ELECTRICITY AS A NATURAL PROPERTY OF RIEMANNIAN GEOMETRY

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Abstract

Field equations of a Riemannian geometry which can be deduced from a Hamiltonian principle have the following general property: Due to the conservation law of Einstein's curvature tensor R_{ik} there appears in the integration of the field equations a free vectorial function ϕ_i, determined, however, by the conservation law. This has been shown by applying a mathematical formulation of Mach's principle to the extremely weak deformation of a given arbitrary metric. Specifying the Hamiltonian function by the evident condition of gauge invariance this free vectorial function ϕ_i has all the fundamental properties of the electro-magnetic vector potential: the law of continuity is strictly fulfilled everywhere, the potential equation is to be deduced in first approximation and also the Lorentz' ponderomotive force of a particle. In this theory the material particle is to be considered as a proper solution of the field equations.

SEVERAL attempts have been made since Einstein's discovery of the theory of general relativity to find a solution of the problem "electricity" in a way analogous to that employed in the successful solution of the problem gravitation. These attempts are mostly characterized by the idea of enlarging the geometrical basis of Riemannian geometry with the tendency to find in this way a geometrical formation corresponding to the vector potential or to the antisymmetrical tensor of electromagnetic field-strength. The following development is characterized by the fact that Riemannian geometry is retained without any modification, discovering in this geometry itself an analytical element, which seems to correspond with a surprising harmony to the electromagnetic vector potential.

1. The Field Equations $R_{ik}=0$ as Characterization of Euclidian Geometry.

The tendency of Einstein has been to generalize the equations $g_{ik}=\text{const}$ by setting up a system of differential equations of second order, usually written in the form:

$$R_{ik} = 0. \tag{1.1}$$

The gravitational field of the sun, the moving of planets around the sun may be perfectly described by these equations.[1] However, we have to remark that such a solution of the field equations, is not without singularities. From the

[1] Even the assumption, the path of a planet may be a geodesic is not a new hypothesis, the dynamics of a mass-point being a consequence of the field equations. Cf. Einstein, Berl. Akad. Ber. 1927, p. 2; Lanczos, Zeits. f. Physik **59**, 514 (1930).

point of view of these equations matter appears as a singularity. Although from a purely mathematical standpoint the use of singularities is not to be rejected, several features suggest the uselessness of singularities in the description of nature. The singularity is always to be considered as the *failing* of a law. For example, to illustrate the situation by a simple familiar example: it makes no difference whether we say that we have the potential equation

$$\Delta\phi = 0 \tag{1.2}$$

permitting singularities or that we have Poisson's equation

$$\Delta\phi = -4\pi\rho \tag{1.3}$$

without singularities. The latter form of the equation shows that in some regions the law $\Delta\phi = 0$ is not fulfilled, since the function ρ in some regions differs from 0. In the same way Einstein's law (1.1) gives a Riemannian space but only if we admit singularities, in other words exceptions form the postulated law, since the singularity is equivalent to the fact that in some regions the tensor R_{ik} differs from 0. If we exclude singularities and demand the fulfilling of the law $R_{ik}=0$, in the *whole space*, adding some natural boundary conditions practically equivalent with the fact that even in infinity the law (1.1) may be fulfilled, we obtain *Euclidian geometry*. And so Einstein's curvature tensor R_{ik} is to be considered as a *fundamental formation* of Riemannian geometry, being the simplest invariant characterization of a geometry. We are accustomed to consider the Riemannian curvature tensor $R_{\alpha\beta\gamma\delta}$ as a perfect characterization of geometry, the vanishing of this tensor being a proof for Euclidian geometry, while the vanishing of R_{ik} is possible also in a curved world. But the consideration of the Riemannian tensor is only necessary considering certain *limited* regions of space. In this case the vanishing of R_{ik} may not be sufficient to introduce Euclidian geometry. But this insufficiency disappears considering the *whole* space without any interruption. The characterization of a geometry by the Riemannian tensor is indeed not of a natural kind, because it shows a high degree of *over-determination*. It is not satisfying to characterize the metric tensor g_{ik} which is a symmetric tensor of second degree by a tensor of fourth degree. Einstein's curvature tensor R_{ik}, however, has just the right degree of determination being also a symmetric tensor of second degree exactly corresponding to the fundamental metric tensor.

