this result, since r_i is used, is of course merely the sum of the weights of the two methods.

(c) There are three available methods for obtaining the point at $n = 3/3$. The first is that of ionization potentials, and the only significant error is that

[22] It was found later that the correct error is ± 0.00419. This corrected value has been used in curve j ahead, but it did not seem necessary to make a recalculation here. See also footnotes 37 and 43.

[23] In the 1929 work I accidentally neglected to compound this error with that due to e. The value that should have been used in 1929 is ± 0.011.

in the measured voltage. The resulting value of h is 6.560 ± 0.0131 ($p = 0.0583$). In the case of the photoelectric effect the various assumed errors are discussed on pp. 53–54 of G.C. 1929. The result is 6.543 ± 0.0073 ($p = 0.1876$). In the third method the only error that concerns us is that due to c_2 itself. With $c_2 = 1.432 \pm 0.003$, one obtains $h = 6.548 \pm 0.0137$ ($p = 0.0533$). The weighted average of these three values of $h_{3/3}$ is 6.5472 and the ratio $r_e/r_i = 0.54$. Here again the results of the various methods, for a given value of n, are more consistent than is to be expected from theory, although the deviation is not unreasonable. Therefore we again base the final probable error on r_i rather than on r_e. The resulting value of $h_{3/3}$ is given in Table III, col. 4, and as before the weight is merely the sum of the three weights just given.

Using the data listed in Table III, cols. 4 and 5, we obtain

$$h_n = 6.5360 \pm 0.0117 + (0.00653 \pm 0.00725)n. \tag{26}$$

In this case $r_e/r_i = 0.89$ and the quoted errors are based on r_i. This equation is plotted as curve e in Fig. 5. The adopted probable errors of each of the

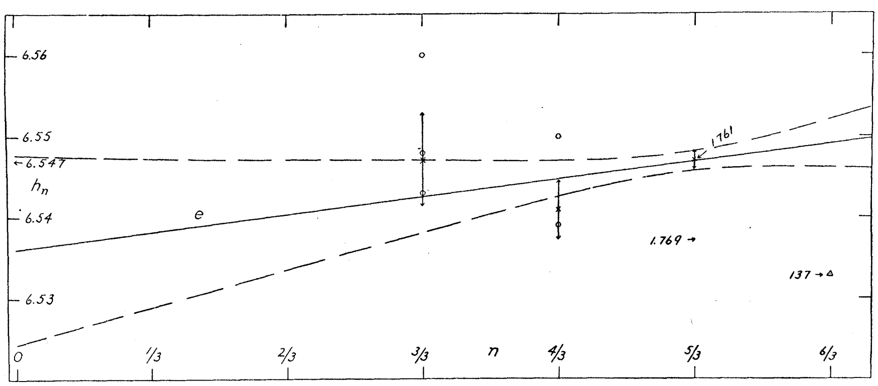

Fig. 5. Solution of Eq. (26). Same data as in Fig. 4, but probable error of each of the six observations computed, and all weights based on probable errors. $h = 6.5360$, $e = 4.7652$.

three points are indicated, but not those of the six observations (circles). It is very gratifying to find that r_e/r_i is less than unity, in the case of the observations and also in the case of the points. The results thus show no indication of constant or systematic errors in any one of the six adopted values of h_n.

The resulting values of the various constants are given in Table II. Since they do not include any direct measurement of e, they cannot be called the results that should have been obtained in 1929. (Such results are given in curve i ahead.) The only fact that seems worthy of mention is the rather large difference in the values of the constants and also of their probable errors, in curves d and e. This difference is due solely to the change from arbitrary equal weighting for all six methods to a more logical system of weighting.

Solutions Based on Present Data, Using Eq. (1)

We proceed now to a re-examination of all available data, beginning with the methods for which $n = 3/3$.

(a) Ionization potentials. There seems to be no new precision measurements since 1929, comparable in accuracy with that by Lawrence,[24] and I therefore again adopt his determination of the ionization potential of mercury, leading as before to

$$h_{3/3}(\text{ion. pot.}) = 6.560 \pm 0.0131 \ (p = 0.0583). \tag{27}$$

(b) Photoelectric effect. In the 1929 work, the only investigation considered was that by Lukirsky and Prilezaev. Their value of $h_{3/3}$, as already quoted, is 6.543 ± 0.0073, exclusive of the error in e. As shown in the discussion on pp. 53–54 of G.C. 1929, it was difficult, due to lack of information, to assign an error in this case, and it seems best now to raise the error slightly to ± 0.010. (This was the 1929 error, *including* the error in e.) Since 1929 there has appeared a new investigation by Olpin.[25] His published result, based on a graphical solution of the voltage: frequency curve, is 6.541. Dr. Olpin has kindly sent me his original data for this curve, and I find for the least squares' solution, $h_{3/3} = 6.561 \pm 0.029$, and this result will be used in the following.

The original precision work on h, from the photoelectric effect, is that by Millikan.[26] A considerable amount of experimental data is given in his article, with several different results. It has seemed desirable to recalculate these results, using least squares' methods for the calculations and the weighting. There is no indication of systematic error in the several results, and my final weighted average of all the data is 6.560 ± 0.037. The actual calculations are rather extensive and it is unnecessary to present them at this time.

For the final value of $h_{3/3}$, as determined from the photoelectric effect, I use the weighted average of these three investigations. This result is

$$h_{3/3}(\text{photoelectric}) = 6.546 \pm 0.0092 \ (p = 0.1181). \tag{28}$$

The ratio r_e/r_i equals 0.337, thus showing a considerably greater consistency of the three individual results than is to be expected on the average. The quoted error is based on r_i.

(c) The radiation constant c_2. There is no new work on this important constant, and I accordingly retain the 1929 value 1.432 ± 0.006, which yields

$$h_{3/3}(c_2) = 6.548 \pm 0.0137 \ (p = 0.0533) \tag{29}$$

as previously quoted.

(d) X-rays. In 1929, the value of h resulting from observations on the continuous spectrum of x-rays was listed under $n = 4/3$. The defining equation in this case is[27]

$$h_n = \frac{e_0 p q V' \lambda 10^8}{c^2}. \tag{30}$$

[24] E. O. Lawrence, Phys. Rev. **28**, 947 (1926).
[25] A. R. Olpin, Phys. Rev. **36**, 251 (1930), see p. 284.
[26] R. A. Millikan, Phys. Rev. **7**, 355 (1916).
[27] Compare Eqs. (11) and (12), p. 51 of G.C. 1929.

