ON PHYSICALLY SIMILAR SYSTEMS; ILLUSTRATIONS OF THE USE OF DIMENSIONAL EQUATIONS.

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1. *The Most General Form of Physical Equations.*—Let it be required to describe by an equation, a relation which subsists among a number of physical quantities of n different kinds. If several quantities of any one kind are involved in the relation, let them be specified by the value of any one and the ratios of the others to this one. The equation will then contain n symbols $Q_1 \cdots Q_n$, one for each kind of quantity, and also, in general, a number of ratios r', r'', etc., so that it may be written

$$f(Q_1, Q_2, \cdots Q_n, r', r'', \cdots) = 0. \tag{1}$$

Let us suppose, for the present only, that the ratios r do not vary during the phenomenon described by the equation: for example, if the equation describes a property of a material system and involves lengths, the system shall remain geometrically similar to itself during any changes of size which may occur. Under this condition equation (1) reduces to

$$F(Q_1, Q_2, \cdots Q_n) = 0. \tag{2}$$

If none of the quantities involved in the relation has been overlooked, the equation will give a complete description of the relation subsisting among the quantities represented in it, and will be a complete equation. The coefficients of a complete equation are dimensionless numbers, *i. e.*, if the quantities Q are measured by an absolute system of units, the coefficients of the equation do not depend on the sizes of the fundamental units but only on the fixed interrelations of the units which characterize the system and differentiate it from any other absolute system.

To illustrate what is meant by a " complete " equation, we may consider the familiar equation

$$\frac{pv}{\theta} = \text{constant},$$

in which p is the pressure, v the specific volume, and θ the absolute temperature of a mass of gas. The constant is not dimensionless but depends, even for a given gas, on the units adopted for measuring p, v, and θ; the equation is not complete. Further investigation shows that the equation may be written

$$\frac{pv}{R\theta} = N,$$

in which the symbol R stands for a quantity characteristic of each gas and differing from one to another, but fixed for any given gas when the units of p, v, and θ are fixed. We thus recognize that R is a quantity that can be measured by a unit derived from those of p, v, and θ. If we do express the value of R in terms of a unit thus derived, N is a dimensionless constant and does not depend on the sizes of the units of p, v, and θ but only on the fixed relation which the unit of R bears to them. The equation is now a "complete" equation.

Every complete physical equation (2) has the more specific form

$$\Sigma M Q_1^{b_1} Q_2^{b_2} \cdots Q_n^{b_n} = 0. \tag{3}$$

Such expressions as log Q or sin Q do not occur in physical equations; for no purely arithmetical operator, except a simple numerical multiplier, can be applied to an operand which is not itself a dimensionless number, because we can not assign any definite meaning to the result of such an operation. The reason why such an expression as Q^2 can appear, is that Q^2 may be regarded as a symbol for the result of operating on Q by Q. For example, when we write $A = l^2$, l^2 is a symbol for the result of operating on a length l by itself. We are directed to take the length l as operand and "operate on it with the length l" by constructing on it as a base, a rectangle of altitude l; and the result of this operation, which fixes an area A, is represented by l^2. Whenever functions that do not have the form of the terms in equation (3) appear to occur in physical equations, it is invariably found upon examination that the arguments of these functions are dimensionless numbers.

2. *Introduction of Dimensional Conditions.*—We have now to make use of the familiar principle, which seems to have been first stated by Fourier, that all the terms of a physical equation must have the same dimensions, or that every correct physical equation is dimensionally homogeneous. Let equation (3) be divided through by any one term and it takes the form

$$\Sigma N Q_1^{a_1} Q_2^{a_2} \cdots Q_n^{a_n} + 1 = 0, \tag{4}$$

in which the N's are dimensionless numbers. In virtue of the principle of dimensional homogeneity the exponents $a_1, a_2, \cdots a_n$ of each term of equation (4) must be such that that term has no dimensions or that a dimensional equation

$$[Q_1^{a_1} Q_2^{a_2} \cdots Q_n^{a_n}] = [1] \tag{5}$$

is satisfied.

Let Π represent a dimensionless product of the form

$$\Pi = Q_1^{a_1} Q_2^{a_2} \cdots Q_n^{a_n}. \tag{6}$$

so that equation (4) may be written more shortly

$$\Sigma N \Pi + 1 = 0. \tag{7}$$

Since Π is dimensionless, Π^x is dimensionless; and furthermore, any product of the form $\Pi_1^{x_1} \Pi_2^{x_2} \cdots \Pi_i^{x_i}$ is also dimensionless. Hence if Π_1, Π_2, $\cdots$ Π_i represent all the separate independent dimensionless products of the form (6) which can be made up in accordance with equation (5) from the quantities Q, equation (7) may be written in the form

$$\Sigma N \Pi_1^{x_1} \Pi_2^{x_2} \cdots \Pi_i^{x_i} + 1 = 0 \tag{8}$$

and still satisfy the requirement of dimensional homogeneity.

Now there are, so far as this requirement is concerned, no restrictions on the number of terms, the values of the coefficients, or the values of the exponents. Hence the Σ merely represents some unknown function of the independent arguments Π_1, $\cdots$ Π_i and equation (8) may more simply be written

$$\psi(\Pi_1, \Pi_2, \cdots \Pi_i) = 0. \tag{9}$$

By reason of the principle of dimensional homogeneity, every complete physical equation of the form (2) is reducible to the form (9) in which

$$[\Pi_1] = [\Pi_2] = \cdots = [\Pi_i] = [1] \tag{10}$$

and the number i, of separate independent arguments of ψ, is the maximum number of independent dimensionless products of the form (6) which can be made by combining the n quantities Q_1, Q_2 $\cdots$ Q_n in different ways.

We have next to find the value of i. Let k be the number of arbitrary fundamental units needed as a basis for the absolute system $[Q_1]$, $\cdots$ $[Q_n]$ by which the Q's are measured. Then in principle and if we disregard the practical considerations connected with the preservation of standards, etc., there is always, among the n units $[Q]$, at least one set of k which may be used as fundamental units, the remaining $(n - k)$ being derived from them.

Now each equation of the form (5) with a particular set of exponents a (corresponding to a particular dimensionless product Π) is an equation to which the dimensions of the units $[Q]$ are subject. But since $(n-k)$ of the units are derivable from the other k and the units are otherwise arbitrary, it is evident that each equation of the form (5) is in reality equivalent to one of these equations of derivation. There are therefore $(n - k)$ equations of the form (5) and the number of products Π which appear as independent variables in equation (9) is

$$i = n - k.$$

Furthermore, if $[Q_1]$, $[Q_2] \cdots [Q_k]$ are k of the n units which might be used as fundamental, the i equations (5) may be written

$$\left.\begin{aligned} [\Pi_1] &= [Q_1^{\alpha_1} Q_2^{\beta_1} \cdots Q_k^{\kappa_1} P_1] = [1] \\ [\Pi_2] &= [Q_1^{\alpha_2} Q_2^{\beta_2} \cdots Q_k^{\kappa_2} P_2] = [1] \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ [\Pi_i] &= [Q_1^{\alpha_i} Q_2^{\beta_i} \cdots Q_k^{\kappa_i} P_i] = [1] \end{aligned}\right\} \tag{11}$$

in which the P's represent $Q_{k+1} \cdots Q_n$, *i. e.*, the quantities that are, temporarily, regarded as derived.

To make use of any one of equations (11) for finding the specific form of the corresponding Π, we replace each of the $[Q]$'s and the $[P]$ by the known dimensional equivalent for it in terms of whatever set of k fundamental units (such as mass, length, time, etc.) we may happen to find convenient. The resulting equation contains the k independent fundamental units, and since both members are of zero dimensions, the exponent of each unit must vanish. We therefore obtain k independent linear equations which suffice to determine the k exponents and so to fix the form of the Π in question. We have still, however, one arbitrary choice left which it is sometimes convenient to make use of. Since the Π's occur in equation (9) as arguments of an indeterminate function ψ and are subject only to the condition of being dimensionless, when we have found the specific form of any one of the Π's, we are at liberty to replace this by any function of it; for this function will also be dimensionless and will be independent of the remaining Π's. This remark enables us to dispense with fractional exponents, when they happen to result from equations of the form (11), and so to simplify the writing down of our results.

3. *Illustration.*—To make the meaning of the foregoing developments more evident we may treat an example. Let us suppose that we have to deal with a relation which involves one quantity of each of the following $n = 7$ kinds:

Name.	Symbol.	Dimensions.	
1. Force	F	$[mlt^{-2}]$	(12)
2. Density	ρ	$[ml^{-3}]$	
3. Length	D	$[l]$	
4. Linear speed	S	$[lt^{-1}]$	
5. Revolutions per unit time	n	$[t^{-1}]$	
6. Viscosity	μ	$[ml^{-1}t^{-1}]$	
7. Acceleration	g	$[lt^{-2}]$	

Three fundamental units are needed, *i. e.*, $k = 3$, but they need not be $[m, l, t]$ for we could also use $[F, \rho, S]$ or $[\rho, n, \mu]$ or several other combinations. On the other hand, such combinations as $[l, S, n]$ or $[S, n, g]$ could not be used.