2. General Form of Field Equations for R_{ik} which have a Hamiltonian Principle

We shall consider Einstein's curvature tensor R_{ik} as a fundamental concept of Riemannian geometry, able to characterize such a geometry. The simplest statement $R_{ik}=0$ gives only the Euclidian geometry, for this reason we expect field equations for the R_{ik} corresponding to the general character of Riemannian geometry which is a differential geometry. We will suppose that these field equations are to be deduced from a Hamiltonian principle. The tensorial problem is then reduced to the consideration of a single invariant, the Hamiltonian function. This Hamiltonian function will be a function of the R_{ik}, but being an invariant it must contain also the g_{ik}. Any non-covariant components may then be deduced to the pure covariants using the g_{ik} writing for example

$$R^{ik} = R_{\alpha\beta} g^{\alpha i} g^{\beta k} \tag{2.1}$$

thus we can write our Hamiltonian function in the following form:

$$H = H(R_{ik}, g_{ik}). \tag{2.2}$$

Our action integral is:

$$I = \int H dv \tag{2.3}$$

dv being the volume element and the condition of minimum is the familiar:

$$\delta I = 0 \tag{2.4}$$

corresponding to an arbitrary variation of g_{ik}. To find our field equations in a useful form we shall consider in the process of variation at first the R_{ik} and the g_{ik} as *independent* variables. Doing that we can write the variation of our integral generally in the following form:

$$\delta I = \epsilon \int (\mu^{ik} \rho_{ik} - v^{ik} \gamma_{ik}) dv \tag{2.5}$$

where ϵ is an infinitesimal parameter, the variation of R_{ik} is denoted by ρ_{ik}, the variation of g_{ik} with γ_{ik}. We do not specify at first the Hamiltonian function and use therefore the general expressions u_{ik} and v_{ik}, which are easy to find if the Hamiltonian function is given.

The variation of R_{ik} and the variation of g_{ik} are related in a covariant way. The relation is given in a former investigation of the author, exploring the extremely weak fields in Einstein's theory.[2] We use the resulting formulae of this work without proof, the details of calculation may be found in the quoted paper.

We use the following notations. The operation $\partial/\partial x_i$ shall mean always a *covariant differentiation*, not the common differentiation. We introduce the invariant Laplace operator:

$$\Delta = g^{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \tag{2.6}$$

and

$$E(\gamma_{ik}) = \Delta\gamma_{ik} + R_i^{\alpha}\gamma_{k\alpha} + R_k^{\alpha}\gamma_{i\alpha} - 2R_{i\alpha k\beta}\gamma^{\alpha\beta} \tag{2.7}$$

$$D(\gamma_{ik}) = E(\gamma_{ik}) - \left(\frac{\partial\chi_i}{\partial x_k} + \frac{\partial\chi_k}{\partial x_i}\right) \tag{2.8}$$

[2] Lanczos, Zeits. f. Physik **31**, 112 (1925).

where the vector χ_i has the significance

$$\chi_i = \frac{\partial \gamma_i{}^\alpha}{\partial x_\alpha} - \frac{1}{2}\frac{\partial \gamma}{\partial x_i} \qquad (\gamma = \gamma_\alpha{}^\alpha). \tag{2.9}$$

We need also the "adjoint" expression of $D(\gamma_{ik})$:

$$F(\gamma_{ik}) = E(\gamma_{ik}) - \left(\frac{\partial \sigma_i}{\partial x_k} + \frac{\partial \sigma_k}{\partial x_i} - \frac{\partial \sigma^\alpha}{\partial x_\alpha} g_{ik}\right) \tag{2.10}$$

with

$$\sigma_i = \frac{\partial \gamma_i{}^\alpha}{\partial x_\alpha}\,. \tag{2.11}$$

The relation between ρ_{ik} and γ_{ik} is expressed by[3]

$$\rho_{ik} = \tfrac{1}{2}D(\gamma_{ik})\,. \tag{2.12}$$

Corresponding to the general properties of the "adjoint" differential expressions we can write

$$\int [u^{ik}D(\gamma_{ik}) - \gamma^{ik}F(u_{ik})]dv = \text{surface int.} \tag{2.13}$$

According to this equation we can transform the term with ρ_{ik} in our expression (2.5) and set equal 0 the resulting coefficient of γ_{ik}. We obtain in this way the field equations in the following form:

$$F\,(u_{ik}) = 2v_{ik} \tag{2.14}$$

3. A General Method of Integration. Appearance of a Free Vectorial Function ϕ_i in the Integrated Form of Field Equations

We can use also another method to find our field equations. If there is a unique connection between ρ_{ik} and γ_{ik} we can also choose the ρ_{ik} as the primary variables reducing the γ_{ik} to the ρ_{ik}. For the vanishing of δI this change cannot make any difference this vanishing being required for *any* variation. The physical meaning of this process corresponds to the idea that Einstein called the "principle of Mach".[4] This principle attempts to determine the metric of a manifold from a given distribution of matter characterized by the material tensor. This principle has a precise significance only in the case of a *weak* deformation of a given metric field and the variation corresponds to just such a supposition. Mathematically we have to solve the differential equation (2.12) determining the γ_{ik} from the ρ_{ik} by integration.