Now, as noted in the 1929 work,[28] there are two methods for determining λ. One is by means of the Bragg law

$$\lambda = 2d \sin \theta \qquad (31)$$

in which the calculated grating space d involves e_0 to the one-third power, so that h_n varies with e_0 raised to the 4/3 power. This is the method[29] used in 1929, and the x-ray result was accordingly listed under $n = 4/3$.

The second method for determining the λ of Eq. (30) is by means of ruled gratings. This is a direct determination, quite independent of any assumed value of e, so that h_n now varies as $e^{3/3}$ and should be listed under $n = 3/3$. The quantity actually observed in the x-ray experiments now under discussion was $\sin \theta$ of Eq. (31), so that it was possible to calculate only a Bragg law wave-length. We may, however, assume that there exists for all wave-lengths the proportional discrepancy found by various investigators for certain lines, as measured with a ruled grating and with a calcite crystal. The two most accurate investigations are those by Bearden[30] and by Cork.[31] The former found the grating λ of two lines to be respectively 0.23 percent and 0.24 percent higher than the crystal values. The latter found, for different lines, discrepancies of 0.288 and 0.305 percent. If we now assume, in agreement with prevailing opinion, that wave-lengths as measured by ruled gratings are actually correct, then all values of h_n, as calculated from x-ray continuous spectra are to be raised by 0.265 percent,—the unweighted average of the above four results. In terms of 10^{-27} erg. sec, the correction is $+0.01740$.

In addition to the x-ray work actually used in 1929, we now have available the result of Feder.[32] This investigation is mentioned in footnote 16[a], page 53 of G. C. 1929, but appeared too late to use there. Since the work is a repetition of that by Wagner, it seems reasonable to use it as a substitute for Wagner's result. Feder's value of h, using my 1929 values of the auxiliary constants, is 6.5463. His work appears to be of accuracy comparable to that of Duane, Palmer and Yeh, and I accordingly give these two investigations equal weight and adopt the arithmetic average $h = 6.55265$. Since both results are based on crystal wave-lengths, we must now add 0.01740, thus getting

$$h_{3/3}(x\text{-rays}) = 6.5700 \pm 0.0063 \; (p = 0.252). \qquad (32)$$

The probable error is calculated directly from the agreement of the two results (i.e., on external consistency), just as was done in 1929.

[28] See pp. 52 and 39–43 of G.C. 1929.

[29] See Eq. (13) p. 52 of G.C. 1929. The discussion following Eq. (16) page 52 is not very logical, again due to a failure to realize that what was being calculated was $h_{4/3}$ and not h itself. Since 4.770 was used in both Eqs. (13) and (15) for what should have been called e_0, the resulting value of h necessarily varied as $e_0^{4/3}$ and should have been denoted $h_{4/3}$.

[30] J. A. Bearden, Proc. Nat. Acad. Sci. **15**, 528 (1929).

[31] J. M. Cork, Phys. Rev. **35**, 1456 (1930).

[32] H. Feder, Ann. d. Physik (5) **1**, 497 (1929).

This completes the discussion of the four methods available for determining $h_{3/3}$. The weighted average of the results listed in Eqs. (27) (28) (29) and (32) is

$$h_{3/3} = 6.5605 \pm 0.0046 \ (p = 0.47). \tag{33}$$

The ratio $r_e/r_i = 0.92$, so that the four results are self-consistent, in spite of the suspiciously high x-ray value. We shall, however, find immediately that this shift of the x-ray result from $n=4/3$ to $n=3/3$ brings about a definite discrepancy in the final collected results. Later in this paper we shall return again to a consideration of the proper value of λ to use in Eq. (30) and we shall see that it is fairly probable that the x-ray method should be left at $n=4/3$.

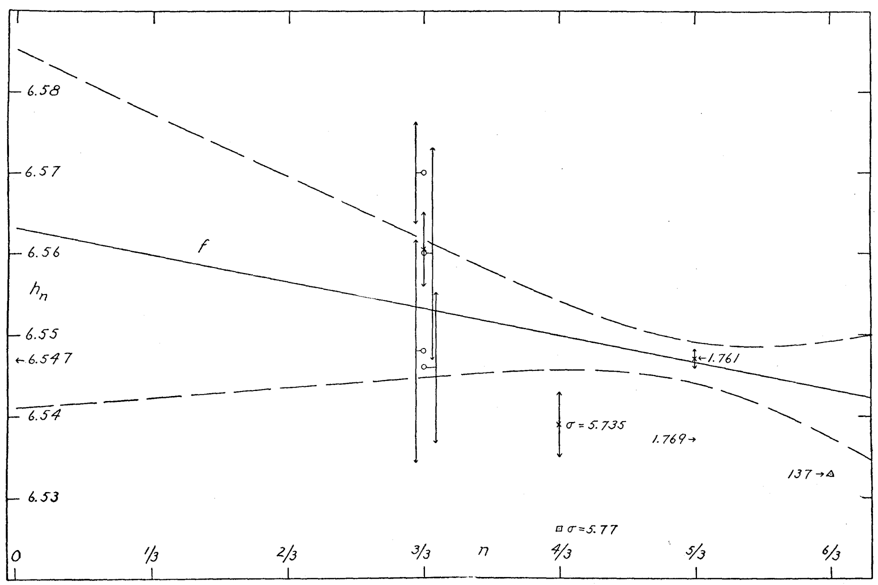

Fig. 6. Solution of Eq. (36). Six observations resulting from a re-examination of all available data. Probable errors of observations and of resulting points shown by arrows. X-ray observation at $n=3/3$, in place of former $n=4/3$. $h=6.563$, $e=4.777$, but data inconsistent.

Assuming for the present that there are four methods for obtaining h that belong under $n=3/3$, there remains only one method at $n=4/3$, and one at $n=5/3$. The one method at $n=4/3$ is based on the total radiation constant σ. There is still considerable uncertainty regarding the best value of this important constant. The reader is referred to pp. 55–57 of G. C. 1929 for a discussion of the situation at that time. As noted on page 57, Hoare's value of 5.735, which was published too late to use in 1929, but which fortunately agreed exactly with the assumed value, appears to be more reliable than any previous determination. Since then Mendenhall[33] has obtained 5.79 (error not stated) and C. Müller, in a general article[34] on this constant,

[33] C. E. Mendenhall, Phys. Rev. **34**, 502 (1929).