We know by section (2) that any relation whatever which involves all the above seven quantities and no others, must be expressible by an equation which can be reduced to the form

$$\psi(\Pi_1, \Pi_2, \Pi_3, \Pi_4) = 0 \qquad (9, a)$$

because $n - k = 7 - 3 = 4$.

To find a specific form of this equation, we select 3 of the quantities as fundamental and proceed to use equations (11).

Let us, to start with, set

$$F = Q_1, \quad \rho = Q_2, \quad D = Q_3$$

these being a possible set of fundamental units sufficient for deriving the others. Then S, n, μ, g, act as P_1, P_2, P_3, P_4 and we have, corresponding to equations (11),

$$\left.\begin{aligned} [F^{\alpha_1}\rho^{\beta_1}D^{\gamma_1}S] &= [1], \\ [F^{\alpha_2}\rho^{\beta_2}D^{\gamma_2}n] &= [1], \\ [F^{\alpha_3}\rho^{\beta_3}D^{\gamma_3}\mu] &= [1], \\ [F^{\alpha_4}\rho^{\beta_4}D^{\gamma_4}g] &= [1], \end{aligned}\right\} \qquad (11, a)$$

from which to determine the α's, β's, and γ's.

Taking the first of these equations and substituting the dimensions of $[F, \rho, D, S]$ in it we have

$$[m^{\alpha_1}l^{\alpha_1}t^{-2\alpha_1} \times m^{\beta_1}l^{-3\beta_1} \times l^{\gamma_1} \times lt^{-1}] = [1]$$

and since m, l, and t are independent, this can be satisfied only if α_1, β_1, and γ_1 are related as shown by the equations

$$\left.\begin{aligned} \alpha_1 + \beta_1 &= 0, \\ \alpha_1 - 3\beta_1 + \gamma_1 + 1 &= 0, \\ 2\alpha_1 + 1 &= 0, \end{aligned}\right\} \text{ or } \left\{\begin{aligned} \alpha_1 &= -\tfrac{1}{2}, \\ \beta_1 &= \tfrac{1}{2}, \\ \gamma_1 &= 1. \end{aligned}\right.$$

We therefore have $\Pi_1 = F^{-\frac{1}{2}}\rho^{\frac{1}{2}}DS$, which will be more convenient to write and satisfy the condition of being dimensionless equally well if we square it and write $\Pi_1 = \rho D^2S^2/F$.

If we follow a similar method with the remaining three equations of the set (11, a) we have

$$\Pi_2 = \frac{\rho D^4n^2}{F}; \quad \Pi_3 = \frac{\mu^2}{F\rho}; \quad \Pi_4 = \frac{\rho D^3g}{F}$$

and equation (9, a) takes the form

$$\psi\left(\frac{\rho D^2S^2}{F}, \frac{\rho D^4n^2}{F}, \frac{\mu^2}{F\rho}, \frac{\rho D^3g}{F}\right) = 0. \qquad (9, b)$$

Our conclusion is that any equation which is the correct and complete expression of a physical relation subsisting among seven quantities of the kinds mentioned is reducible to the form (9, b).

If $[F, \rho, D]$ were the only triad that could be used as fundamental units for the seven kinds of quantity, equation (9, b) would be the only general form of the equation; but in reality several other triads can be used, so that other equations may be found which, while essentially equivalent to (9, b), present a different appearance. If, for instance, we select the triad $[\rho, D, S]$, a process like that which led to equation (9, b) gives us the equation

$$\psi\left(\frac{\rho D^2S^2}{F}, \frac{Dn}{S}, \frac{\rho DS}{\mu}, \frac{Dg}{S^2}\right) = 0, \qquad (9, c)$$

to which we shall have occasion to refer later.

4. *The General Form to Which Any Physical Equation is Reducible.*—Equation (9), subject to equations (11), gives the necessary form of any relation which subsists among n quantities of different kinds: it is the final form to which the dimensional conditions reduce equation (2). Now equation (2) describes a particular form of the more general relation described by equation (1), in which several quantities of each of the n kinds may be involved,—all but one of each kind being specified by their ratios to that one. Dimensional reasoning can not furnish any information regarding the influence of these dimensionless ratios on the phenomenon which is characterized by the relation in question, nor can it tell us how they are involved in the equation which describes the relation. But we can not assume that they are without influence, and the possibility of their entering into the relation must be indicated in the final equation which corresponds to (1) as equation (9) does to (2). Since equation (9) follows from equation (2), it is correct for any fixed values of the r's, and it may therefore be generalized so as to be applicable to any and all values of the ratios r by introducing the r's as independent arguments of the unknown function ψ, which is then a function of *all* the independent dimensionless combinations of powers of *all* the quantities of all the n kinds which are involved in the relation to be described.

The general conclusion from the principle of dimensional homogeneity may therefore be stated as follows: If a relation subsists among any number of physical quantities of n different kinds, and if the symbols $Q_1, Q_2, \cdots Q_n$ represent one quantity of each kind, while the remaining quantities of each kind are specified by their ratios r', r'', $\cdots$, etc., to the particular quantity of that kind selected, then: any equation which describes this relation completely is reducible to the form

$$\psi(\Pi_1, \Pi_2, \cdots \Pi_i, r', r'' \cdots) = 0. \tag{13}$$

If k is the number of fundamental units required in an absolute system for measuring the n kinds of quantity, the number of the dimensionless products Π is

$$i = n - k.$$

If $[Q_1]$, $[Q_2]$ $\cdots$ $[Q_k]$ are any k of the units for measuring the Q's, which are independent and so might be used as fundamental units; and if the remaining units needed are denoted by $[P_1]$, $[P_2]$ $\cdots$ $[P_i]$, each of the Π's may be determined from a dimensional equation

$$[\Pi] = [Q^{\alpha}{}_1 Q^{\beta}{}_2 \cdots Q^{\kappa}{}_k P] = [1] \tag{14}$$

after substituting in this equation the known dimensions of the $[Q]$'s and of $[P]$ in terms of any suitable set of k fundamental units.

5. *Remarks of the Utilization of the Foregoing Results.*—Equation (13), representing a single relation connecting a number of variables, can, in principle at least, be solved for any one of them and put into the form

$$\Pi_1 = \varphi(\Pi_2, \Pi_2, \cdots, \Pi_i, r', r'', r''', \cdots) \tag{15}$$

in which Π_1 is any one of the Π's; or into the form

$$r' = \varphi_1(\Pi_1, \Pi_2, \cdots \Pi_i, r'', r''', \cdots) \tag{16}$$

in which r' is any one of the r's.

Although the form of φ is unknown so that neither of these equations gives any definite general information, they may nevertheless be useful in particular circumstances. Equation (15), for example, tells us that if Π_2, Π_3, $\cdots$, Π_i, r', r'', etc., are all kept constant, Π_1 is also constant, regardless of the form of the unknown function φ. And since Π_1 is a product of known powers of Q_1, Q_2, $\cdots$ Q_k, P_1, we know how any one of these $(k + 1)$ quantities varies with the others under the given conditions. To illustrate; equation (9, b) may be put into the form

$$\frac{\rho D^2 S^2}{F} = \varphi\left(\frac{\rho D^4 n^2}{F}, \quad \frac{\mu^2}{F\rho}, \quad \frac{\rho D^3 g}{F}\right) \tag{17}$$

and if $\rho D^4 n^2/F$, $\mu^2/F\rho$ and $\rho D^3 g/F$ are kept constant we have

$$S^2 = \text{const.} \times \frac{F}{\rho D^2}. \tag{18}$$

If we wish to treat some one quantity X as the unknown and get an equation of the form (15) with this quantity absent from the second member, X must be a factor of only one of the Π's. This means that in selecting the variables which are to act as Q's and P's in equations (11) or (14), X must be one of the P's.

To illustrate the application of the above remarks, let us consider the screw-propeller problem. Let F be the thrust exerted by a screw propeller of a particular shape specified by a number of ratios of lengths r', r'', etc. and of a size specified by the diameter D. The thrust must be supposed to depend on the number of revolutions per unit time n, the speed of advance S, and the density ρ and viscosity μ of the liquid. It may safely be assumed that the very slight compressibility of the liquid has no sensible effect on the thrust, but unless the propeller is very deeply immersed, there will be surface disturbances and we must expect the thrust to be affected by the weight of the liquid, *i. e.*, by the intensity of gravity g. It does not appear that any other circumstance, except the depth of immersion which may be specified by its ratio to D and represented by an extra r, can influence the thrust, and if we are right in this assumption there must be an equation

$$f(F, \rho, D, S, n, \mu, g, r', r'', \cdots) = 0 \tag{19}$$

corresponding to equation (1). This equation must be reducible to the form (13), and we have already given in equations (9, *b*) and (9, *c*), two of the forms which it might have in the case of constant r's, *i. e.*, for a propeller of fixed shape and immersion.

