TEMPERATURE DISTRIBUTION IN AN INCANDESCENT LAMP FILAMENT IN THE NEIGHBORHOOD OF A COOLING JUNCTION.

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Introduction.

The filaments of incandescent lamps are subject to end-losses which result from the cooling effects of supports. These losses are sometimes of considerable magnitude and must be determined if it is desirable to study certain of the properties of the filament. The variations existing near such a cooling junction may, indeed, afford the opportunity for the convenient measurement of certain other properties, such as the thermal conductivity or the Thomson electromotive force, subjects on which the writer hopes to present papers soon. A knowledge of the temperature distribution in such cases is of considerable value.

Theoretical Considerations.

The theory which is involved here assumes a homogeneous, cylindrical filament of uniform cross-section and surface condition, in an evacuated chamber, the filament being heated by an electric current which is uniformly distributed over any cross-section. A small amount of speculation leads one to expect, as one proceeds from the cooling junction, a rapid rise in temperature which approaches a maximum value according to some exponential law. The isothermal surfaces within might be expected to be somewhat similar to portions of paraboloids of revolution belonging to a family whose axis is the axis of the filament. The fundamental differential equation as used here is an expression in cylindrical coordinates stating that, for the condition of a steady state, the net rate of conduction of heat into a small element of such a filament plus the rate of development of heat electrically within it equates to zero. Consider a hollow cylindrical element of such a filament of length Δl, of radius r and of radial thickness Δr, whose axis coincides with the axis of the filament. There follows:

$$(1) \quad 2\pi r\left[k\frac{\partial^2 T}{\partial l^2}+\frac{dk}{dT}\left(\frac{\partial T}{\partial l}\right)^2\right]\Delta l\Delta r + 2\pi r\left[k\frac{\partial^2 T}{\partial r^2}+\frac{k}{r}\frac{\partial T}{\partial r} +\frac{dk}{dT}\left(\frac{\partial T}{\partial r}\right)^2\right]\Delta l\Delta r + 2\pi r i^2\rho\Delta l\Delta r = 0.$$

The meanings of the above symbols together with others to be used later are given in Table I. The first, second and third terms represent respectively the net rate of heat conduction into the element of volume through the ends, the net rate of heat conduction into the element of volume radially, and the rate of heat production in the element of volume electrically. (1) leads directly to

$$(2)\qquad \frac{\partial^2 T}{\partial l^2}+\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{k}\frac{dk}{dT}\left[\left(\frac{\partial T}{\partial l}\right)^2+\left(\frac{\partial T}{\partial r}\right)^2\right]+\frac{i^2\rho}{k}=0.$$

Table I.

T	Temperature.
l	Distance parallel to filament axis measured from the cooling support.
r	Radial distance.
ρ	Resistivity.
k	Thermal conductivity.
i	Current density.
r_0	Radius of the filament.
T_0	Temperature at the filament surface.
T_{0m}	Maximum surface temperature.
k_m	Thermal conductivity at the maximum filament temperature.
k', T', T_0', T_{0m}'	Quantities similar to k, T, T_0, T_{0m} in a system where $(dk'/dT')=0$.
c, a, μ, l_0, ν, A	Constants whose meanings will be defined later.

(2) may be changed to a more suitable form by assuming a scale of temperature T', in which

$$(3)\qquad \frac{dk'}{dT'}=0;$$

we then have

$$(4)\qquad \frac{\partial^2 T'}{\partial l^2}+\frac{\partial^2 T'}{\partial r^2}+\frac{1}{r}\frac{\partial T'}{\partial r}+\frac{i^2\rho}{k'}=0.$$

In case

$$(5)\qquad \frac{i^2\rho}{k'}=a^2T'$$

we finally have

$$(6)\qquad \frac{\partial^2 T'}{\partial l^2}+\frac{\partial^2 T'}{\partial r^2}+\frac{1}{r}\frac{\partial T'}{\partial r}+a^2T'=0.$$

By means of the well-known device of assuming

$$(7)\qquad T'=RL,$$

where R and L are respectively functions only of r and of l, two independent equations may be obtained from (6):

$$(8)\qquad \frac{d^2L}{dl^2}-\mu^2L=0,$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + (\mu^2 + a^2)R = 0, \tag{9}$$

where μ^2 is a constant.