[34] Wien-Harms Handbuch d. Exp. Physik **9**, 427–455 (1929).

lists a recent value of his own, 5.77 ± 0.03, of which I have seen no further reference. He adopts 5.77 as the most probable value, in agreement with Ladenburg's earlier estimate.[35] A value of 5.77 corresponds to $h_{4/3} = 6.5256$, and this point is plotted on Fig. 6. It obviously is inconsistent with the results from all other methods. Even $\sigma = 5.735$ seems too high, and the indirect value[36] of σ, as calculated in 1929, is 5.7139×10^{-5}. As discussed then, the chief experimental uncertainty seems to lie in the corrections to be applied for incomplete absorption of the receiver. Hoare claims that his method eliminates such corrections, and it appears to the writer, as it did in 1929, that 5.735 is still the most reliable experimental value.[36a] The probable error is, under the circumstances, very uncertain, and merely for convenience I shall retain the 1929 value. Hence[37]

$$h_{4/3} = 6.539 \pm 0.0040 \ (p = 0.625). \tag{34}$$

In the case of $n = 5/3$, the only known method for obtaining h is from the Rydberg constant formula. The error in $h_{5/3}$ is due almost solely to the error in e/m, and as shown in curve e, based on the 1929 data, this point is, apparently, so accurately known that the final values of e and h depend primarily upon the adopted value $e/m = 1.761 \pm 0.001$. The three new determinations of e/m, since 1929, have been listed in connection with curve c. They are all 1.761 or lower. Two of them are obtained from the acceleration of free electrons in an electric field, and constitute the first "low" values of e/m obtained by a non-spectroscopic method. It is still extremely important to obtain a really reliable value of e/m from magnetic deflection. In the meantime the presumption of evidence is that there is only one value[38] of e/m, and that this is the so-called "spectroscopic" value. It is quite possible that this value is 1.760, or even lower, but it has seemed better to retain, for the present, the 1929 value, 1.761 ± 0.001. Hence

$$h_{5/3} = 6.5471 \pm 0.0012 \ (p = 6.94) \tag{35}$$

as used in curve e.

[35] Geiger and Scheel, Handbuch d. Physik **23**, 305 (1926). See p. 56 of G.C. 1929.

[36] See Table b, p. 61, G.C. 1929. With the values of the principal constants, including e, adopted in 1929, one has $\sigma = 16.03418 \times 10^{-84}/h_{4/3}^3$ and $h_{4/3} = 2.521635 \times 10^{-28}/\sigma^{1/3}$.

[36a] In a very recent investigation, Phil. Mag. **13**, 380 (1932) Hoare gets 5.737, from 50 extremely consistent determinations.

[37] The writer must apologize for several small but annoying discrepancies in this article. Many solutions have been obtained that, for lack of any significant features, are not published, and the total calculations are quite extensive. As a result, there have been several instances where brief calculations were unintentionally repeated, with slightly different results due to the employment of a different number of significant figures. Thus, in a previous section, Eq. (34) was quoted as ± 0.00405 ($p = 0.610$), and as stated in footnote 22, it happens that both these results are slightly in error.

[38] Several writers have implicitly criticized the assumption of two different values of e/m, as made in G.C. 1929. I think every one admits that there can be but one *correct* value, and the sole reason for adopting the two values, in 1929, was to call attention, as emphatically as possible, to the obvious discrepancy in the experimental results. The extensive discussion and investigations of e/m that have since appeared in the literature constitute sufficient evidence that this object has been attained.

(1) Curve *f*. This completes the discussion of present available data. The three resulting points are given in Eqs. (33) (34) and (35), and are listed in cols. 6 and 7 of Table III. The solution of these data is

$$h_n = 6.563 \pm 0.022 - (0.010 \pm 0.014)n. \tag{36}$$

This is plotted as curve f, Fig. 6, and the resulting constants are given in Table II. The value of the ratio r_e/r_i is 2.14, a deviation from unity of 4.14 times the probable error. There is only one chance in 190 of such an excess, and we thus have here clear evidence of an incompatibility of the three h_n points. A glance at curve f shows that this unwelcome result is due jointly to the abnormally low value of $h_{4/3}$, and the abnormally high value of that one of the four observations composing $h_{3/3}$ deduced from the x-ray method. It is difficult to explain the low value of $h_{4/3}$, and it has just been noted that most reviewers favor $\sigma = 5.77$, which leads to the even lower value $h_{4/3} = 6.5256$. On the other hand, the high value of $h_{3/3}$ from the x-ray method is due mainly to the correction computed from the assumption that the true x-ray wave-lengths are those given by ruled gratings. The writer believes that the situation shown by curve f constitutes definite independent evidence against such an assumption.

(2) Curve *g*. The next logical step, under the circumstances, seems to be to ignore entirely both the σ and the x-ray result. This leaves three observations at $n = 3/3$, given by Eqs. (27) (28) and (29). The new weighted average is

$$h_{3/3} = 6.5500 \pm 0.0066 \ (p = 0.23). \tag{37}$$

The ratio $r_e/r_i = 0.425$, as contrasted with 0.92 when four methods were included.

The one remaining point is $h_{5/3}$, as given by Eq. (35). With only two points and two undetermined constants we now abandon least squares' methods and make a direct calculation of the intercept and slope of the $h_n{:}n$ curve. These constants, and their errors, are given by Eqs. (27) to (30) of L. S. 1932, and the error in the function itself by Eq. (32). In the present case the $y_1 \pm r_1$ of L. S. 1932 may be denoted $h_3 \pm r_3$, where h_3 stands for $h_{3/3}$, and $y_2 \pm r_2$ may be denoted $h_5 \pm r_5$, where h_5 stands for $h_{5/3}$. Also $x_1 = 3/3$ and $x_2 = 5/3$. Hence Eqs. (27) to (32), *L. S.* 1929, become respectively

$$a = 2.5h_3 - 1.5h_5 \tag{38}$$

$$b = 1.5(h_5 - h_3) \tag{39}$$

$$r_a = [(2.5r_3)^2 + (1.5r_5)^2]^{1/2} \tag{40}$$

$$r_b = [(1.5r_5)^2 + (1.5r_3)^2]^{1/2} \tag{41}$$

$$y = h_n = h_3(2.5 - 1.5n) + h_5(1.5n - 1.5) \tag{42}$$

$$r_n = ([(2.5 - 1.5n)r_3]^2 + [(1.5n - 1.5)r_5]^2)^{1/2}. \tag{43}$$

With $h_3 \pm r_3$ given by Eq. (37) and $h_5 \pm r_5$ by Eq. (35) one gets

$$y = h_n = 6.5543 \pm 0.0166 - (0.00435 \pm 0.0100)n. \tag{44}$$

This equation is plotted as curve g in Fig. 7, and the errors in the function, calculated by Eq. (43), are shown as usual by broken lines. This error of the function must of course equal the error of any observed point, at the corresponding value of n. Hence in Fig. 7 the broken lines touch the arrows indicating the assumed errors r_3 and r_5. The magnitude of $1/\alpha$ and its error are calculated in the usual way. In the case of e/m, however, we have only the value originally assumed, i.e., 1.761 ± 0.001. The various constants and their errors are listed in Table II.