Now suppose that we wish to find out how the thrust depends on the density of the liquid, the diameter of the propeller, and the speed of advance. Then in using equations (11) we must use F as one of the P's and take $[\rho, D, S]$ as $[Q_1, Q_2, Q_3]$. We then get the Π's which appear in equation (9, *v*) and the equation corresponding to (13) is

$$\psi\left(\frac{\rho D^2 S^2}{F}, \frac{Dn}{S}, \frac{\rho DS}{\mu}, \frac{Dg}{S^2}, r', r'', \cdots\right) = 0.$$

By solving for $\rho D^2 S^2/F$, this may be put into the form

$$F = \rho D^2 S^2 \varphi\left(\frac{Dn}{S}, \frac{\rho DS}{\mu}, \frac{Dg}{S^2}, r', r'', \cdots\right) \tag{20}$$

which tells us without any experimentation at all, that if we can keep Dn/S, $\rho DS/\mu$, and Dg/S^2 constant, the thrust of a propeller of any given shape (r', r'', etc., constant) is proportional to the density of the liquid, the square of the diameter, and the square of the speed of advance. The meaning of this result and the value of the infor mation willbe discussed later: at present we may return to generalities.

6. *Physically Similar Systems.*—Equation (13) is a convenient expression of the conclusions to be drawn directly from the principle of dimensional homogeneity. It is useful in various ways, as will be illustrated later, but at present we may develop from it the notion of similar systems.

Let S be a physical system, and let a relation subsist among a number of quantities Q which pertain to S. Let us imagine S to be transformed into another system S' so that S' "corresponds" to S as regards the essential quantities. There is no point of the transformation at which we can suppose that the quantities cease to be dependent on one another; hence we must suppose that some relation will subsist among the quantities Q' in S' which correspond to the quantities Q in S. If this relation in S' is of the same form as the relation in S and is describable by the same equation, the two systems are "physically similar" as regards this relation. We have to enquire what sort of transformation would lead to this result, *i. e.*, what are the conditions which determine that two systems shall be similar as regards a given physical relation.

The original relation subsisting in S is reducible to the form (13), or

$$\psi(\Pi_1, \cdots, \Pi_i, r) = 0, \tag{13}$$

r representing *all* the independent ratios of quantities of the same kind which enter into the relation. The changes of the Q's during the transformation will, in general, result in a change of the numerical value of each Π or r. But these expressions remain dimensionless, so that to each of the arguments of ψ there corresponds, after the transformation, an expression Π' or r' of the same form in terms of the transformed quantities Q'; and these are all the independent dimensionless products of powers that can be made up of the quantities Q'. Hence the equation which describes the relation subsisting in S' among the quantities Q' is reducible to the form

$$\psi'(\Pi'_1, \cdots \Pi'_i, r') = 0. \tag{13'}$$

The requirement that S and S' shall be similar as regards this relation, means that the operators ψ and ψ' must be identical, and this will occur if the transformation leaves the numerical values of all the Π's and r's unchanged. For ψ and ψ' will then be applied to identical operands: and while two different functions of the same variables may vanish simultaneously for discrete sets of values of the variables, they can not do so for a continuous infinity of sets, and yet equations (13) and (13′) are satisfied without restriction. It follows that ψ and ψ' can not be different, or in other words; the systems S and S' are similar as regards this relation if corresponding Π's and r's are equal in the two systems. The nature of the transformation which leaves a system similar to itself may therefore be specified as follows:

(*a*) Any k quantities of independent kinds may be changed in completely arbitrary ratios, after which we must

(*b*) Change one quantity of each of the $(n - k) = i$ remaining kinds in such a ratio as to keep the numerical value of its Π unchanged; and finally we must

(*c*) Change the remaining quantities of each of the n kinds in the same ratio as the one quantity of that kind already mentioned, thereby keeping the ratios r unchanged.

The last and simplest of these conditions means that the system must remain similar to itself as regards each separate kind of quantity. If, for example, the sizes and shapes of some of its parts are essentially involved in the relation, the transformation must leave the system geometrically similar to itself as regards these parts, although other and unessential parts may change in any way. We have, in all, k arbitrary choices of ratios of change and since each of these may be made in an infinite number of ways there is a k-fold infinity of systems S' which are similar to any given system S as regards any particular physical relation. If the above conditions are fulfilled for all possible physical quantities which can pertain to a physical system, the transformed system will be similar to the original one as regards any possible relation between physical quantities and the two will be physically similar in *all* respects.

When absolute units are used, the validity of a complete physical equation is unaffected by changes in the fundamental units. Hence in changing from a system S to a similar system S' it is immaterial to the validity of the equation in question whether we do or do not retain our original fundamental units. If we alter the sizes of the fundamental units $[Q_1] \cdots [Q_k]$ in the same ratios as the kinds of quantity $Q_1 \cdots Q_k$ which they measure, the numerical value of any quantity of one of these kinds will be the same in both systems. And if we do not change the relations of the derived and fundamental units of our absolute system, every derived unit $[P]$ will change in the same ratio as every quantity P of that kind, so that the numerical value of every quantity in the system S will be equal to the numerical value of the corresponding quantity in the similar system S'.

This change of units will occur if the concrete primary standards which preserve the units partake of the transformation. To an observer whose quantitative information was all obtained by measurements based on such standards, not only all physical relations, but the numerical values of individual quantities, would appear the same in two similar systems: he could not distinguish the two systems nor detect a transformation of one into the other.

The foregoing theorem may, if we choose, be applied to an imaginary transformation of the whole physical universe, but in this grandiose general form it is of only metaphysical interest; for it is merely a statement about what would happen if we were to bring about certain changes which it is obviously quite beyond our powers to effect. Nevertheless in particular elementary instances the notion of physical similarity is useful and it is convenient to have the conditions of physical similarity formulated in a general way. One of these is always that of similarity with respect to each separate kind of quantity, such as length, speed, density, etc., which may enter into the physical phenomenon with which we happen to be concerned.

Let us suppose that a relation subsists among certain physical quantities. Dimensional reasoning suffices to tell us that if the relation is complete, the equation which describes it is reducible to the form (13); and if $Q_1 \cdots Q_k$ are the quantities which are used as independent in finding the Π's of equation (13), the equation can always be solved in the form

$$P_1 = Q_1^a Q_2^b \cdots Q_k^k \varphi(\Pi_2, \Pi_3, \cdots \Pi_i, r)$$

the form of the operator φ remaining to be found by other means. Without going into any abstruse consideration of conceivable modifications of the universe to which the various quantities may be regarded as pertaining, it is obvious that so long as we can, experimentally, control enough of these quantities to keep all the Π's and r's constant, any function of these arguments must also remain constant, no matter what its form may be. The practical application of the notion of similarity is based on this remark. The conditions necessary for this simplification are given by setting each Π and r equal to a constant: and when this is done the nature of the possible simultaneous variations which fulfil the requirements at once becomes evident.

It usually happens that some of the quantities concerned in the relation are obviously attached to or are properties of some body or some material system of limited extent and can be changed in value only by changing to a different body or system. If this second system is similar to the first as regards each separate kind of quantity all the r's which pertain to the system are the same. If we can so arrange the circumstances in which the system is placed that upon substituting one body or system for the other, $\Pi_2, \Pi_3, \cdots \Pi_i$ as well as any remaining r's, retain their values unchanged, the equation

$$P_1 = Q_1^a Q_2^b \cdots Q_k^k \times \text{const.}$$

is satisfied for both systems with the same value of the constant. The phenomenon characterized by the relation then occurs in a similar manner for both systems, and we say that the bodies or systems are similar with respect to this phenomenon. If the relation is a dynamical one, all essential parts of the two systems must be geometrically similar and have similar distributions of density, elasticity, etc., so far as these properties affect their behavior. If, in addition, the Π's of the relation are kept the same for one system as for the other, the systems are said to be "dynamically similar," though they might, of course, not be similar as regards some other dynamical relation nor behave similarly in some different sort of experiment.

The notion of physical similarity does not appear to have been developed and used to any extent except in this most obvious form of dynamical similarity. But the more general conception of a similarity which extends to other than merely dynamical relations, evidently follows directly from the dimensional reasoning, based on the principle of homogeneity, which culminated in equations (13) and (14).

7. *Remarks.*—In his article entitled "The Principle of Similitude," appearing in the April, 1914, number of the PHYSICAL REVIEW, Mr. Richard C. Tolman announces the discovery of a new principle and illustrates its value in reasoning about the forms of physical equations, by treating several examples. The statement of the principle is couched in such general terms that I have difficulty in understanding just what the postulate is, but it seems to me to be merely a particular case of the general theorem given in the foregoing section. Mr. Tolman selects length, speed, quantity of electricity, and electrostatic force as the four independent kinds of quantity which suffice for his purposes, and after subjecting them to four arbitrary conditions, he proceeds to find the conditions to which several other kinds of quantity are subject in passing from the actual universe to a miniature universe that is physically similar to it. Now I do not know whether the developments set forth above have ever been published in just this form, but it is certain that they are merely consequences of the principle of dimensional homogeneity, which is far from being either new or unfamiliar. The unnecessary introduction of new postulates into physics is of doubtful advantage, and it seems to me decidedly better, from the physicist's standpoint, not to drag in either electrons or relativity when we can get on just as well without them. Accordingly, my object in publishing the foregoing sections, which are a

fragment of a longer paper that I have had in hand for some time, is to call attention to the fact that in the present instance no new postulate seems to be needed. My feeling that Mr. Tolman's " Principle of Similitude " is not really new may, of course, be mistaken. But for the purposes to which he puts it, it is, at all events, superfluous, and this I shall proceed to prove by treating some of the problems he has used as illustrations.