There are three boundary conditions which must be fulfilled.

$$\left[\frac{T'}{T_{0m}'} = \frac{J_0(ar)}{J_0(ar_0)}\right]_{l=\infty}, \tag{10}$$

where J_0 represents a Bessel function of zero order.

$$\left[\frac{T'}{T_{0m}'} = f(l)\right]_{r=r_0}, \tag{11}$$

$$\left[\frac{T'}{T_{0m}'} = F(r)\right]_{l=0}. \tag{12}$$

The first of these only is theoretical and merely states the particular solution of (6) for the condition of no conduction of heat along the filament, a condition which is fulfilled at a considerable distance from a cooling junction. The second boundary condition gives the surface temperature distribution. It is necessarily one that must be experimentally determined. The third boundary condition naturally specifies the temperature distribution across some surface near the cooling junction. In practice this distribution must be assumed.

In case $f(l)$ is expressed in terms of a hyperbolic sine or a hyperbolic cosine series, a solution may be obtained which represents accurately the distribution of temperature by means of an infinite series whose terms are of the type

$$A \sinh (\mu l)\ J_0(\sqrt{\mu^2 - a^2}\sqrt{-1}\, r).$$

In case $f(l)$ is expressed in terms of an exponential series, as has been done by the writer for some of his determinations, an expression may be obtained for the temperature distribution which although it theoretically fails under certain conditions, represents the temperature distribution to a high degree of accuracy in all practical cases which the writer has considered. Assume thus the following solution of (8)

$$f(l) = 1 - A_1e^{-\mu(l+l_0)} + A_2e^{-2\mu(l+l_0)} - \cdots, \tag{13}$$

where l_0 is a constant depending on the materials and dimensions at the cooling support and on the conditions of operation. For $F(r)$ in practice one can not, as already stated, make any experimental determination. It is evident from simple considerations and from the later determinations of temperature difference between filament axis and filament surface that such an assumption as

$$F(r) = \text{const.} \tag{14}$$

is warranted. A double series suffices to give the resultant temperature distribution.

$$\frac{T'}{T_{0m}'} = S_1 + S_2 = \frac{J_0(ar)}{J_0(ar_0)} - A_1 e^{-\mu(l+l_0)} \frac{J_0(\sqrt{\mu^2 + a^2}\, r)}{J_0(\sqrt{\mu^2 + a^2}\, r_0)}$$

$$(15) \quad + A_2 e^{-2\mu(l+l_0)} \frac{J_0(\sqrt{(2\mu)^2 + a^2}\, r)}{J_0(\sqrt{(2\mu)^2 + a^2}\, r_0)} - \cdots + B_1 e^{-\nu(l+l_0)} J_0(\sqrt{\nu_1^2 + a^2}\, r)$$

$$+ B_2 e^{-\nu_2(l+l_0)} J_0(\sqrt{\nu_2^2 + a^2}\, r) - \cdots,$$

where $B_1, B_2 \cdots \nu_1, \nu_2 \cdots$ represent undetermined constants. If $\nu_1, \nu_2 \cdots$ be considered particular values of μ, it is readily seen that each term of (15) is a solution of (8) and of (9) and therefore of (6). The second of the two series S_2 in (15) represents a correction term which enters, due to the failure of the first series S_1 by itself to fulfill the boundary condition (14). In determining the constants of S_2 it is to be considered as a separate series. Its boundary conditions as such, since the first two of the three main boundary conditions are fulfilled by S_1, are

$$(16) \qquad [S_2 = 0]_{l=\infty},$$

$$(17) \qquad [S_2 = 0]_{r=r_0},$$

$$(18) \qquad [S_2 = -S_1]_{l=0}.$$

The method of evaluating the constants here may be found in appropriate texts dealing with Bessel's functions.[1] It is of course necessary to first evaluate S_1 for the position $l = 0$. Evidently the sum of S_1 and S_2 fulfills the boundaries (10), (11) and (12).