If the grating values of x-ray lines are correct, with the result that the value of h_n belongs at $n=3/3$ and is with justification omitted, then curve g represents the best independent determination of e. This curve gives also the clearest illustration of the dependence of e and h upon the value adopted

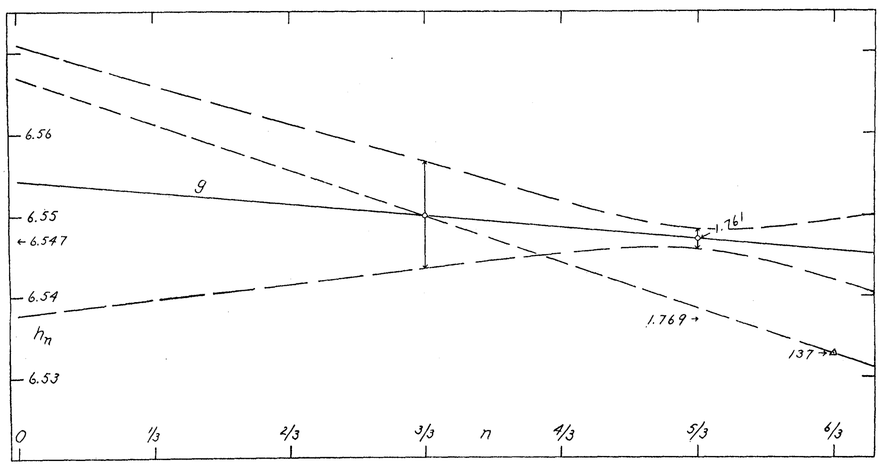

Fig. 7. Solution of Eq. (44). Same data as Fig. 6, but with x-ray and σ observations discarded. $h=6.5543$, $e=4.7732$. Broken straight line represents necessary solution ($h=6.5670$, $e=4.7824$) if $1/\alpha=137$.

for e/m. Taking $h_{3/3}=6.5500$ as one fixed point, we may for instance assume $h_{3/3}=6.53295$ (corresponding to $1/\alpha=137$) for the other. The resulting curve is indicated by broken lines in Fig. 7. Its equation is

$$h_n = 6.56705 - 0.01705n \tag{45}$$

and the resulting constants are $e/m=1.7679$, $e=4.7824$, $h=6.5670$.

It thus seems highly probable that if Eddington's theory concerning $1/\alpha$ is true, the correct value of e/m must be approximately 1.768, and as already discussed, there is now very little experimental evidence in favor of such a value. If, however, future work should establish the correctness of Eddington's theory, then it follows almost inevitably that the correct value of e is about 0.26 percent higher than our previously adopted $e_0=4.770$, and that h is about 0.30 percent higher than $h_0=6.547$.

Returning once more to the assumptions made in deriving curve g, we note that the probable errors in the resulting constants are greater than in the case of most of the previous solutions. This is due primarily to the fact that the curve is based on two points only, instead of on three. The value of e (4.7732 ± 0.0072) agrees, however, with the oil-drop value (4.768 ± 0.005) well within the probable errors, and we have therefore achieved our main objective, i.e., we have obtained a new and entirely independent value of e and have shown that it is compatible with the previously accepted value. Hence we may now proceed to the calculation of "most probable" values of e and h, by the use of Eq. (2) which allows the inclusion of a directly measured value of e.

Solutions Based on Eq. (2)

Eq. (2) is of the same form as Eq. (1), with h and e merely interchanged. Hence the least squares' treatment of Eq. (2) leads to a linear equation like Eq. (10), with e and h interchanged, i.e.,

$$e_m = e - \left(e_0 \frac{\Delta h}{h_0}\right) m \tag{46}$$

where e_m, in analogy with Eq. (9), is a new parameter defined by

$$e_m \equiv a_m h_0{}^m . \tag{47}$$

Also, as stated in Eq. (3), $m = 1/n$, and $a_m = (A_n)^{-1/n}$.

Ordinarily one would evaluate e_m by means of the value of a_m that can be calculated from the experimental data. But in this case we know all the needed values of h_n and from them it is possible to get, by a simple calculation, the corresponding values of e_m. Thus, from Eqs. (9) (47) and (3) we obtain

$$A_n = \frac{h_n}{e_0{}^n} = \frac{h_0}{e_m{}^n} \cdot \tag{48}$$

Hence

$$\frac{e_m}{e_0} = \left(\frac{h_n}{h_0}\right)^{-1/n} . \tag{49}$$

Then, writing $e_m = e_0 + \delta e$, and $h_n = h_0 + \delta h$, where δe and δh are small quantities, Eq. (49) becomes

$$\frac{\delta e}{e_0} = -\frac{1}{n}\frac{\delta h}{h_0} \cdot \tag{50}$$

Putting in the numerical values of e_0 and h_0 (4.770 and 6.547), we get

$$\delta e = -\frac{0.72857\,\delta h}{n} \tag{51}$$

where

$$\delta h = h_n - h_0 \text{ and } e_m = e_0 + \delta e . \tag{52}$$

This gives the value of e_m corresponding to any assumed value of h_n. Also, in Eq. (51), if δh represents the probable error in h_n, then δe represents the corresponding probable error in e_m.

As in the case of the h_n:n curve, it is convenient to obtain explicit formulas for the calculation of $1/\alpha$ and e/m from the e_m:m curve. By analogy with Eqs. (14) (15) and (19) we write

$$1/\alpha = \left(\frac{ch_0}{2\pi}\right)\frac{1}{e_{1/2}{}^2} = \frac{3.12385 \times 10^{-17}}{e_{1/2}{}^2} \tag{53}$$

and the proportional error in $1/\alpha$ is therefore twice the proportional error in $e_{1/2}$, which in turn is the error in the function e_m at $m=1/2$. For $1/\alpha=137$, we have $e_{1/2}=4.77512\times10^{-10}$, and this point is shown on the plotted e_m curves.

Similarly, e/m may be calculated from the value of e_m at $m=3/5$. Corresponding to Eq. (20) we write

$$e/m = \frac{2\pi^2 e_{3/5}{}^5}{R_\infty c^2 h_0^3} = (7.13175 \times 10^{53})e_{3/5}{}^5 \tag{54}$$

and the proportional error in e/m is five times that in $e_{3/5}$. If $e/m=1.761\times10^7$, $e_{3/5}=4.76996\times10^{-10}$, and if $e/m=1.769\times10^7$, $e_{3/5}=4.77427\times10^{-10}$. These two points also are shown on the plotted e_m curves.