This process is indeed possible with a certain natural restriction. An arbitrary deformation γ_{ik} can be of two kinds: a real and an apparent deformation. The latter results from pure transformations of coordinates and has no importance. Its form is:[5]

[3] Reference 2, Eq. (16).
[4] Einstein, Ann. d. Physik **55,** 241 (1918).
[5] Reference 2, Eq. (30).

$$\gamma_{ik} = \frac{\partial \Phi_i}{\partial x_k} + \frac{\partial \Phi_k}{\partial x_i} \tag{3.1}$$

where Φ_i is an arbitrary vector. It is evident that we should exclude such apparent deformations, which neither produce matter nor influence our invariant integral.

We do that by *normalizing* our coordinate system. We can always add to a given γ-field an apparent field of the character (3.1) by making a suitable transformation of coordinates. We can now determine the vector Φ_i so that the vector χ_i, defined by (2.9) becomes 0 for the resulting field.

$$\chi_i = \frac{\partial \gamma_i{}^\alpha}{\partial x_\alpha} - \frac{1}{2}\frac{\partial \gamma}{\partial x_i} = 0 \tag{3.2}$$

that gives a vectorial differential equation for the Φ_i which is to be solved. The only condition is that the homogeneous equation does not have "proper solutions", i.e., any regular solution besides 0. This homogeneous equation is:

$$\frac{\partial \gamma_i{}^\alpha}{\partial x_\alpha} - \frac{1}{2}\frac{\partial \gamma}{\partial x_i} = 0 \tag{3.3}$$

with the assumption (3.1) for γ_{ik}. That gives[6]

$$\Delta \Phi_i - R_i{}^\alpha \Phi_\alpha = 0. \tag{3.4}$$

In order to apply Mach's principle we require that this equation does not have any solution besides $\Phi_i = 0$.

After normalising our coordinates we have between ρ_{ik} and γ_{ik} the simple connection

$$E(\gamma_{ik}) = 2\rho_{ik} \tag{3.5}$$

which differential equation is now *self-adjoint*. We can integrate this equation uniquely, and thus find a unique correspondence between γ_{ik} and ρ_{ik}, if the homogeneous equation

$$E(\gamma_{ik}) = 0 \tag{3.6}$$

has no proper solutions besides 0. However, a special proper solution cannot be avoided:

$$\gamma_{ik} = \kappa g_{ik} \tag{3.7}$$

where κ is a constant—disturbing the general claim of our method. But the significance of this exception is very easy to discover. The interpretation of (3.7) is a change in the *gauge*. That is not a real deformation and therefore does not produce any matter. On the other hand, it does not belong to the pure transformations of coordinates, because the value of ds^2 is changed, owing to the fact that the measure of a length is dimensioned and so depends upon the gauge used. Since this exceptional deformation is trivial and disappears if we normalize our gauge, its existence is not to be considered as an

[6] Reference 2, Eq. (32).

objection against Mach's principle. We see that our program to use Mach's principle becomes possible by normalizing the coordinates and normalizing the gauge. While the first normalization can be produced in a natural way by annuling the vector χ_i, the latter normalization remains artificial. But the change in the gauge is certainly without any influence on the result if we have a Hamiltonian integral which is "gauge-invariant", that means it does not depend on the gauge. We will use later just this principle to determine our Hamiltonian function. And it is therefore sufficient for our purpose to use any *arbitrary* normalization.

We have now attained the desired unique correspondence between ρ_{ik} and γ_{ik} and will only state that the possibility of accomplishing our program is given through the conditions that the equation (3.4) does not have any solution besides 0 and the equation (3.6) does not have any solution besides (3.7) with a constant κ. We will add the remark that a proper solution imposes always a condition for the right side of a differential equation: the condition of orthogonality. The homogeneous equation (3.6) having the proper solution (3.7) we get the following scalar condition which must be fulfilled by the material tensor ρ_{ik}:

$$\int \rho dv = 0 \quad (\rho = \rho_\alpha{}^\alpha). \tag{3.8}$$

The solution of the differential equation (3.5) by integration will be symbolized thus:

$$\gamma_{ik} = T(\rho_{ik}) \tag{3.9}$$

where T is a certain integral operator, the inverse operator of E.