The most satisfactory comprehension of a temperature distribution results from a consideration of $1/T_{0m}'\, \partial T'/\partial l$ and $1/T_{0m}' \partial T'/\partial r$ and through them of the axial and the radial components of the temperature gradient $\partial T/\partial l$ and $\partial T/\partial r$ and of the angle which the temperature gradient makes with the axis of the filament. Differentiation of (15) gives

$$(19) \quad \frac{1}{T_{0m}'}\frac{\partial T'}{\partial l} = A_1\mu e^{-\mu(l+l_0)} \frac{J_0(\sqrt{\mu^2 + a^2}\, r)}{J_0(\sqrt{\mu^2 + a^2}\, r_0)} - 2A_2\mu e^{-2\mu(l+l_0)} \frac{J_0(\sqrt{(2\mu)^2 + a^2}\, r)}{J_0(\sqrt{(2\mu)_2 + a^2}\, r_0)} + \cdots$$

and

$$(20) \quad \frac{1}{T_{0m}'}\frac{\partial T'}{\partial r} = \frac{\partial/\partial r\ J_0(ar)}{J_0(ar_0)} - A_1 e^{-\mu(l+l_0)} \frac{\partial/\partial r\ J_0(\sqrt{\mu^2 + a^2}\, r)}{J_0(\sqrt{\mu^2 + a^2}\, r_0)} + \cdots$$

For computations using (19) and (20) often the quantities represented

[1] Byerly's Fourier's Series and Spherical Harmonics, p. 228.

by Bessel's functions may be taken as unity, while for expressions such as $\partial/\partial r\ J_0\ (cr)$ may be taken the approximation,

$$\frac{\partial}{\partial r} J_0(cr) \doteq - cJ_1(cr) \doteq - \frac{c^2 r}{2} . \tag{21}$$

The method of procedure for obtaining the corresponding values for $\partial T/\partial l$ and $\partial T/\partial r$ will be evident when the relation between T and T' is once determined.

The Surface Temperature Distribution.

The method of study was based on the Holborn-Kurlbaum optical pyrometer principle and is very similar to that employed by Hyde, Cady and Worthing[1] in a study of energy losses in electric incandescent lamps and by the writer in a study of the variations from Lambert's cosine law.[2] All of the precautions which have been enumerated by Worthing and Forsythe[3] were made use of. The arrangement of apparatus is shown in Fig. 1. The filaments studied were of tungsten mounted

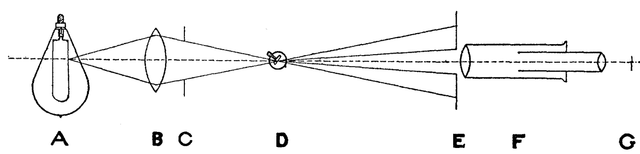


Fig. 1.

Diagram showing arrangement of apparatus. *A*, background; *B*, objective lens; *C*, entrance cone diaphragm; *D*, pyrometer filament; *E*, eyepiece diaphragm; *F*, eyepiece; *G*, monochromatic glass filter.

as U-shaped loops in evacuated bulbs. The lamps, which were well aged in order to remove gases, were located at *A*. The necessary measurements consist in balancing in brightness the pyrometer filament *D* against the background filament *A* in the neighborhood of the cooling junction as that filament, heated to incandescence by a constant current, is raised or lowered. Next with exactly the same arrangement of apparatus, excepting that a black body is substituted at *A*, and is operated at various temperatures, a calibration of the pyrometer filament current as a function of the black body temperature of the background is obtained. Then with the aid of a black body temperature—true temperature calibration of the background filament, taking account of the glass of the bulb of the lamp placed at *A*, one easily obtains the surface temperature distribution along the filament placed at *A*. The writer, however, used

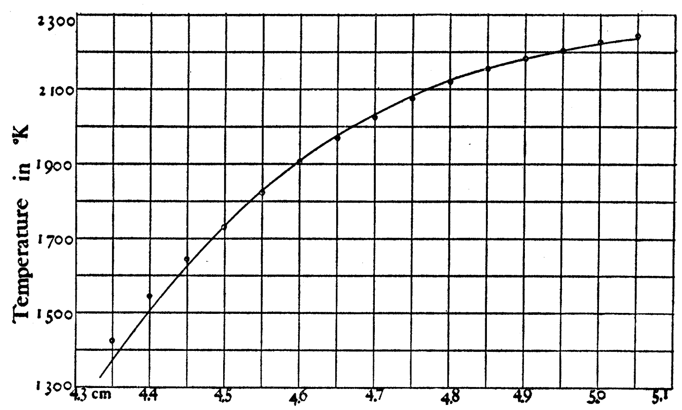
[1] Trans. of Ill. Eng. Soc. (U.S.), 6, p. 238, 1911. Illum. Eng. (Lond.), 4, p. 389, 1911.
[2] Astrophys. Jour., 36, p. 345, 1912.
[3] Phys. Rev. II, 4, p. 163 (1914).