(1) Curve *h*. For the first solution of Eq. (46) I use the data of curve g, *plus* the best directly measured value of e (to be called e_d). This latter is the oil-drop value[39] 4.768 ± 0.005, and not the 4.770 value adopted in 1929, for the following reason. This latter value is a weighted average of the oil-drop result, 4.768, and a value calculated from the grating measurements of x-ray wave-lengths. The more recent work of Bearden and of Cork, as already noted, indicates a discrepancy of 0.265 percent in the crystal and grating wave-lengths. The discrepancy in the resulting value of e is three times as great, or 0.8 percent. It is generally agreed at the present time, and the calculations in this article support the opinion, that the observed discrepancy in the x-ray wave-lengths cannot be due to an error of 0.8 percent in the adopted value of e (4.770), but must be due to some other cause. In that situation it is not possible to calculate a value of e from grating wave-lengths, and the only directly determined value that is available is the oil-drop result 4.768.

The needed data are, accordingly,

$$e_{3/3} = 4.7678 \pm 0.0048 \ (p = 0.43), \text{ from } h_{3/3} = 6.5500 \pm 0.0066,$$
$$e_{3/5} = 4.76996 \pm 0.00052 \ (p = 37.0), \text{ from } h_{5/3} = 6.5471 \pm 0.0012,$$
$$e_d = 4.768 \pm 0.005 \ (p = 0.40).$$

This last point is of course to be plotted at $m=0$, since it is independent of the value of h_0. The least squares' solution of these three points is

$$e_m = 4.7696 \pm 0.0041 + (0.00045 \pm 0.00686)m. \tag{55}$$

[39] See G.C. 1929, pp 36–40.

The ratio $r_e/r_i = 0.389$, indicating remarkable consistency of the data. The errors in Eq. (55) are accordingly based on r_i, and it is worthy of notice that if they were based on r_e they would be only 39 percent as large. Eq. (55) is plotted as curve h of Fig. 8, and the various resulting constants are listed in Table II. This figure shows also the curve necessary if $1/\alpha = 137$ and $e_{3/3}$ has the value given above. The intercept at $m=0$ is 4.7824, as already given in connection with Eq. (45).

The values of e and h resulting from curve h are for all practical purposes identical with those adopted in 1929. In order to facilitate comparison, I give in Table II, on the next line below the curve h values, the 1929 constants. With the probable exception of curve k, to be discussed later, this curve h solution represents, I believe, the most probable values to be obtained from

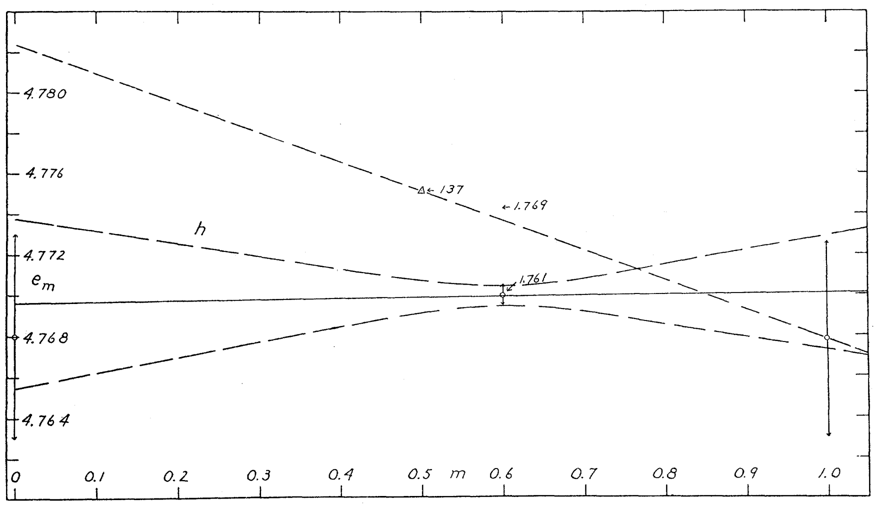

Fig. 8. Solution of Eq. (55). Data of Fig. 7, transformed to e_m values, plus the directly determined $e_d = 4.768$, at $m=0$. $h = 6.5464$, $e = 4.7696$. (The letter h should have been placed on the continuous line).

the data at the present time. The chief difference between the 1929 and the 1932 values lies in the stated probable errors. As has already been noted, the 1929 errors were calculated in a very crude way, and as shown in more detail in the concluding section, some of them are too large, and some are too small.

(2) Curve *i*. It is now of interest to calculate the values of the constants that should have been obtained in 1929, had they been correctly calculated. For this purpose we use merely the data of curve e (Table III, col. 4), transformed to e_m values, plus $e_d = 4.770 \pm 0.005$ ($p = 0.40$), since that is the 1929 adopted value from all direct determinations of e. The $e_{3/5}$ point is the same as for curve h. The other points are

$$e_{3/4} = 4.7759 \pm 0.0036 \; (p = 0.77)$$

$$e_{3/3} = 4.7699 \pm 0.0034 \; (p = 0.87).$$

From these four points we obtain

$$e_m = 4.7688 \pm 0.0035 + (0.0021 \pm 0.0058)m. \qquad (56)$$

This solution is plotted as curve i, Fig. 9.

The ratio $r_e/r_i = 0.76$ and the data are accordingly, as a whole, quite consistent, although the $e_{3/4}$ point is obviously too high, just as the $h_{4/3}$ point in curve e is low. The various constants calculated from Eq. (56) are listed in Table II, where they may be compared with the published 1929 values placed directly above them. It is of interest to note that the best present values, as given by curve h, agree rather better with the published 1929 values than with the values that should have been obtained in 1929. However, all three sets of values agree with each other far within the limits of error, and it is quite immaterial which set is used.