Using this equation we are ready to consider the ρ_{ik} as the primary quantities, considering γ_{ik} as a function of the ρ_{ik}. However, we must notice an important fact. The γ_{ik} have been treated as completely arbitrary quantities. We made later a restriction excluding the apparent deformations, but this restriction does not influence the result. Regarding the ρ_{ik}, however, the situation is quite different. We cannot consider the material tensor as completely arbitrary, it must always obey the general vectorial conservation law of momentum and energy. This law is a mathematical identity in Einstein's theory of relativity and expresses a fundamental property of Riemannian geometry. As it is necessarily fullfilled also for the deformed geometry, we must observe it even during the variation. The condition which must be fulfilled in every point of the manifold is given by the following equation:[7]

$$\frac{\partial \rho_i{}^\alpha}{\partial x_\alpha} - \frac{1}{2}\frac{\partial \rho}{\partial x_i} = P_{\alpha\beta i}\gamma^{\alpha\beta} \tag{3.10}$$

putting

$$P_{\alpha\beta i} = \frac{1}{2}\left(\frac{\partial R_{i\alpha}}{\partial x_\beta} + \frac{\partial R_{i\beta}}{\partial x_\alpha} - \frac{\partial R_{\alpha\beta}}{\partial x_i}\right). \tag{3.11}$$

[7] Reference 2, Eq. (25). It is supposed that we use the normalized system of coordinates $\chi_i = 0$.

We have to consider this equation as an *auxiliary condition* of our variation problem, corresponding to the ingenious general method of Lagrange: the method of undetermined multipliers. Since in our case the condition is vectorial, the multiplier will be a *vector*, we denote it with $2\phi_i$. According to Lagrange's method we add to our variation integral the expression

$$2\int\left[\phi^i\left(\frac{\partial\rho_i{}^\alpha}{\partial x_\alpha}-\frac{1}{2}\frac{\partial\rho}{\partial x_i}\right)-P_{\alpha\beta i}\gamma^{\alpha\beta}\phi^i\right]dv. \tag{3.12}$$

In the same way we must treat the condition (3.8) which also imposes a restriction on the ρ_{ik}. This latter condition introduces a constant λ as Lagrangian factor and we have to add the expression

$$\lambda\int\rho_{ik}g^{ik}dv \tag{3.13}$$

to our variation. We are now authorized to treat our problem completely like a free variation problem without any restrictions, and we can equate according to the usual method the coefficients of ρ_{ik} to 0. But, since the derivatives of ρ_{ik} appear in the first term of (3.12) we will make a partial integration in the familiar way, finding,

$$-\int\rho^{ik}\left(\frac{\partial\phi_i}{\partial x_k}+\frac{\partial\phi_k}{\partial x_i}-\frac{\partial\phi^\alpha}{\partial x_\alpha}g_{ik}\right)dv. \tag{3.14}$$

Thus we find our field equations in the following form:[8]

$$U_{ik}=T(V_{ik}) \tag{3.15}$$

if we put

$$U_{ik}=u_{ik}+\lambda g_{ik}-\left(\frac{\partial\phi_i}{\partial x_k}+\frac{\partial\phi_k}{\partial x_i}-\frac{\partial\phi^\alpha}{\partial x_\alpha}g_{ik}\right) \tag{3.16}$$

$$V_{ik}=v_{ik}+2P_{ik\alpha}\phi^\alpha. \tag{3.17}$$

Finally we can transform the obtained Eq. (3.15) in a pure differential equation. We invert the integral operator T in the corresponding differential operator E, observing the two Eqs. (3.5) and (3.9) which are equivalent. We obtain then our field equations in the following form:

$$\left.\begin{aligned}U_{ik}&=2w_{ik}\\E(w_{ik})&=V_{ik}\end{aligned}\right\}. \tag{3.18}$$

The Eqs. (3.15) and (3.18) are equivalent. Compared with the original form (2.14) they represent a new formulation of the field equations. We have in fact but a new *form*, not a new *statement*, using in both cases the same Hamiltonian principle, only changing the variables, which process cannot influence the result. The difference is that the new formulation is an *integrated* form of our original field equations (2.14). This integration would certainly

[8] We employ in the deduction the property of T to be a *symmetric* operator: $\bar{T}=T$.

be possible also by a direct investigation of the field equations in its original form. But the derivation in an indirect manner, using the variation principle itself by means of Mach's principle, is much shorter, giving the result immediately.

The striking and important result is, that the integrated form of the field-equations shows the *appearance of a free vectorial function* ϕ_i. The necessity to introduce it lies in the conservation law which demands that we regard it as an auxiliary condition for the variation. We have to expect, therefore, that the same condition must give us a determination for the undetermined Lagrangian factor ϕ_i. That is indeed the fact. We have to consider our field-equations first of all as determining equations for the fundamental curvature tensor R_{ik}. To find a metric belonging to this tensor is only possible if the conservation law is fulfilled. It is not to be expected that an arbitrary solution of our field equations will possess this quality but the possibility of attaining this must exist. And that can occur by the use of the free function ϕ_i which has to be determined in such a manner that the conservation law becomes fulfilled. This gives just the right determination for ϕ_i, namely a vectorial differential equation in every point of our manifold. Anticipating the later physical interpretation we will call the vectorial function ϕ_i appearing as a free function in the integration of field equations and determined by the conservation law: the "vector potential".