The relations that involve temperature may be passed over, because Mr. Tolman's reasoning is based on the assumption that absolute temperature has the dimensions of energy, and this assumption is not permissible. If by " absolute temperature " is meant temperature measured by what is commonly called the thermodynamic scale, then the ratio of two temperatures is, by definition, the ratio of two quantities of heat. In a similar way, two intervals of time, measured by our ordinary time scale, have the same ratio as two angles through which the earth has rotated about its axis during these intervals; or two forces have the same ratio as the lengths by which they can stretch a given spiral spring. We do not, however, conclude that time has the dimensions of angle and force the dimensions of length; nor can we say that temperature has the dimensions of energy. The units needed for measuring thermal quantities can not all be derived from mass, length, and time or from any other set of three fundamental units which suffice for mechanics, and a fourth unit is indispensable. In practice, this special thermal unit is nearly always temperature; it is fixed by the arbitrary selection of the interval between the freezing and boiling points of water and the arbitrary assignment of a particular numerical value to this interval. We do not at present know of any method by which this or any other interval of temperature can be fixed by, *i. e.*, derived from, purely mechanical quantities without some further act of arbitrary choice than the selection of three mechanical units. We may therefore turn to Mr. Tolman's electromagnetic problems.

For the measurement of electric and magnetic quantities, one new fundamental unit is needed, beyond the three of mechanics, so that there must, in general, be four in all. In the electromagnetic system the new unit is permeability $[\mu]$ and the four are $[m, l, t, \mu]$. In the electrostatic system it is dielectric inductivity $[\epsilon]$, and the four are $[m, l, t, \epsilon]$. Other sets are sometimes more convenient, for example $[l, t, C, R]$, $[C]$ being current and $[R]$ resistance: this system corresponds to the " international " units, a system in which the ampere and the ohm are, by definition, fundamental units. Various systems might be used in dimensional reasoning without altering anything but a little algebra; the only important thing is the number of fundamental units required as the basis of the system.

8. *Illustrations of the Treatment of Electromagnetic Problems by the Method of Dimensions*: (*A*) *Energy Density of an Electromagnetic Field.*—We assume that the energy density u is completely determined by the field strengths E and H, and by the permeability μ and dielectric inductivity ϵ of the medium. If, or when, this assumption is valid we have

$$f(u, E, H, \mu, \epsilon) = 0. \tag{21}$$

The dimensions of these quantities on the $[m, l, t, \mu]$ system are

$$\left.\begin{aligned} [E] &= [m^{\frac{1}{2}}l^{\frac{1}{2}}t^{-2}\mu^{\frac{1}{2}}], & [u] &= [ml^{-1}t^{-2}], \\ [H] &= [m^{\frac{1}{2}}l^{-\frac{1}{2}}t^{-1-}\mu^{\frac{1}{2}}], & [\epsilon] &= [l^{-2}t^{2}\mu^{-1}]. \end{aligned}\right\} \tag{22}$$

We wish to get a relation that can be solved for u; hence u must not be one of the Q's of equation (14). Although in general, electrogmagnetic units require four fundamental units, three are enough in this instance: for example we may take $[E]$, $[\mu]$, and $[\epsilon]$ as fundamental units and from them derive the remaining two units

$$\left.\begin{aligned} [H] &= [E\epsilon^{\frac{1}{2}}\mu^{-\frac{1}{2}}], \\ [\mu] &= [E^{2}\epsilon]. \end{aligned}\right\} \tag{23}$$

With $n = 5$ and $k = 3$, $n - k = i = 2$ and there are only two of the Π's. To determine them we have by equation (14)

$$\left.\begin{aligned} [\Pi_1] &= [E^{\alpha_1}\epsilon^{\beta_1}\mu^{\gamma_1}H] = [1], \\ [\Pi_2] &= [E^{\alpha_2}\epsilon^{\beta_2}\mu^{\gamma_2}u] = [1]. \end{aligned}\right\} \tag{24}$$

To determine the exponents, we might substitute the dimensions given in equations (22); but since we already have the dimensions of $[H]$ and $[u]$ in terms of $[E, \epsilon, \mu]$ by equations (23), it is easier not to refer back to the complicated $[m, l, t, \mu]$ equations but use the $[E, \epsilon, \mu]$ system at once, Equations (24) then give us

$$\left.\begin{aligned} [\Pi_1] &= [E^{\alpha_1}\epsilon^{\beta_1}\mu^{\gamma_1}E\epsilon^{\frac{1}{2}}\mu^{-\frac{1}{2}}] = [1] \\ [\Pi_2] &= [E^{\alpha_2}\epsilon^{\beta_2}\mu^{\gamma_2}E^{2}\epsilon] = [1]. \end{aligned}\right\} \tag{25}$$

From the first of these we obtain the values

$$\alpha_1 = -1, \qquad \beta_1 = -\tfrac{1}{2}, \qquad \gamma_1 = \tfrac{1}{2},$$

so that we may write

$$\Pi_1 = \frac{\mu^{\frac{1}{2}}H}{\epsilon^{\frac{1}{2}}E}$$

or more conveniently

$$\Pi_1 = \frac{\mu H^2}{\epsilon E^2}.$$

From the second of equations (25) we have

$$\alpha_2 = -2, \qquad \beta_2 = -1, \qquad \gamma_2 = 0,$$

whence

$$\Pi_2 = \frac{u}{\epsilon E^2},$$

and the equation which corresponds to (9) or (13) therefore has the form

$$\psi\left(\frac{\mu H^2}{\epsilon E^2}, \quad \frac{u}{\epsilon E^2}\right) = 0. \tag{26}$$

Solving this for $u/\epsilon E^2$ and multiplying by ϵE^2 we have, finally,

$$u = \epsilon E^2 \varphi_1\left(\frac{\mu H^2}{\epsilon E^2}\right). \tag{27}$$

For the sake of illustration we have chosen to obtain this result by means of the general process described in the earlier sections: but the result is obvious without the aid of any such elaborate machinery. For since by equation (23) $[u] = [\epsilon E^2]$, it is evident directly from the principle of dimensional homogeneity that if u is to be expressed as a function of ϵ and E it can only be in the form ϵE^2 multiplied by a dimensionless number.

By taking $[H, \epsilon, \mu]$ as fundamentai, instead of $[E, \epsilon, \mu]$ we should have got the obviously equivalent result

$$u = \mu H^2 \varphi_2\left(\frac{\epsilon E^2}{\mu H^2}\right).$$

Assuming that the complete formula is

$$u = \frac{1}{8\pi}(\epsilon E^2 + \mu H^2)$$

we have

$$\varphi_1(x) = \varphi_2(x) = \frac{1 + x}{8\pi}.$$

If the medium is not isotropic, certain angles which fix the directions of E and H with respect to the principal axes of ϵ and μ must also appear as arguments of the unknown functions φ_1 and φ_2.

(*B*) *Relation between Mass and Radius of an Electron.*—There is no object in limiting our considerations to a particular kind of disembodied charge moving in free space, and we may as well make the treatment more general.

Let T be the energy of a charge e of any fixed distribution and of a size specified by any one of its linear dimensions D, when it is moving at the speed S through a medium of permeability μ and inductivity ϵ. If r', r'', etc., are ratios of lengths which specify the distribution of the charge, and if we assume that, for a fixed distribution, T does not depend on any other quantities than those already named, we must have

$$f(T, e, D, S, \mu, \epsilon, r', r'', \cdots) = 0. \tag{28}$$

For measuring all these quantities, an absolute system requires four fundamental units, the only new electrical quantity not mentioned in equations (22) being e, which has the dimensions

$$[e] = [m^{\frac{1}{2}} l^{\frac{1}{2}} \mu^{-\frac{1}{2}}].$$

Since we wish finally to solve for T, this must be one of the $[P]$'s of equation (14) and for the $[Q]$'s we must take four of the five units $[e, D, S, \mu, \epsilon]$. The five different combinations of these units, taken four at a time are

$$D, S, e, \epsilon;\ D, S, e, \mu;\ D, S, \epsilon, \mu;\ D, e, \epsilon, \mu;\ S, e, \epsilon, \mu.$$

But since $[\mu\epsilon] = [S^{-2}]$ as is seen from equations (22) S, ϵ, and μ can not be used simultaneously as fundamental units, so that our choice is limited to the combinations

$$D, S, e, \epsilon;\ D, S, e, \mu;\ D, e, \epsilon, \mu.$$

The three separate sets of units are then as follows:

$$\left.\begin{array}{lllll} 1. & [D, S, e, \epsilon], & [\mu] = [S^{-2}\epsilon^{-1}], & [T] = [D^{-1}e^2\epsilon^{-1}], \\ 2. & [D, S, e, \mu], & [\epsilon] = [S^{-2}\mu^{-1}], & [T] = [D^{-1}S^2e^2\mu], \\ 3. & [D, e, \epsilon, \mu], & [S] = [\epsilon^{-\frac{1}{2}}\mu^{-\frac{1}{2}}], & [T] = [D^{-1}e^2\epsilon^{-1}], \end{array}\right\} \tag{29}$$

and we can get a solution by using any one of these sets in equation (14) for finding Π_1 and Π_2.