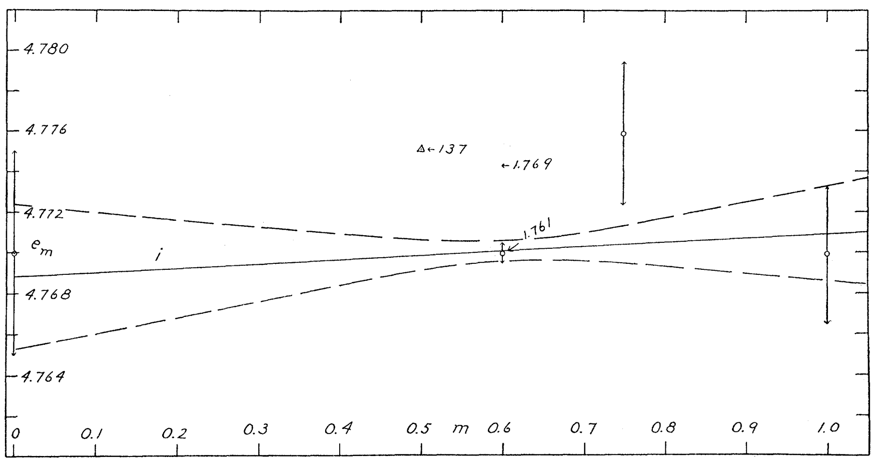

Fig. 9. Solution of Eq. (56). Data of Fig. 5, transformed to e_m, plus $e_d = 4.770$. The resulting constants ($h = 6.5441 \pm 0.0079$, $e = 4.7688 \pm 0.0035$) are the values that should have been obtained in 1929, had the calculations been made correctly.

New Solutions with X-ray Method at $n = 4/3$

The calculations outlined above were presented to the American Physical Society, at the Berkeley meeting, Dec. 1931.[40] Since then two more solutions have been made, based on the very recent work of Bearden.[41] It has been noted, at the end of the discussion of curve f, that that curve furnished independent evidence that the true x-ray wave-lengths are *not* those given by ruled gratings. Bearden has independently been led to this same conclusion from an entirely different source. Using the measured index of refraction of quartz, for various wave-lengths, he has been able to *calculate* values of the respective wave-lengths. In order to do this it is necessary to assume values of various general constants, including e/m. For this last constant Bearden

[40] R. T. Birge, Phys. Rev. 39, 547 (1932) abstract 6.

[41] J. A. Bearden, Phys. Rev. 39, 1 (1932), and further details by private communication.

adopted[42] 1.761. The resulting wave-lengths show remarkable agreement with those obtained by crystals, and are in definite disagreement with those obtained by ruled gratings. To get agreement with the latter, it would be necessary to adopt a value of e/m some 0.5 percent *lower* than 1.761.

I do not feel competent to make a critical estimate of the possible errors in the theoretical equation used by Bearden. If it is a sufficiently correct equation, and if the various constants appearing in it have been correctly evaluated, we are led to the surprising result that it is the grating wave-lengths that are wrong, and not the crystal wave-lengths, contrary to all previous opinion.

(1) Curve *j*. With this new condition of affairs, it is of interest to leave the x-ray value of h_n at $n=4/3$, and to adopt the crystal wave-lengths, just as was done in 1929. I have accordingly recalculated curve *f*, with this change. For $n=3/3$ we now have left three methods, with a weighted average $h_{3/3}=6.5500\pm0.0066$, as adopted for curve *g*. For $n=4/3$ there are two methods. The x-ray value of h_n is now 6.5526 ± 0.0063. This is the quantity to which 0.01740 was added, when transferring the x-ray observation from $n=4/3$ to $n=3/3$, in curve *f*. This x-ray observation is now to be averaged with $h_n=6.539\pm0.0042$[43] as obtained from σ. The weighted average is $h_{4/3}=6.5431\pm0.0042$ $(p=0.57)$, and $r_e/r_i=1.21$. The expected deviation from unity is 34 percent, and so we have here no definite indication of inconsistency in the two methods. The $h_{5/3}$ point is as usual 6.5471 ± 0.0012 $(p=6.94)$, based on $e/m=1.761\pm0.001$.

The solution of these three points is

$$h_n = 6.5443 \pm 0.0133 + (0.00162 \pm 0.0081)n \tag{57}$$

with $r_e/r_i=0.685$ and with the errors consequently based on r_i. This is certainly a far different situation from that found in curve *f*, where $r_e/r_i=2.14$, but the improvement is partly an illusion. It merely happens that the x-ray value of h_n, which even without the 0.017 correction is rather high, is here averaged with the low value of h_n derived from σ, to give a point $h_{4/3}$ that agrees well with the other two points. In curve *f* the low $h_{4/3}$ point came from σ alone, and the very high x-ray value helped to raise the resulting $h_{3/3}$ point and to produce the inconsistency of all points. It is certainly true, however, that the retention of the x-ray value of h at $n=4/3$ improves the consistency of the data, and there is now no sufficient reason for rejecting any one of the six methods.

Eq. (57) is plotted as curve *j*, Fig. 10, and the derived constants are listed in Table II. They are almost identical with the constants given by curve *i*. The value of e is 4.7688 ± 0.0059, and with the assumption that the crystal wave-lengths of x-rays are the correct values, this becomes the best inde-

[42] In previous work on the index of refraction of x-rays, by Stauss, Bearden and others, the wave-lengths as given by gratings were assumed to be correct, and a value of e/m was calculated.

[43] Here we use the correct probable error, of which mention has been made in footnotes 22 and 37.

pendent evaluation of e by the new method. It is really remarkable that it should agree so well with the oil-drop value, and this again is confirmation of Millikan's work.

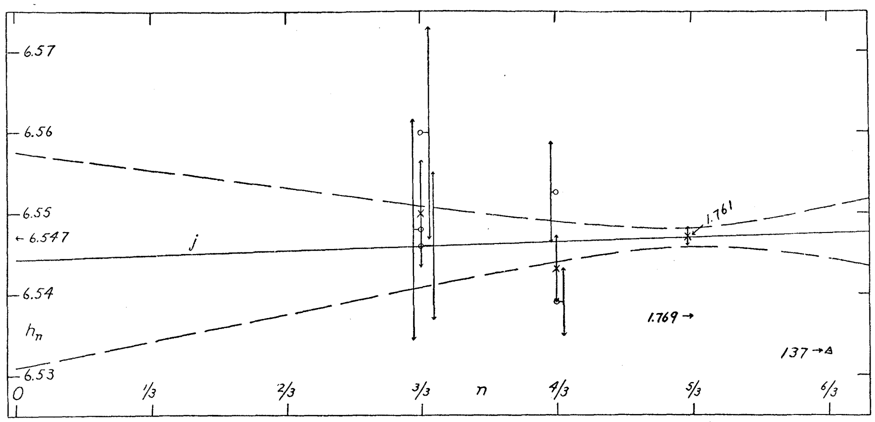

Fig. 10. Solution of Eq. (57). Data of Fig. 6, but with x-ray observation transferred back to $n=4/3$. $h=6.5443\pm0.0133$, $e=4.7688\pm0.0059$. This is the best value of e, as calculated by the new method.