It suggests itself to raise the question: What may be the reason that the original field equations of gravitation $R_{ik}=0$ do not show the appearing of any vector potential although according to the general theorem the vector potential appears always upon integrating our equations. The only condition for this theorem, viz., the existence of Hamiltonian principle, is fulfilled in the case of the equations $R_{ik}=0$.

The reason of this fact is very peculiar. The general method leads in this case to the following connection between curvature tensor and vector potential:[9]

$$R_{ik} = \frac{\partial \phi_i}{\partial x_k} + \frac{\partial \phi_k}{\partial x_i} \cdot \tag{3.19}$$

This connection corresponds perfectly to the connection (3.1) between an apparent deformation γ_{ik} produced by a pure transformation of coordinates, and the arbitrary vector ϕ_i. The conservation law, the divergence equation of the tensor $R_{ik}-\frac{1}{2}Rg_{ik}$ corresponds completely to the homogeneous Eq. (3.3) upon substituting the expression (3.2). One of the conditions for carrying out Mach's principle was that this equation may not have any solution besides 0. As a consequence this follows

[9] Lanczos, Zeits. f. Physik **32,** 163 (1925); Eq. (33). This apparent *enlargement* of Einstein's equations baffled the author the first time when he discovered the above explained method in the just mentioned paper. He thought that using Mach's principle we are able to *complete* the field equations by the vector potential. This erroneous idea was the hindrance to discover the general importance and real meaning of this method, namely to be a general method of *integration* without any change in the field equations.

$$\phi_i = 0. \tag{3.20}$$

And so just in the case of the gravitational equations the vector potential *remains latent*, disappearing owing to the conditional equation which it has to fulfill.

4. Gauge-Invariance of the Hamiltonian Intergral

The fact that the Riemannian ds^2 is a *dimensioned* quantity and therefore furnished with an arbitrary factor cannot be avoided. It is, however, a natural assumption that our action integral should not be affected by this arbitrariness. We have to seek a minimum for a certain function viz., the action integral. If there is an arbitrary gauge-factor influencing the value of this function, the minimum problem loses its meaning because it is possible to obtain any value by a proper choice of gauge. It is natural to require that the action-integral should not depend on the gauge, i.e. it has to be a *non-dimensioned* quantity, a pure number. In this case the arbitrariness of gauge is overcome; ("gauge-invariance"). In 4 dimensions the unit of volume element dv is $[L]$.[4] On the other hand the unit of R_{ik} is $[L]^{-2}$; (irrespective of the number of dimensions of the space). We see that the dependency of the Hamiltonian function on R_{ik} in 4 dimensions has to be of a *quadratic* kind.

We find only *two* invariants of the required quality built up upon the R_{ik}:[10]

$$H_1 = R_{\alpha\beta}R^{\alpha\beta} \tag{4.1}$$

$$H_2 = (R_{\alpha\beta}g^{\alpha\beta})^2 = R^2. \tag{4.2}$$

Therefore we should choose a linear combination of these two invariants.

$$H = H_1 + CH_2 \tag{4.3}$$

as our Hamiltonian function. C is a numerical constant whose value is not to be decided at present. It seems that the quality of the two possible functions is connected with the duality of electricity and gravitation in nature. The first by itself leading, as we will show, to the familiar phenomena of electricity, the second by itself leading to the pure gravitational equations. The combination of both seems to be necessary to build up a material particle as a "proper solution" of the field equations.

We are now able to replace in the field equations the previously undetermined u_{ik} and v_{ik} by the specified values which we obtain from the specified Hamiltonian function. We only need to carry out the variation with respect to R_{ik} and g_{ik}, considering these two kinds of quantities as independent of each other. We find without any difficulty the following expressions. Using the first invariant:

$$u_{ik} = 2R_{ik} \tag{4.4}$$

$$v_{ik} = 2(R_{i\alpha}R_k^{\ \alpha} - \tfrac{1}{4}R_{\alpha\beta}R^{\alpha\beta}g_{ik}) \tag{4.5}$$

[10] The invariant $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is likewise gauge-invariant but does not correspond to our point of view, because the Riemannian curvature tensor $R_{\alpha\beta\gamma\delta}$ cannot be reduced to Einstein's curvature tensor R_{ik}.