In any case, it is easily seen that $S^2\mu\epsilon$ will be one of the two Π's. The remaining Π is to be found from one of the equations

$$\begin{aligned} [\Pi] &= [D^\alpha S^\beta e^\gamma \epsilon^\delta D^{-1} e^2 \epsilon^{-1}] = [1], \\ [\Pi] &= [D^\alpha S^\beta e^\gamma \mu^\delta D^{-1} S^2 e^2 \mu] = [1], \\ [\Pi] &= [D^\alpha e^\beta \epsilon^\gamma \mu^\delta D^{-1} e^2 \epsilon^{-1}] = [1]. \end{aligned} \tag{30}$$

From these three equations we get successively

$$\Pi = \frac{D\epsilon T}{e^2}, \quad \Pi = \frac{DT}{S^2 e^2 \mu}, \quad \Pi = \frac{D\epsilon T}{e^2}.$$

Since the first and third solutions are identical we have only two different forms of equation corresponding to (13), and they are

$$\psi_1\left(\frac{D\epsilon T}{e^2},\ S^2\mu\epsilon,\ r',\ r''\ \cdots\right) = 0, \tag{31}$$

$$\psi_2\left(\frac{DT}{S^2e^2\mu},\ S^2\mu\epsilon,\ r',\ r''\ \cdots\right) = 0, \tag{32}$$

or solving for T,

$$T = \frac{e^2}{D\epsilon}\varphi_1(S^2\mu\epsilon,\ r',\ r''\ \cdots), \tag{33}$$

$$T = \frac{S^2e^2\mu}{D}\varphi_2(S^2\mu\epsilon,\ r',\ r'',\ \cdots). \tag{34}$$

It is interesting to consider the physical meaning of these results. We did not restrict T to being merely energy due to motion: it is the total energy and it must therefore reduce to T_0, the electrostatic energy of the charge, when $S = 0$. Equation (33) accordingly gives

$$T_0 = \frac{e^2}{D\epsilon}\varphi_1(0,\ r',\ r'',\ \cdots)$$

and the agrees, as it should, with the known fact that the work done in collecting a charge e, in a medium of inductivity ϵ, into a distribution of linear size D is proportional to $e^2/D\epsilon$, the proportionality factor depending on the shape of the distribution, *i. e.*, on the ratios r', r'', etc. Since equation (34) must give the same result, φ_2 must contain the factor $1/S^2\mu\epsilon$ in order to keep T finite as S vanishes. If this factor is taken out the two equations become identical.

If we had originally let T represent only T_e, the part of the energy due to motion or the work required to start the charge going from rest, nothing in the reasoning would have been changed and the resulting equations would have had the same form as above. If we define $T_e/S^2 = m_e$ as the electrical mass of the moving charge, equation (34) gives us

$$m_e = \frac{e^2\mu}{D}\varphi_3(S^2\mu\epsilon,\ r',\ r''\ \cdots),$$

and if we set $\mu\epsilon = c^{-2}$, we have

$$m_e = \frac{e^2\mu}{D}\varphi_4\left(\frac{S}{c},\ r',\ r'',\ \cdots\right). \tag{35}$$

If we now let the charge be an electron of fixed quantity moving in free space so that ϵ, μ and c are constant, the equation takes the simpler form

$$m_e = \frac{1}{D}f(S,\ r',\ r'',\ \cdots) \tag{36}$$

or in words: the electrical mass of an electron of fixed distribution is inversely proportional to its linear dimensions; it also depends on the speed and on the distribution, but the nature of this dependence is not determinable from dimensional reasoning.

(*C*) *Radiation from an Accelerated Electron.*—Let a disembodied charge e be moving with the velocity S through a medium of permeability μ and inductivity ϵ. Let the distribution of the charge be specified by the length ratios r', r'', etc., and its size by some linear dimension D. Let the charge have a resultant acceleration a which makes an angle θ with S, θ being dimensionless like the r's. Let R be the total time rate of loss of energy by radiation.

So far as we can tell a priori, R may depend on all of the circumstances mentioned above, and if it does not depend on any others, we may write

$$f(R, D, S, a, e, \mu, \epsilon, \theta, r', r'' \cdots) = 0. \tag{37}$$

We have, in this instance, $n = 7$, $k = 4$ hence $i = 3$ and the equation will be reducible to the form

$$\psi(\Pi_1, \Pi_2, \Pi_3, \theta, r', r'', \cdots) = 0. \tag{38}$$

It shortens the process of solution to remark that $S^2\mu\epsilon$ and Da/S^2 are both dimensionless, so that as there can be at most only three such *independent* dimensionless products, only one remains to be found and these two can be used for two of the Π's, whatever form of solution we choose.

Since we wish ultimately to solve for R we must make $[R] = [P]$ in finding the third Π, and the four $[Q]$'s are to be selected from among $[D, S, a, e, \epsilon, \mu]$. Since the two relations $[S^2\mu\epsilon] = [1]$ and $[DaS^{-2}] = [1]$ are already given among the 6 units, these two combinations can not occur and there are only 6 combinations of 4 independent units each, instead of the 15 which there would otherwise be. Using these 6 combinations, together with $[R] = [P]$, successively in an equation of the form (14), we find that the resulting forms obtained for our third Π are

$$\Pi = \frac{e^2S^3\mu}{RD^2}, \quad \frac{e^2S}{RD^2\epsilon}, \quad \frac{e^2a^{\frac{3}{2}}\mu}{RD^{\frac{1}{2}}}, \quad \frac{e^2a^{\frac{1}{2}}}{RD^{\frac{3}{2}}\epsilon}, \quad \frac{e^2a^2\mu,}{RS}, \quad \frac{e^2a^2}{RS^3\epsilon}. \tag{39}$$

The 6 resulting forms of equation (38) are all equivalent, and it is sufficient to consider any one of them, *e. g.*,

$$\psi\left(\frac{e^2a^2\mu}{RS}, \quad S^2\mu\epsilon, \quad \frac{Da}{S^2}, \theta, ,r', r'', \cdots\right) = 0, \tag{40}$$

which we may put into the form

$$R = \frac{e^2a^2\mu}{S}\varphi\left(S^2\mu\epsilon, \frac{Da}{S^2}\theta, r', r'', \cdots\right). \tag{41}$$

Since D, a, and S appear in the unknown function, this equation gives us no definite information except that R is proportional to e^2, and we can not tell anything about how R may depend on a or S. It is therefore evident that we must assume a more limited range of dependence of R if we are to get any more definite results. If we assume that R does not depend on the linear size of the charge, φ must be independent of D, therefore of Da/S^2, and therefore of a. Hence on this hypothesis equation (41) reduces to

$$R = \frac{e^2a^2\mu}{S}\varphi_1\,(S^2\mu\epsilon, \theta, r', r'', \cdots). \tag{42}$$

If we assume that R is independent of S but make no assumption as to D, we can not eliminate a from the unknown function because S appears, in equation (41), in two of the independent arguments of φ. If we assume that R does not depend on either S or D, φ_1 of equation (42) must contain S as a factor and therefore $S\mu^{\frac{1}{2}}\epsilon^{\frac{1}{2}}$, so that the equation reduces to

$$R = e^2a^2\mu^{\frac{3}{2}}\epsilon^{\frac{1}{2}}\varphi_2(\theta, r', r'', \cdots). \tag{43}$$

Either (42) or (43) might have been obtained by making the necessary exclusion of variables from equation (37) and working the result out separately. It is, however, much more instructive to include, at the start, all the quantities which we can reasonably suppose might be of importance, and then carry out our exclusion after the general equation (41) has been obtained.

If we now restrict our considerations to a charge of fixed amount and of fixed shape and size; and if we further suppose it to move always in the same medium, e, D, μ, ϵ, r', r'', etc., are all constant and equations (41), (42), and (43) degenerate into

$$R = a^2f_1(S, a, \theta), \tag{41, a}$$
$$R = a^2f_2(S, \theta), \tag{42, a}$$
$$R = a^2f_3(\theta). \tag{43, a}$$

These are the equations for the radiation of an electron of specified shape and size moving in free space. As we see, the simple form (43, *a*) can not be obtained without assumptions which are far from plausible.

9. *Thermal Transmissivity*.—For variety, we may illustrate the use of the same general method by applying it to a thermal problem, namely that of the transmission of heat between the wall of a metal pipe and a

stream of fluid which is flowing through it and is hotter or colder than the pipe. Although a great many experiments have been made on this important practical subject, our information is still very incomplete, and the method of dimensions may be of service, both in planning experiments, and in analyzing and interpreting the results obtained.