(2) Curve k. The last solution to be presented is based on the data of curve j, converted to e_m values, and with the addition of $e_d=4.768\pm0.005$. The needed data are just those of curve h, with the added point $e_{3/4}=4.7738$

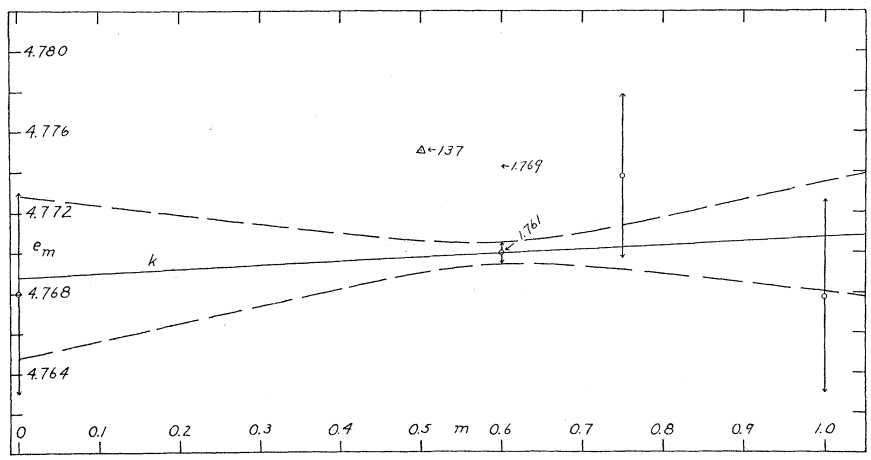

Fig. 11. Solution of Eq. (58). Data of Fig. 10, converted to e_m values, and $e_d=4.768$ added. The resulting constants ($h=6.5443\pm0.0091$, $e=4.7688\pm0.0040$) are considered the most reliable values, on the basis of present data.

±0.0041 ($p=0.60$), from $h_{4/3}=6.5431\pm0.0042$. The solution of the four points is

$$e_m = 4.7688 \pm 0.0040 + (0.00196 \pm 0.0066)m \qquad (58)$$

for which $r_e/r_i=0.51$. The equation is plotted as curve k, Fig. 11, and the resulting constants are given in Table II. They are nearly identical with those obtained from curve i. These numerical coincidences are devoid of scientific significance, but they are nevertheless very interesting. Thus if one assumes that the x-ray method belongs at $n=3/3$, and must be ignored, together with the σ method, because of inconsistency, then the best resulting set of constants is given by curve h, and this set is almost identical with the constants published in 1929. On the other hand, if one assumes that the x-ray method belongs at $n=4/3$, the resulting set of constants is given by curve k, and this set is almost identical with the constants that would have been obtained in 1929, if correctly calculated (curve i).

General Discussion and Conclusions

Table II shows, in summary form, the various sets of constants resulting from the data and curves plotted in Figs. 1 to 11. The most interesting of these constants is probably $1/\alpha$. The source of interest lies in the fact that $1/\alpha$ is one of the very few dimensionless ratios formed from general physical constants. Its numerical value should therefore have theoretical significance, and the theories that have already been proposed in this connection are tabulated in *G. C.* 1929, pp. 71–72, and by Bond,[2] pp. 995–996. Of these theories the one most deserving of serious consideration seems to be that put forward by Eddington,[10] which predicts $1/\alpha=137$.

The tabular material of Table II shows that if $1/\alpha=137$, then e/m equals 1.768 or more. The broken line drawn in Figs. 7 and 8 and given by Eq. (45) corresponds to $e/m=1.7679$, as already noted. The only solution of the experimental material yielding a value of $1/\alpha$ equal to 137 within the probable error is that given by curve a, Fig. 1. This solution is based in Bond's 36 observations, yielding 3 points, with each point weighted according to the number of observations composing it. The point at $n=5/3$ comes from 13 observed values of e/m, of which only two equal 1.761. The other values are all higher, running up to 1.773. In fact the first observed "low" value of e/m was that by Babcock[44] in 1923. Since that date no new experimental value of e/m, with the exception of the magnetic deflection work by Wolf,[45] has been higher than 1.761. The most important conclusion of Bond's work is that $1/\alpha$ is really 137, from purely experimental evidence. We see now that this conclusion was reached only by using every value of e/m listed in certain compilations, and by giving the same weight to each investigation. With numerous older "high" values of e/m Bond thus got a high average. At the present time one may say with some assurance that either there is a serious constant error of essentially the *same* magnitude in the recent determinations of e/m by three totally different methods (Zeeman effect, fine structure of hydrogen and helium spectral lines, and acceleration of free electrons by an electric field), or else $1/\alpha$ is *not* 137. The two most probable sets of data (curves h and k) give 137.305 ± 0.048 as the average value. These solutions

[44] H. D. Babcock, Astrophys. J. **58**, 149 (1923). $e/m=1.761$ from Zeeman effect. See page 44 of G.C. 1929.

[45] F. Wolf, Ann. d. Physik **83**, 849 (1927). See p. 43, *G.C.* 1929.

have been obtained on the assumption that the experimental value of e/m equals 1.761 ± 0.001, and the value calculated from the solutions is identical with this, as shown in Table II.

With such an assumption regarding e/m, the resulting values of e and h fall within certain rather narrow limits. These limits may be brought out more clearly by a brief review of the various solutions that have been obtained. Curve b results from Bond's 36 observations, but with the three points weighted according to their probable errors. Curve c is based on the same method of weighting but with the addition of three new determinations of e/m. The resulting values of e and h are slightly higher than those advocated in 1929, but this result is due to the relatively high averate value of e/m obtained from the inclusion of all the older work. Solution d is based on an arbitrary equal weighting of the six observations adopted in 1929, and has no real merit. A more logical weighting of these same six observations gives the quite different solution shown by curve e, and this solution represents the values that should have been obtained in 1929, as based on Eq. (1). Since, however, Eq. (1) does not include any directly determined value of e, the solution merely shows that the 1929 data give a value of e (4.765) by the new method that is quite consistent with the oil-drop value (4.768).