Using the second invariant:

$$u_{ik} = 2Rg_{ik} \tag{4.6}$$

$$v_{ik} = 2R(R_{ik} - \tfrac{1}{4}Rg_{ik}). \tag{4.7}$$

The construction of v_{ik} produced by the first invariant shows a great analogy to Maxwell's stress-tensor being built up in the same way upon the symmetric tensor R_{ik} as Maxwell's tensor upon the antisymmetric tensor F_{ik} the electromagnetic field-strength. This analogy becomes even greater considering the fact, which we shall deduce later, that to the first approximation we have

$$R_{ik} = \frac{\partial \phi_i}{\partial x_k} + \frac{\partial \phi_k}{\partial x_i}$$

while on the other hand F_{ik} is the same combination but antisymmetrically:

$$F_{ik} = \frac{\partial \phi_i}{\partial x_k} - \frac{\partial \phi_k}{\partial x_i}.$$

We observe the remarkable quality that the scalar $v = v_\alpha{}^\alpha$ *vanishes* in the first as well as in the second case.

$$v = v_\alpha{}^\alpha = 0. \tag{4.8}$$

5. First Integral of our Field-Equations

The last mentioned quality of the v_{ik} permits us to find a scalar integral of our field equations. We consider the field equations in their original form (2.14). We build the scalar equation

$$g^{ik}F(u_{ik}) = 2g^{ik}v_{ik}. \tag{5.1}$$

The right side vanishes in consequence of (4.8). The left side is reduced to a single term if we utilize the fundamental conservation law:

$$\frac{\partial R_i{}^\alpha}{\partial x_\alpha} - \frac{1}{2}\frac{\partial R}{\partial x_i} = 0 \tag{5.2}$$

and it yields

$$\Delta R = 0. \tag{5.3}$$

This equation has undoubtedly the solution

$$R = \text{const.} \tag{5.4}$$

But that is also the *only* solution. Make the assumption given in (3.7) for the homogeneous Eq. (3.6). That leads to the condition

$$\Delta \kappa = 0. \tag{5.5}$$

Any solution of this equation besides $\kappa = \text{const.}$ would produce a possible solution of (3.6) which we have excluded beforehand. The solution (5.4) of (5.3) is therefore not only possible but also *necessary*. In this way we have found a first integral of our field equations of a scalar kind: the *scalar Riemannian curvature must be constant in every point of our manifold.*

6. A Further First Integral: the Equation of Continuity for the Vector Potential

We now use the *integrated* form of our field equations, the equations (3.18), building in the same way the corresponding scalar equation on multiplying by g^{ik}. The second equation gives

$$w = w_\alpha^\alpha = \text{const.} \tag{6.1}$$

Introducing that in the first equation we find

$$2R(1 + 4C) + 4\lambda + 2\frac{\partial\phi^\alpha}{\partial x_\alpha} = \text{const.} \tag{6.2}$$

We are surprised that we do not come back to our former result $R=\text{const.}$ But the difference is that in the earlier deduction we used not only the field equations but also the divergence condition of R_{ik} which is not utilized for the deduction of (6.2). But we can introduce now our former result (5.4) and attain in this way

$$\frac{\partial\phi^\alpha}{\partial x_\alpha} = \text{const.} \tag{6.3}$$

This constant cannot differ from 0, otherwise the solution $\phi_i=0$ would not be possible and the magnitude of the vector potential would increase more and more. The only possibility is therefore

$$\frac{\partial\phi^\alpha}{\partial x_\alpha} = 0. \tag{6.4}$$

That is the well-known divergence condition of the vector potential corresponding to the conservation law of electricity in the electromagnetic field theory.

7. The Potential Equation in First Approximation

The constant of integration in (5.4) must be extremely small. Otherwise the average curvature-radius of the universe could not be so extremely great. For this reason the second invariant recedes to the background in significance compared with the first because the u_{ik} and v_{ik} produced by it, contain this very small value as a factor. To suppose that the constant C could compensate this small value and make the second invariant dominant by a choice of a very high value for it seems to be very improbable. In this case the gravitational force would have to dominate the electrical force but in reality the reverse is true. The importance of the second invariant may be connected with the existence of proper solutions[11] which are to be considered as the representation of a material particle. But beyond the very central parts of matter with which there is possibly connected a very high curvature its influence seems to be negligible, or at most to be regarded as a correction. The essen-

[11] That means non-trivial solutions of the field equations which are everywhere regular.

tial part of the Hamiltonian function is H_1, and we will content ourselves with considering only this part.