Let the pipe be of uniform section and long compared with its greatest diameter. Let the shape of its section be specified by a number of length ratios, which we will represent by a single symbol r, and let D be any one dimension, such as the diameter if the pipe is round. Let S be the mean linear speed of the fluid at any section, as measured by the rate of discharge. Let θ be the absolute temperature of the wall surface at any section, and $\Delta\theta$ the difference between this and the mean temperature of the fluid at that section. There will be a flow of heat between the pipe and the fluid in one direction or the other, according to the direction of the temperature drop, and until the contrary is shown we must assume that this rate of heat transmission may depend on D, S, θ, $\Delta\theta$, and the properties of the fluid.

We shall suppose that the part played by radiation is negligible, thereby excluding the consideration of such cases as flame in boiler tubes; and the thermal properties of the fluid which need attention are then its thermal conductivity λ and its specific heat C. The rate of transmission will, in general, be affected by convection, so that we must take account of the mechanical properties which determine the nature of the motion of the fluid, namely its density ρ and viscosity μ. If the fluid is a gas, the compressibility may also need to be taken into consideration; but it appears that at speeds which are less than one half that of sound in the medium, this element may be disregarded, gases behaving sensibly like liquids of the same density and viscosity. We shall limit our considerations to these moderate speeds, so that such results as are obtained will not be applicable without modification to the transmission of heat between a steam-turbine nozzle and the jet flowing through it, or to similar cases where the speed is very high.

Let $\tau\Delta\theta$ be the heat transmitted per unit time through unit area of wall surface, τ being known as the transmission coefficient or "transmissivity." To obviate the need of introducing the mechanical equivalent of heat, we may suppose quantities of heat to be measured in absolute work units derived from the fundamental mechanical units $[m, l, t]$ which, together with the temperature unit $[\theta]$, will suffice for all the quantities with which we have to deal. We do *not* assume that the transmissivity τ is independent of $\Delta\theta$; that question is left open.

If we have not overlooked any of the circumstances which have a sensible effect on the heat transmission, we may now write

$$f(\tau, D, S, \Delta\theta, \rho, \mu, \lambda, C, \Delta\theta/\theta, r) = 0 \qquad (44)$$

θ and $\Delta\theta$ being quantities of the same kind, so that only one of them appears in the list of variables while the other is represented by a ratio, $\Delta\theta/\theta$. The number of different kinds of quantity is $n = 8$; the number of fundamental units required is $k = 4$; hence $i = 4$ and the equation, whatever its precise form, must be reducible to

$$\psi(\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Delta\theta/\theta, r) = 0, \qquad (45)$$

We wish to find an expression for the transmissivity; hence in finding the Π's by means of equation (14), τ must be one of the P's. In the process of solution let us set

$$[\rho, D, S, \Delta\theta] = [Q_1, Q_2, Q_3, Q_4],$$
$$[\tau, \mu, \lambda, C] = [P_1, P_2, P_3, P_4].$$

The dimensions of these quantities on the $[m, l, t, \theta]$ system are

$$[\rho] = [ml^{-3}], \qquad [\tau] = [mt^{-3}\theta^{-1}],$$
$$[D] = [l], \qquad [\mu] = [ml^{-1}t^{-1}],$$
$$[S] = [lt^{-1}], \qquad [\lambda] = [mlt^{-3}\theta^{-1}],$$
$$[\Delta\theta] = [\theta], \qquad [C] = [l^2t^{-2}\theta^{-1}],$$

and if we use these values in solving equations (14) for the four Π's, the usual routine procedure gives us the equation

$$\psi\left(\frac{\tau\Delta\theta}{\rho S^3}, \frac{\mu}{\rho DS}, \frac{\lambda\Delta\theta}{\rho DS^3}, \frac{C\Delta\theta}{S^2}, \frac{\Delta\theta}{\theta}, r\right) = 0 \qquad (46)$$

or

$$\tau = \frac{\rho S^3}{\Delta\theta}\varphi\left(\frac{\mu}{\rho DS}, \frac{\lambda\Delta\theta}{\rho DS^3}, \frac{C\Delta\theta}{S^2}, \frac{\Delta\theta}{\theta}, r\right) \qquad (47)$$

as a form to which the equation for τ must be reducible if our initial assumptions regarding the dependence of τ on the other quantities were correct.

Equation (47) conveys no definite information whatever, but we may give a few indications of how such an equation may, nevertheless, be utilized in supplementing incomplete experimental data or in planning new experiments.

Since dimensional reasoning can give us no further help, we turn to experiment. It is known that while at low speeds we may have stream-line motion of a fluid through a smooth straight pipe, this form of motion becomes unstable at higher speeds and breaks up into turbulent motion. We shall suppose that the speed is high enough and the pipe sufficiently rough that the motion of the fluid is very turbulent. It is known, further, that under these conditions the mechanical behavior of the fluid and the nature of its motion are nearly independent of the value of the viscosity. And since in turbulent motion, convection will certainly play an important part in the phenomenon of heat transmission, and the nature of the fluid motion will therefore be important, it is legitimate to assume, as an approximation at all events, that μ does not appear in the equation for τ which applies to these conditions of flow. The variable $\mu/\rho DS$ will therefore be absent and equation (47) will assume the simpler form

$$\tau = \frac{\rho S^3}{\Delta\theta}\varphi\left(\frac{\lambda\Delta\theta}{\rho DS^3}, \frac{C\Delta\theta}{S^2}, \frac{\Delta\theta}{\theta}, r\right). \tag{48}$$

We must now resort to experiments on transmission, for information about the form of φ, varying the arguments of φ separately and determining corresponding values of τ. To vary one of these arguments we have to vary its separate factors, which are the physical quantities over which we have direct control; and it is usually most convenient in practice to vary these separate factors one at a time,—for instance, to find the relation of τ to S when everything else is kept constant. If we are to vary a single one of the arguments of φ by varying a particular one of the physical quantities in question, that quantity must appear in only one of the arguments. This is true, in equation (48), of ρ, λ, C, and θ, so that we could proceed at once to investigate the form of φ by making experiments on the relations of τ to these quantities separately. On the other hand, D, S, and $\Delta\theta$ appear in more than one argument, so that we could not at once interpret the results of experiments in which one of these quantities was varied.

Now it is not practicable to vary the density, conductivity, and specific heat of the fluid arbitrarily and independently, though we may keep them all constant by making all our experiments on the same fluid. Furthermore, while θ may be varied independently of D, S, and $\Delta\theta$, θ inevitably influences the properties of the fluid, which can not be kept entirely constant during variations of temperature; and, in addition, attempts to vary θ over a wide range may encounter formidable difficulties. The quantities ρ, λ, C, and θ are thus precisely the ones which we do not want to use as independent variables. In practice, the most natural and convenient mode of experimentation is to vary S or $\Delta\theta$; and if we have various pipes available, D also may be varied. Hence equation (48) is not at present in a suitable form for our purpose and the arguments of φ must be replaced by others which are still independent and dimensionless but in which D, S, and $\Delta\theta$ are, if possible, separated.

Before proceeding to this transformation, we shall first limit our considerations to pipes of a particular shape, *e. g.*, round pipes, which are the the most important. The ratios r are then constant, and so long as it is understood that we refer only to round pipes, r may be omitted from the equations, the effect of varying shape being left for separate investigation, after the study has been completed for round pipes.

Since the speed S is the easiest of our quantities to vary arbitrarily, we attend to it first: it appears in two of the arguments of φ and we will therefore replace one of these by another which does not contain S. Since the form of φ is unknown, we may raise any one of its arguments to any desired power. We take the 2/3 power of the first and notice that

$$\left(\frac{\lambda\Delta\theta}{\rho D S^3}\right)^{\frac{2}{3}} = \frac{C\Delta\theta}{S^2}\left(\frac{\lambda^2}{\rho^2 C^3 D^2 \Delta\theta}\right)^{\frac{1}{3}}.$$

Let

$$\frac{\lambda^2}{\rho^2 C^3} = K,$$

K being then a quantity which involves only properties of the fluid and may be regarded as one of its characteristic constants. We may now write

$$\left(\frac{\lambda\Delta\theta}{\rho D S^3}\right)^{\frac{2}{3}} = \frac{C\Delta\theta}{S^2}\left(\frac{K}{D^2 \Delta\theta}\right)^{\frac{1}{3}}.$$

But any function of xy and y may be expressed as a function of x and y or of x^m and y^n: hence we may replace (48) by the equivalent equation, referring to a fixed shape of cross section,

$$\tau = \frac{\rho S^3}{\Delta\theta}\varphi_1\left(\frac{K}{D^2\Delta\theta}, \frac{C\Delta\theta}{S^2}, \frac{\Delta\theta}{\theta}\right), \tag{49}$$

in which D, S and θ are separated, though $\Delta\theta$ remains involved in all three arguments. Equation (49) is suitable for the interpretation of experimental data on τ obtained by varying D, S, and θ separately, for the variations will vary the three arguments of φ_1 separately and so tell us how φ_1 varies with the *whole* of any one of its three arguments.