A re-examination of all available data then leads to curve f, which however could not be used because of definite inconsistency of the data ($r_e/r_i = 2.14$). Solution g, obtained by omitting the x-ray and σ methods, is properly self-consistent and the resulting value $e = 4.7732 \pm 0.0072$ represents the best independent determination of e by the new method, *provided* we assume that the x-ray observation belongs at $n = 3/3$. The use of Eq. (2) and the addition of the direct determination $e = 4.768$, then gives curve h, which is one of our two most probable solutions. The other of these most probable solutions is curve k, and this differs so little from curve h that it is really immaterial which is accepted. Curve k results from the assumption that the crystal wave-lengths of x-rays are correct, an assumption that places the x-ray method at $n = 4/3$. Two independent sources of evidence,—Bearden's recent work,[41] and the increased consistency of the data resulting from the transfer of the x-ray method from $n = 3/3$ to $n = 4/3$,—unite to indicate the correctness of the assumption. This was discussed at the opening of the section entitled "New Solutions with X-ray Method at $n = 4/3$." Curve j is the solution based on Eq. (1), corresponding to curve k, based on Eq. (2). Curve i is the solution, based on Eq. (2), of my 1929 data, and represents the values that should have been obtained at that time.

This brings us to the final conclusions resulting from our analysis of the data. The crystal x-ray wave-lengths are probably correct, and curve k represents the most reliable solution. The best value of e, obtained by the new method, is given by curve j, viz. $e = 4.7688 \pm 0.0059$. The best final value of e is 4.7688 ± 0.0040, from curve k. As was stated in the published abstract of this paper,[40] the possibility of the simultaneous evaluation of e and h, from the totality of experiments designed to measure h, is of the utmost importance. The fact that the new value of e thus obtained is almost identical with the oil-drop value is, in my opinion, very strong evidence that there is no unsuspected error of any significance in Millikan's work. On the other hand, the preceding analysis has shown that the most probable values of e, h and $1/\alpha$ depend primarily on the value adopted for e/m. The present evidence favors $e/m=1.761$, and with this one gets, almost inevitably, a value of e agreeing well with the oil-drop value, and a value of $1/\alpha$ definitely different from 137. If $1/\alpha$ is to equal 137, it is necessary that e/m equal approximately 1.768. As discussed in connection with Eq. (45), the resulting values of e and h are then about 0.26 percent and 0.30 percent higher than the values adopted in 1929. The direct and necessary dependence of the values of e, h and $1/\alpha$ on the value adopted for e/m is the most important result brought out in this paper.

With curve k assumed as the most probable solution, the resulting values of e, h, $1/\alpha$ and e/m differ so little from those suggested by the writer in 1929 that, for practical purposes, *no* change is advocated. It would seem good policy to recommend a change in the value of a fundamental physical constant *only* when the new value differs from the old by more than the probable error of the new value. When two values agree within less than the probable error, their difference has no scientific significance. In view of the wide use of fundamental constants, and of the numerous constants that may be derived from them[46] it is desirable, as a purely practical matter, that the values in common use be changed as infrequently as possible. This has been the policy of all atomic weight committees. The writer feels that, as a result as much of good luck as good management, he advocated in 1929 values of e, h, e/m and $1/\alpha$ that are still quite acceptable. In special tests of theoretical relations one should of course use the latest and most reliable values of all constants. For such work the constants listed on the last line of Table II (solution k) are recommended, but for all ordinary work the 1929 values may well be retained.

The only significant error in these 1929 values lies in the stated probable error of all ratios of the type h^x/e^y. The method adopted in 1929 for calculating the error in such cases is outlined in footnote 2, page 60 of G. C. 1929. The preceding discussions of the present paper have shown the correct method. Thus, from Eqs. (14) and (15)

$$\frac{2\pi}{\alpha c}=\frac{h}{e^2}=\frac{h_2}{e_0{}^2}\,. \tag{59}$$

The proportional error in h/e^2, or in α, is merely the proportional error in h_2, as calculated from the adopted $h_n\!:\!n$ curve. Similarly, from Eq. (53)

$$\frac{2\pi}{\alpha c}=\frac{h_0}{e_{1/2}{}^2} \tag{60}$$

and the proportional error in α is twice the proportional error in $e_{1/2}$, as calculated from the adopted $e_m\!:\!m$ curve. This last curve always yields the most

[46] Table *c* of G.C. 1929 gives an incomplete list of such derived constants.

reliable values of e and h. Hence Eq. (60) is to be used, rather than (59). We may combine and generalize Eqs. (59) and (60) as follows,

$$\frac{h}{e^n} = \frac{h_n}{e_0{}^n} = \frac{h_0}{e_{1/n}{}^n} \,. \tag{61}$$

Replacing n by its equivalent $1/m$ and forming the mth power, Eq. (61) becomes

$$\frac{h^m}{e} = \frac{h_{1/m}{}^m}{e_0} = \frac{h_0{}^m}{e_m} \,. \tag{62}$$

This last equation is most convenient for use with the $e_m\!:\!m$ curve, and states that the ratio h^m/e has a proportional error equal to the proportional error in the calculated value e_m. Similarly Eq. (61) states that the proportional error in h/e^n equals that in the calculated h_n. Writing

$$h^x/e^y = (h^m/e)^y = (h/e^n)^x \tag{63}$$

where $m=x/y$, or $n=y/x$, we obtain from Eqs. (61) (62) and (63) the final generalization that the proportional probable error in h^x/e^y is y times the proportional probable error in e_m, as obtained from the adopted $e_m\!:\!m$ curve, or x times the proportional probable error in h_n, as obtained from the adopted $h_n\!:\!n$ curve. The $e_m\!:\!m$ curve includes the most data and is therefore usually the more reliable.

As examples of the use of Eq. (63) we may note that the correct probable error in h/e^2 (or in $1/\alpha$) is less than one-half as large[47] as that given in 1929. On the other hand, the ratio h/e is, from solution k, $(1.3723 \pm 0.0008) \times 10^{-17}$ in place of 1.3725 ± 0.0005 given in 1929. The error is thus larger than before. Many of the constants listed in Tables b and c of G.C. 1929 contain the factor h^x/e^y, and the probable errors in all such cases require more or less revision.[48] It is, however, not my purpose here to present any extended revision of the general physical constants.[49]

This paper may well conclude with the remark made in the introduction,—that the primary object at this time is to present a mathematically correct method for the (necessarily) simultaneous evaluation of e and h and of the probable errors in all ratios of the type h^x/e^y. It is only in respect to these points that, so far as I am now aware, the methods used in G.C. 1929 are open to real criticism.

[47] 137.307 ± 0.048, in place of $137.29_4 \pm 0.11$.

[48] The method used in 1929 involved the assumption of a certain *constant* error for h_n or e_m, instead of the varying error shown in Figs. 1 to 11 of this article. Hence the true probable error, for values of m or n near the "center of gravity" of the data, is *less* than that given in 1929, while for more distant values it is greater.

[49] There is now in existence a National Research Council Committee on Physical Constants, of which the writer is chairman. This committee will issue a detailed report whenever any significant changes in the *values* (not the probable errors) of the fundamental constants seem required.