a somewhat more roundabout method in this work, due to his interest in certain other considerations. The effects of the Thomson electromotive force were eliminated by averaging the results as to temperature distribution obtained with direct currents flowing through the filament first in one direction then in the opposite direction. The black-body temperature—true temperature calibration used was that obtained by Mendenhall and Forsythe.[1]

As has already been stated an exponential series relation between surface temperature and distance was found,[2]

$$(22)\quad \frac{T_0}{T_{0m}} = [1 - e^{-\mu(l+l_0)}]^n = 1 - ne^{-\mu(l+l_0)} + \frac{n(n-1)}{2}e^{-2\mu(l+l_0)} - \cdots.$$

How nearly this represents the distribution experimentally obtained in a particular case may be seen from Fig. 2. The average of a great



Distance along the filament in cm. from an arbitrarily chosen zero.

Fig. 2.

number of such determinations has shown all of the deviations to be noticed, with one exception, to be accidental. Deviations in a direction to make the computed value of T_0/T_{0m}, too small in the immediate neighborhood of the cooling junction always occur. They may result from a noticeable progressive change in the surface condition as one proceeds toward the junction. These deviations are neglected in this paper. Measurements on several different sized filaments operated at different maximum temperatures, indicate that n is constant throughout and that μ varies so that

$$(23)\qquad \frac{i}{\mu} = \text{constant (approx.)}$$

[1] Astrophys. Jour., 37, p. 380, 1913.

[2] See following paper.

The deviation from constancy occurs only with a change in the maximum operating temperature, the ratio decreasing slightly with decreasing maximum temperatures. For the case of a change in μ due only to a change in filament size, (23) may be shown, on assuming the same general surface distribution in the two cases, to be theoretically true. Of two filaments of radii r_1 and r_2 operated at the same maximum temperature, consider two corresponding elements of filament of equal length and the same mean temperature. We have for both elements the identity,

(24) Electrical input = radiation output + conduction output.

Consequently

$$(25)\qquad \pi r_1^2 \Delta l\, i_1^2 \rho = 2\pi r_1 \Delta l f(T) + \pi r_1^2 \Delta l\left[k\left(\frac{\partial^2 T}{\partial l^2}\right)_1 + \frac{dk}{dT}\left(\frac{\partial T}{dl}\right)_1^2\right],$$

$$(26)\qquad \pi r_2^2 \Delta l\, i_2^2 \rho = 2\pi r_2 \Delta l f(T) + \pi r_2^2 \Delta l\left[k\left(\frac{\partial^2 T}{\partial l^2}\right)_2 + \frac{dk}{dT}\left(\frac{\partial T}{\partial l}\right)_2^2\right],$$

where $f(T)$ represents the radiation intensity.
The well-known relation

$$(27)\qquad \left[\frac{i_1}{i_2} = \left(\frac{r_2}{r_1}\right)^{\frac{1}{2}}\right]_{T_1=T_2},$$

When combined with (25) and (26) gives

$$(28)\qquad \frac{r_2}{r_1} = \frac{\left(\frac{\partial^2 T}{\partial l^2}\right)_1 + \frac{1}{k}\frac{dk}{dT}\left(\frac{\partial T}{\partial l}\right)_1^2}{\left(\frac{\partial^2 T}{\partial l^2}\right)_2 + \frac{1}{k}\frac{dk}{dT}\left(\frac{\partial T}{\partial l}\right)_2^2}.$$

If for filament 1 we have

$$(29)\qquad T_1 = f_1(l),$$

and for filament 2 the assumed same general surface distribution such that

$$(30)\qquad \left[T_2 = f_2(l) = f_1\left(\frac{\left(\frac{\partial T}{\partial l}\right)_2}{\left(\frac{\partial T}{\partial l}\right)_1} l\right)\right]_{T_1=T_2},$$

there follow

$$(31)\qquad \frac{\left(\frac{\partial^2 T}{\partial l^2}\right)_1}{\left(\frac{\partial^2 T}{\partial l^2}\right)_2} = \frac{\left(\frac{\partial T}{\partial l}\right)_1^2}{\left(\frac{\partial T}{\partial l}\right)_2^2}$$

and

$$\left(\frac{\partial T}{\partial l}\right)_1 i_2 = \left(\frac{\partial T}{\partial l}\right)_2 i_1. \tag{32}$$

When $f(l)$ has the form given in (22), (23) results.