A sufficiently complete and accurate experimental investigation of this sort would, in principle, always enable us to find the complete form of the operator φ_1, and the use of dimensional reasoning has the advantage that it enables us to plan the experiments rationally. It may turn out that the form of φ_1 is so complicated and the investigation so laborious that a complete solution of the problem is virtually impossible, or the result to be obtained not of sufficient importance to warrant the labor involved. In such instances the use of the principle of similarity, *i. e.*, reducing the unknown function to a constant by keeping all its arguments constant, sometimes permits of our securing partial information which suffices for particular practical purposes, and this will be illustrated in the following section. On the other hand, especially in cases where the quantities in question can not be measured very accurately or where no great accuracy in the results is required, it may happen that, to the approximation needed, the form of the unknown function is very simple. The method of procedure in such an instance may be illustrated by continuing the consideration of transmissivity.

Returning to equation (49), let us consider variations of the speed S. It appears from experiment that the transmissivity is nearly proportional to the 0.8 power of the speed when other things are constant; and merely to illustrate how such a result might be used, we shall suppose this relation to be exact. It follows that S can not be involved in φ_1 except as a factor $S^{-2.2}$. And since S appears only in the argument $C\Delta\theta/S^2$, φ_1 must contain the factor $(C\Delta\theta/S^2)^{1.1}$. Equation (49) must therefore have the more specific form

$$\tau = \rho S^{0.8} C^{1.1} \Delta\theta^{0.1} \varphi_2(K/D^2\Delta\theta,\ \Delta\theta/\theta). \tag{50}$$

This is simpler than before and suitable for continuing the work by varying the diameter and wall temperature of the pipe, or by using various fluids so as to vary K, the values of ρ, λ, and C being assumed to be known for the fluids used.

We will suppose, however, that it is not practicable to vary θ through any wide range and that we prefer to make experiments with various values of the temperature drop $\Delta\theta$ before altering D, which requires the dismantling of the apparatus and the substitution of a new pipe of different diameter. We must then transform equation (50) in such a way that $\Delta\theta$, which we are to use experimentally as the independent variable, appears in only one of the arguments of the unknown function. This is evidently accomplished by writing

$$\tau = \rho S^{0.8} C^{1.1} \Delta\theta^{0.1} \varphi_3(K/D^2\Delta\theta,\ K/D^2\theta), \tag{51}$$

which is suited to the interpretation of experiments on the dependence of τ on $\Delta\theta$.

It is commonly assumed that so long as $\Delta\theta$ is small, τ is independent of $\Delta\theta$. If experiment were to show that this relation was a general one, it would thereby be proved that φ_3 must contain $(K/D^2\Delta\theta)^{0.1}$ as a factor, and equation (51) would receive the still more specific form

$$\tau = \frac{\rho S^{0.8} C^{1.1} K^{0.1}}{D^{0.2}} \varphi_4\left(\frac{K}{D^2\theta}\right). \tag{52}$$

Having reached this point, the investigation might be completed by varying D, or changing the fluid so as to vary the value of K, the two methods providing a mutual check. If θ also were varied, a second check would be provided.

If, to take a purely hypothetical case, it were found that, in pipes for which τ is sensibly proportional to $S^{0.8}$, and within temperature limits such that τ is sensibly independent of $\Delta\theta$, τ was also sensibly independent of the diameter D, we should know that within these limits φ_4 could be represented with sensible accuracy by $(D^2\theta/k)^{0.1} \times$ constant and τ by the equation

$$\tau = \text{const.} \times \rho S^{0.8} C^{1.1} \theta^{0.1}.$$

Or if, to take another imaginary result, it were found that the transmissivity, beside being independent of the viscosity, proportional to the 0.8 power of the speed, and independent of the temperature difference, was also independent of the temperature of the wall surface, we should know that the expression for τ must be

$$\tau = \text{const.} \times \frac{\rho^{0.8} S^{0.8} C^{0.8} \lambda^{0.2}}{D^{0.2}}.$$

It would be out of place here to pursue this subject into an analysis of the numerous but unhomogeneous data which have been published concerning transmissivity. Enough has been said to illustrate the procedure and to show that the utility of the dimensional method is by no means confined to its applications to hydrodynamics or electromagnetic theory.

10. *An Illustration of Dynamical Similarity.*—The application of dimensional reasoning to mechanical problems is often useful in the interpretation of model experiments designed to furnish, at a comparatively small expense, information about the performance to be expected from full-sized machines. Advantage is then taken of the idea of dynamical similarity—a particular case of physical similarity in general. Since this subject seems to be less familiar to physicists than it deserves to be, a single illustrative example may, perhaps, be worth giving.

It was found, in section 5, that if the thrust F of a screw propeller of given shape and immersion can be assumed to depend only on the diameter D, the speed of advance S, the number of turns per unit time n, the density and viscosity of the liquid ρ and μ, and the acceleration of gravity g, we must have the relation given by equation (20) or

$$F = \rho D^2 S^2 \varphi\left(\frac{Dn}{S}, \frac{\rho DS}{\mu}, \frac{Dg}{S^2}, r', r'', \cdots\right) \tag{53}$$

in which the ratios r specify the shape and immersion of the propeller.

The principle of dynamical similarity states that in passing from one screw propeller to a second, in the same or in another liquid, any three kinds of quantity, such as (ρ, D, S), which can provide fundamental units, may be changed in any ratios whatever; and that the equation which connects the thrust with the other quantities will remain precisely the same if the values of the arguments of φ remain unchanged. This means, in simpler language, that if we find the value of the constant N in the equation

$$F = N\rho D^2 S^2$$

from an experiment in which the arguments of φ have a certain fixed set of values, the same constant is applicable to any values of (ρ, D, S) if the values of Dn/S, $\rho DS/\mu$, Dg/S^2, and the r's are the same in the second case as in the first.

The simplest of the requirements for the useful application of equation (53) is that the r's shall be constant; hence the two propellers, whatever their diameters, must be geometrically similar and similarly immersed; and the smaller may be called the model while the larger is called the original. The next simplest condition is that Dn/S shall remain constant. Now πDn is the speed of the circumferential motion of a point on the tip of one of the blades, and $\pi(Dn/S)$ is the tangent of the angle between the actual helical path of such a point and the direction of advance of the screw as a whole, which is supposed to coincide with the axis of the screw. The blades being of a fixed shape, the condition that Dn/S shall be constant is the same as the condition that the "angle of attack" of the blades on the still water into which they are advancing shall be constant. If p is the pitch of the propeller so that pn is the so-called "speed of the screw" or the speed at which it would advance if the water acted like a solid nut, $(pn - S)$ is the "slip" and $(pn - S)/pn$ is the "slip ratio." It is easily seen that if Dn/S is constant for propellers of a given shape, the slip ratio is constant. Our two conditions may now be expressed by saying that for two screw propellers to be dynamically similar, they must first of all have the same shape and be run at the same relative immersion and at the same slip ratio.

When the foregoing preliminary conditions are fulfilled, equation (53) reduces to the form

$$F = \rho D^2 S^2 \varphi\left(\frac{\rho DS}{\mu}, \frac{Dg}{S^2}\right) \tag{54}$$

and the next question is whether we can obtain any information about the thrust to be expected from a screw of diameter D run at the speed S, by experiments on a model screw of diameter D', run at the speed S' and at the same immersion and slip ratio as the original. The answer depends on our ability to arrange matters so that $\rho DS/\mu$ and Dg/S^2 shall be the same in the model experiment as in the practical operation of the full-sized original, and we at once encounter difficulties. In the first place, the intensity of gravity g is sensibly constant so that D/S^2 must also be kept constant. But on the other hand, we are virtually limited to experimenting in water for which ρ/μ is sensibly constant. Hence DS as well as D/S^2 must be kept constant, so that neither D nor S can be varied: in other words, we can not, in practice, run a reduced-scale model screw propeller so that it shall be dynamically similar to its original. We must therefore limit ourselves to a less ambitious program and attempt to obtain an approximate result which may be of some value, even though it is recognized as incomplete; and to do this we must find a plausible pretext for omitting one of the two arguments of φ from equation (54).