The field equations are to be written:[12]

$$\left.\begin{aligned} R_{ik} &= \frac{1}{2}\left(\frac{\partial \phi_i}{\partial x_k} + \frac{\partial \phi_k}{\partial x_i}\right) + w_{ik} \\ E(w_{ik}) &= 2(R_{i\alpha}R_k^{\alpha} - \tfrac{1}{4}R_{\alpha\beta}R^{\alpha\beta}g_{ik} + P_{ik\alpha}\phi^{\alpha}) \end{aligned}\right\}. \tag{7.1}$$

We observe that w_{ik} is a quantity of *second* order if we consider R_{ik} to be of the first order. We can therefore write as a first approximation:

$$R_{ik} = \frac{1}{2}\left(\frac{\partial \phi_i}{\partial x_k} + \frac{\partial \phi_k}{\partial x_i}\right) \tag{7.2}$$

this connection between curvature tensor and vector potential corresponds to the connection (3.19), deduced for the case of Einstein's gravitational equations as mentioned above. However, in our case this connection is only a *first approximation* not a strict law. The correctious terms of second degree which have to be added become high in the central field and remove the necessity that the vector potential has to vanish.

On constructing the divergence equation we find in first approximation the following determining equation for ϕ_i:

$$\Delta \phi_i = 0 \tag{7.3}$$

which is the familiar potential equation. The notation "vector potential" is therefore indeed justified.

8. The Ponderomotive Lorentz Force in First Approximation

A formation like the antisymmetric tensor of the electromagnetic field-intensity

$$F_{ik} = \frac{\partial \phi_i}{\partial x_k} - \frac{\partial \phi_k}{\partial x_i} \tag{8.1}$$

nowhere appears in our treatment. The question arises then how can we give an explanation for this quantity. If we find a possibility of deducing a dynamical law of motion for a material particle like the phenomenological Lorentz law, we will be indeed justified to identify the vectorial function, ϕ_i appearing in the integration of our field equations, with the electromagnetic vector potential. Because all the fundamental facts of the electromagnetic field are embraced by the potential equation and the law of continuity for the vector potential, and also the Lorentzian force expressing the action of the field upon the material particle. The first two are already deduced. Finally, we consider the last problem, viz. the action of the field on a particle.

[12] We neglect also the "cosmological term" λg_{ik} which is only important for large regions of the universe. It is interesting to remark that the field equations themselves do not allow any artificial assumption to introduce the cosmological constant, it appears as *constant of integration*.

A special dynamical law independent on the field equations does not exist in a theory built up on the idea of proper solutions excluding any singularities. The field equations are able to determine every event and we do not have the arbitrariness connected with the behavior of singularities. The influence of an external field on the motion of a particle has to be considered as a consequence of the field equations.

We cannot, however, proceed directly in this way to find a dynamical law. We do not at present have sufficient mathematical tools for the treatment of such a complicated system of equations especially those concerning the problem of proper solutions. On the other hand: we do not need a *detailed* knowledge about the influence of an external field on the field of a proper solution. Our desire is only to obtain the resulting influence on the particle *as a whole*. For this purpose it seems much more convenient to avoid the direct use of the field equations and to attain the goal by a treatment which has also proved to be the best way to integrate the field equations without directly using them. This treatment is the direct application of Hamilton's principle. We know that this principle is valid for *every* variation. We are thus justified in applying it to the special variation which sufficies to obtain the dynamical law of a material particle. We do not need for this purpose the detailed knowledge of the construction of a proper solution which represents the material particle. Some general features are sufficient to find an approximate solution of our problem. That is the great advantage of this process.

The second invariant H_2 is without importance for our purpose. In consequence of the intermediary integral $R=\text{const.}$ it becomes a constant. We can then observe this condition also during the variation so that the variation of this part of the action-integral disappears and we have to consider only the first integral.

We consider our field as a superposition of the proper field of the particle and a weak external field. The proper field produces an integral which can approximately be calculated on supposing that the perturbance of the external field is only weak and the reaction of the particle on itself is negligible ("quasi-static" condition). We know that the central high part of the field is chiefly important in its contribution to the integral and this central field is to be considered as moving with the particle unchanged and not essentially modified by the weak external field. Supposing that the average distribution of the field may have a spherical symmetry we can write the resulting integral in the following form:

$$m\int ds \tag{8.2}$$

with the constant m on which turns out later to be the "electro-magnetic mass"; ds is the line-element of the world-line of the particle.

We have to calculate now the interaction between the internal and the external field. We cannot survey the situation in the high central field. But it seems probable to consider this part of interaction as to be weak as compared with the interaction in the "ether", meaning with this word the very large *exterior* part of space in which the field becomes weak, and where the linear approximation is permitted and the principle of superposition is valid. The predominance of this part of space with respect to the interaction is a consequence of its huge extension compared with the "matter", this word meaning the high central field of small extension in which no superposition is possible. The duality "field-matter" is in this sense retained although obviously without precise distinction.