This relation is important in that it enables one to go directly from a surface distribution for one filament operating at a definite maximum temperature to that of another filament operating at the same maximum temperature.

The T' Scale of Temperature.

Results on the thermal conductivity of tungsten[1] indicate that

$$k = k_m + C_1(T - T_m). \tag{33}$$

The new temperature scale T' having the necessary condition (3) of constant thermal conductivity was further arbitrarily fixed by making

$$k' = k_m \tag{34}$$

and

$$[T' = 0]_{T=0}. \tag{35}$$

Consequently

$$T' = \left(1 - C_1\frac{T_m}{k_m}\right) T + \frac{C_1}{2k_m} T^2. \tag{36}$$

The substitution of (22) in (36) gives the previously assumed relation (13) in which A_1, A_2, etc., are functions only of n and of the coefficients of (36).

Application of Foregoing to a Special Tungsten Filament.

The value of certain constants relating to the particular filament used and the coefficients of (22) and (33) are incorporated in Table II.

Table II.

Certain Important Constants.

Constant.	Value.	Constant.	Value.
r_0	.01045 cm.	μ	$3.80 \frac{1}{\text{cm}}$
i	$12600 \frac{\text{amp}}{\text{cm}^2}$	n	1.87
ρ_m	75.8×10^{-6} ohms×cm.	k_m	$1.385 \frac{\text{watts}}{\text{cm} \times \text{deg}}$
T_{0m}	2315°K.	C_1	$5.10^{-4} \frac{\text{watts}}{\text{cm} \times \text{deg}^2}$
l_0	.3 cm (roughly)	a	$2.60 \frac{1}{\text{cm}}$

The value given for a represents an average value for the interval 1500° K. to 2450° K., throughout which range there is a decrease of approximately

[1] See later paper.

10 per cent. in going from the lower to the higher temperature. The assumption of constancy is here allowable in case one is concerned chiefly with the magnitude and the direction of the temperature gradient. In case one is interested in the differences in temperature between the surface and the axis at given cross sections, only values of the right order of magnitude will be obtained. The values thus found, however, will be directly related to the true differences approximately as the square of the a used in the computations is related to the squares of the true values of a for the regions under consideration.

Another method of obtaining the temperature differences consists in computing the distributions for various values of a and then applying to the regions in question the appropriate distributions. This consideration is based on computations which show that, for positions where the axial components of the temperature gradients are large in comparison with the radial components, small changes in the temperature distribution at one cross-section appreciably affect only a very limited region thereabout.

The equation for the special case resulting from (15) is

$$\frac{T'}{T_{0m}'} = S_1 + S_2 = \frac{J_0(2.60r)}{J_0(2.60r_0)} - 3.21e^{-3.8(l+l_0)}\frac{J_0(4.61r)}{J_0(4.61r_0)} + 3.91e^{-7.6(l+l_0)}\frac{J_0(8.04r)}{J_0(8.04r_0)} - \cdots - 6.4\times 10^{-5}e^{-230(l+l_0)}J_0(230r) + \cdots . \tag{37}$$[1]

As has already been noted, this solution fails under certain conditions,—*e. g.*, when $J_0(\sqrt{\mu^2_n + a^2}\, r_0)$ is zero. In ordinary cases the series in (13) may consist of a finite number of terms so terminated that this condition is impossible. The first term only of S_2 in (37) produces any noticeable effect on the temperature distribution.

The most interesting results for a particular application based on a constant value for a are indicated in Table III.

On account of a slightly different relation obtained for k as a function of T in the preliminary work published in an abstract,[1] values there given will not be found to be entirely consistent with those here given.