This presents no difficulty. For it is apparent from various hydrodynamic experiments that when a fluid is in very turbulent motion its mechanical behavior is little influenced by viscosity, density being much more important. Now the motion of the water about the blades of a screw propeller at ordinary working speeds is certainly very turbulent indeed, so that we may safely assume that if μ, *i. e.*, $\rho DS/\mu$, occurs at all in equation (54), it is only in terms with very small exponents. It is therefore a legitimate approximation to omit it altogether and write the equation in the simpler form

$$F = \rho D^2 S^2 \varphi\left(\frac{Dg}{S^2}\right). \tag{55}$$

Since gravity is sensibly constant, we can now make two propellers dynamically similar, if they satisfy the preliminary conditions regarding shape, immersion, and slip ratio, by running them at speeds such that D/S^2 is constant. The condition for "corresponding speeds" is therefore

$$\frac{S'}{S} = \sqrt{\frac{D'}{D}},$$

accented letters referring to the model and unaccented to the original. When the two are run at corresponding speeds we therefore have, by equation (55),

$$\frac{F}{F'} = \frac{\rho}{\rho'}\left(\frac{D}{D'}\right)^3.$$

If the model is run in water of the same density as that in which the full-sized propeller is to run, $\rho = \rho'$ and we have

$$F = F'\left(\frac{D}{D'}\right)^3.$$

If a propeller is very deeply immersed so that no disturbance of the water surface is produced, the weight of the water can have no influence on the thrust and g can not appear in equation (55). The unknown function then degenerates into a mere constant and the equation reduces to

$$F = N\rho D^2 S^2.$$

Any two propellers are then dynamically similar, whatever their speeds, if they have the same shape and are run at the same slip ratio, so that we have, for very deep immersion in a given liquid,

$$\frac{F}{F'} = \left(\frac{DS}{D'S'}\right)^2.$$

By disregarding viscosity we have, in effect, disregarded the effect of skin friction on the action of the propeller; and we have also left aside the question of cavitation. But without venturing further into the chaos of screw-propeller theory, the foregoing example will serve to illustrate the sort of use that may be made of dimensional reasoning in attacking mechanical problems which are—like most of those that occur in practical hydro- and aerodynamics—too difficult to be handled at all by ordinary methods.

11. *The Relation of the Law of Gravitation to Our Ordinary System of Mechanical Units.*—In our reasoning up to the present point, it has been assumed that three fundamental, *i. e.*, independent, units are required in an absolute system for measuring all the kinds of quantity needed in the description of purely mechanical phenomena, two more being required for thermal and electromagnetic quantities. If this assumption is permissible, a purely mechanical system may be kept similar to itself when any three independent kinds of mechanical quantity pertaining to it are varied in arbitrary ratios, by simultaneously changing the remaining kinds of quantity in ratios specified by equation (14), as described in section 6. We must now examine this assumption.

When we say that one quantity is derived from another or others which act as fundamental, we mean that by using or combining particular examples of these other kinds of quantitity in some specified manner, we can fix a quantity of the derived kind which has a particular definite magnitude. For instance, we derive a unit of force from independent fundamental units of mass, length, and time, by using these units in a certain way which is fixed by definition, and we thereby determine a definite force which is reproducible and may be used as a unit. Now by Newton's law of gravitation it is, in principle, possible to derive one of the three fundamental units of mechanics from the other two. Let two free masses be placed at rest at a distance apart which is very large compared with their linear dimensions. Let them be released and allowed to approach each other by a certain measured distance, and let the time required to cover this distance be observed. This interval of time is fixed by the masses and the distances: in other words, an interval of time can be derived from masses and lengths, and by adopting a suitable form of definition, a unit of time can be derived from the units of mass and length. It is, of course, immaterial which one of the three units is derived from the other two; the point is that if we utilize the law of gravitation, only two fundamental units are needed for mechanical quantities, instead of the three which physicists ordinarily use. By carrying out this process or some other equivalent to it, we should eliminate one of our three primary standards,—the international kilogram, the international meter, or the standard clock, namely the rotating earth which preserves the mean solar second. For practical purposes we should still use these three standards, but one of them would be reduced to the rank of a secondary or working standard.

One reason for not proceeding in this manner is that we do not yet know the value of the gravitation constant accurately enough to bring the proposition within the range of practicability. But since we must admit the theoretical possibility of such a procedure if we recognize the law of gravitation, it is incumbent upon us to consider what bearing this possibility may have on our dimensional reasoning and on our applications of the theorem of physical similarity; for the number of fundamental units needed is a matter of vital importance to our conclusions regarding any practical problem. For example; in treating the screw propeller, we assumed that $[m, l, t]$ were independent units and therefore that two propellers could be made to constitute dynamically similar systems when three quantities ρ, D, and S were varied in arbitrary ratios upon passing from one system to the other. The question now evidently presents itself: ought we not to have limited the arbitrary variations to two; are we not *bound* to treat mechanical quantities as derived from only two and not from three independent fundamental quantities?

To see the answer to this question, we may read over again the definition of physically similar systems given in section 6. It was found that a physical system remains similar to itself, as regards any relation among a number of kinds of quantity, when certain of these kinds—equal in number to the fundamental units required for the absolute measurement of all the quantities involved in the relation—are subject to variation in arbitrary ratios, if we fix the ratios in which the remaining kinds of quantity shall then change by imposing the condition that the Π's shall remain invariable. We now see that the answer to the question: how many fundamental mechanical units are to be used? *i. e.*, to the question whether we are or are not at liberty to ignore the law of gravitation, depends on the nature of the relation in question. If the relation with which we happen to be concerned refers to and characterizes some phenomenon which does not involve and is not affected by the form of the law of gravitation, we can carry out a complete investigation of the phenomenon and represent our results by a complete equation without ever knowing of the existence of the law of gravitation: this law does not concern us, and our knowledge of the phenomenon under investigation does not depend on our knowing the correct expression for the law of gravitation. We are therefore plainly at liberty to ignore it altogether, and if we do so, three fundamental units are indispensable because the only means of eliminating one of them is to use the law of gravitation. It is not necessary that the phenomenon be unaffected by the weight of material bodies, but merely that it be not sensibly dependent on the fact that weight is proportional to the mass of the Earth and to the inverse square of the distance from its center.

In the most general case, when we include within the field of our reasoning all kinds of physical quantity and *all possible relations* among them, we must admit our familiarity with the law of gravitation and limit ourselves to two fundamental mechanical units. But if for "all possible relations" we substitute "all relations that do not involve the law of gravitation," we may ignore the law and proceed as if it were non-existent.

With this single proviso all our foregoing reasoning retains its full validity. The limitation is seldom felt, because, in practice, physicists are seldom concerned with the law of gravitation: for all our ordinary physical phenomena occur subject to the attraction of an earth of constant mass and most of them occur under such circumstances that the variation of gravity with height is of no sensible importance. In precise geodesy and still more in astronomy, the observed phenomena do involve the operation of the law of gravitation in such a way that they can not be completely described without making explicit use of it. If the physical relations which characterize such phenomena are under discussion, we *must* recognize the law of gravitation, we *must* regard all mechanical units as derivable from two and not three independent fundamental units, and if a physical system is to remain similar to itself only four and not five arbitrary changes are possible, or if we exclude thermal and electromagnetic quantities, only two. The geodesist and the astronomer must therefore, in using dimensional reasoning, submit to one restriction from which the physicist is usually free, though this formal restriction is offset by the power of using the law of gravitation explicitly.

To take an illustration, let us suppose that we have to consider a phenomenon which involves mechanical and electromagnetic but not thermal quantities, and that the law of gravitation in its general form does not influence the phenomenon. The physical system in which this phenomenon occurs may remain similar to itself while four independent kinds of quantity Q are changed in any four arbitrary ratios, if all the other kinds P involved in the phenomenon are changed in the ratios specified by equation (14) taken with the arbitrary changes of the Q's. We may, for example, divide all lengths by x, divide all times by x, multiply all masses by x, and leave all electrical charges unchanged: the altered system will be similar to the original one as regards all phenomena that do not depend on the law of gravitation, if the remaining kinds of quantity are changed as shown by equation (14). But if the phenomenon involves the law of gravitation we can impose only three arbitrary ratios of change, of which one must refer to purely electromagnetic quantities: we can no longer impose arbitrary conditions on lengths, times, and masses but only on two of these kinds of magnitude. To put it in another way, and omitting electromagnetic quantities, which so far as we know have nothing to do with the case in hand, we may keep a gravitational system similar to itself while we change its size and its time intervals in any arbitrary ratios; but after the change, corresponding gravitational forces must stand in a determinate ratio which is not arbitrary. Or to make it less abstract, if we construct a miniature universe by multiplying all actual lengths by a, and if we change the densities in such a way that the mass of every volume element of the miniature universe is b times the mass of the corresponding volume element of the actual universe, then if the miniature universe is to be mechanically similar to the actual universe, the gravitational forces in the miniature universe must bear to the corresponding gravitational forces in the actual universe a ratio fixed by the law of gravitation. And if the speeds at which gravitational phenomena occur in the miniature universe are to have the same numerical values as corresponding speeds in the actual universe, the unit or time or speed can not be fixed arbitrarily but must have a particulaf relation to our actual unit.

Conclusion.—A convenient summary of the general consequences of the principle of dimensional homogeneity consists in the statement that any equation which describes completely a relation subsisting among a number of physical quantities of an equal or smaller number of different kinds, is reducible to the form

$$\psi(\Pi_1, \Pi_2, \cdots, \text{etc.}) = 0,$$

in which the Π's are all the independent dimensionless products of the form Q_1^x, Q_2^y, $\cdots$, etc. that can be made by using the symbols of all the quantities Q.

While this theorem appears rather noncommittal, it is in fact a powerful tool and comparable, in this regard, to the methods of thermodynamics or Lagrange's method of generalized coordinates. It is hoped that the few sample illustrations of its use which have been given will prove interesting to physicists who have not been in the habit of making much use of dimensional reasoning; but if this paper merely helps a little toward dispelling the metaphysical fog that seems to be engulfing us, it will have attained its object.

BUREAU OF STANDARDS,
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