The formation of an expression for $\partial T'/\partial r$ from the S_1 terms of (37) shows that it, so far as first order terms are concerned, is proportional to the radial distance. Evidently also at any cross-section

$$\frac{\partial T}{\partial r} = -C_2 r, \tag{38}$$

where C_2 is a constant. Integration gives

[1] PHYS. REV., II., 3, p. 67 (1914).

TABLE III.

Temperature Distribution Results in a Filament of Tungsten (r_0 = 0.01045 cm.) Heated to a Maximum Temperature of 2315° K.

l	T_0	Relative Emission Intensity.	Temperature Gradient at Surface. Radial Component.	Axial Component.	Angle Made with Filament Axis.	Temperature Difference Between Axis and Surface.
0.0 cm	1149°K	0.023	$0 \frac{\text{deg}}{\text{cm}}$	$3750 \frac{\text{deg}}{\text{cm}}$	0.0°	0.00°
0.2	1711	.200	15	2130	0.3	.08
0.5	2114	.619	31	756	2.3	.16
0.7	2225	.808	39	364	6.1	.20
1.2	2301	.967	46	54.7	40	.24
1.7	2314	.9952	47	8.3	80	.25
∞	2315	1.0000	47	0.0	90	.25

$$T - T_0 = \frac{C_2}{2}(r_0^2 - r^2). \tag{39}$$

Therefore

$$[T - T_0]_{r=0} = \frac{C_2}{2} r_0^2 = -\frac{r_0}{2}\left(\frac{\partial T}{\partial r}\right)_{r=r_0}. \tag{40}$$

Equation (40) was used in computing the last column of Table III. It readily follows with slight approximation that the isothermal surfaces are portions of paraboloids of revolution which become more and more convex toward the cooling junction as one recedes along the filament from it. Further (39) and (40) show that the difference in temperature between a point on the axis and another point on the same cross-section varies as the square of the radial distance.

When, however, corresponding points on two filaments of the same material but of different radii and operated at the same maximum temperature, are compared, (23) together with the derivative $\partial T'/\partial r$, which may be obtained with the aid of (20), indicate that the differences in temperature, particularly when not too far from the maximum temperature, between corresponding axial points and other points on the same cross-section vary as the radii of the filaments. This fact would appear as evident from the equation given by Angell[1] for the portion of the filament where the temperature gradient is wholly axial.

A close approximation to the temperature distribution in a tungsten filament near a cooling junction for a certain definite heating current, when once the distribution is known for some other heating current may likewise be obtained quite simply. A consideration of (19) in which the coefficients A_1, A_2, etc., are practically constant shows, for corresponding

[1] Phys. Rev., 33, p. 421 (1911).

points chosen from the standpoint of T_0'/T_{0m}', that the expression $1/T_{0m}' \, \partial T'/\partial l$ varies as μ, which in turn according to (23) varies as i. The axial components of the temperature gradient may evidently be everywhere determined for the unknown surface distribution. The variation in $1/T_{0m}' \, \partial T'/\partial r$ may be obtained from similar considerations By (5) we see that a varies as i. From (20) and (21) we see that each term of $1/T_{0m}' \, \partial T'/\partial r$ and that therefore the quantity as a whole, since both a and μ vary as i, varies as i^2. The deviations mentioned in both (5) and (23) are such as to counteract each other. An accurate knowledge of the temperature distribution for one tungsten filament, together with the above considerations will enable one to determine quite accurately the distribution in any case for a filament of the same material.

Summary.

A solution has been obtained which represents with considerable accuracy the temperature distribution in a cylindrical tungsten filament of uniform cross-section and surface condition, heated electrically in an evacuated chamber.

This solution has been applied to a special case in which a tungsten filament (radius = 0.01045 cm.) is heated by a current to a maximum central temperature of 2315° K. (See Table III.)

The method of obtaining from the known distribution the temperature distribution in another filament of the same material, but of different radius heated under the same conditions to the same maximum temperature, is indicated.

The method of obtaining from the known distribution the temperature distribution in the same filament heated under the same conditions to a different maximum temperature is also indicated.

The surface temperature distribution for a tungsten filament for a given case and the approximate laws of variation with changes in filament size and maximum operating temperature have been experimentally determined. See equations (22) and (23).

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