In this "ether" we are able to accomplish our calculation. We have here corresponding to (7.2)

$$R_{\alpha\beta}R^{\alpha\beta} = \frac{1}{4}\left(\frac{\partial\phi_\alpha}{\partial x_\beta} + \frac{\partial\phi_\beta}{\partial x_\alpha}\right)^2 \tag{8.3}$$

the metric being in first approximation Euclidian. The interaction is given by

$$\begin{aligned} I_{12} &= \frac{1}{2}\int\left(\frac{\phi_\alpha{}^{(1)}}{\partial x_\beta} + \frac{\partial\phi_\beta{}^{(1)}}{\partial x_\alpha}\right)\left(\frac{\partial\phi_\alpha{}^{(2)}}{\partial x_\beta} + \frac{\partial\phi_\beta{}^{(2)}}{\partial x_\alpha}\right)dv \\ &= \int\left(\frac{\partial\phi_\alpha{}^{(1)}}{\partial x} + \frac{\partial\phi_\beta{}^{(1)}}{\partial x_\alpha}\right)\frac{\partial\phi_\alpha{}^{(2)}}{\partial x_\beta}\,dv. \end{aligned} \tag{8.4}$$

The index 1 means the internal, the index 2 the external field. With aid of the potential equation and the continuity law for $\phi_i{}^{(2)}$ we can transform this volume integral into a surface integral to be taken over the boundary surface of the considered region:

$$I_{12} = \int\phi_\alpha{}^{(2)}\nu_\beta\left(\frac{\partial\phi_\alpha{}^{(1)}}{\partial x_\beta} + \frac{\partial\phi_\beta{}^{(1)}}{\partial x_\alpha}\right)dF \tag{8.5}$$

dF is the surface-element, (the "surface" is here, of course, a 3-dimensional region) ν_i the normal of the surface at the considered point. The boundary surface is a small tube around the material particle separating the ether from the matter. The positive direction of ν_i shows towards the inside of the tube.

We can accomplish our integral in two steps, integrating at first around the small cross-section of the tube, and secondly, over the world-line of the particle. During the first step we can consider $\phi_i{}^{(2)}$ in this small region of the space as a constant putting it in front of the integral sign. We have only to consider the first integral, the second vanishing, owing to the law of continuity which is fulfilled also on the inside of the tube. This first integral can be written in the following form

$$\int\phi_\alpha{}^{(2)}\sigma_\alpha ds \tag{8.6}$$

where σ_i is a vector, depending only on the particle itself. If we suppose that the field of the particle has a spherical symmetry, the direction of σ_i can only be the direction of the world-line. And going in a "rest-system" we see that the length of σ_i is nothing else but the electric *charge* of the material particle:

$$\bar{\sigma}_4 = \int \frac{\partial \bar{\phi}_4^{(1)}}{\partial x_\alpha} \nu_\alpha df = e. \tag{8.7}$$

The dash should mark that we are in a rest system. df is the surface-element of a common surface (of our 3-dimensional space) enclosing the electron.

We therefore have to put

$$\sigma_i = e\dot{x}_i. \tag{8.8}$$

(The point over a letter means: differentiation with respect to ds) and our whole action integral becomes

$$m\int ds + e\int \phi_\alpha^{(2)} \dot{x}_\alpha ds. \tag{8.9}$$

Using the familiar variation-equations of Euler-Lagrange we obtain immediately

$$m\ddot{x}_i + e\left(\frac{\partial \phi_i^{(2)}}{\partial x_\alpha} - \frac{\partial \phi_\alpha^{(2)}}{\partial x_i}\right)\dot{x}_\alpha = 0 \tag{8.10}$$

which is nothing else but the *ponderomotive law of Lorentz.* Here, for the first time, appears the antisymmetric combination (8.1) which never appeared earlier and has no fundamental significance.

It seems that the vector ϕ_i, originally a free vector of integration, arising and also determined by the fact that a fundamental quantity of Riemannian geometry, viz., the curvature tensor of Einstein obey the law of conservation, corresponds completely to the fundamental quantity of the electromagnetic field viz., the vector potential. We get the impression that it is not only a mistake to believe that the Riemannian geometry contains nothing like electricity, but on the contrary the existence of electricity, seems to be a direct test for a fundamental property of Riemannian geometry the conservation law of Einstein's curvature tensor R_{ik}.

It should be necessary to investigate whether there are proper solutions of the field-equations with the charges $-e$ and $+e$ but different masses, representing the fundamental material particles electron and proton.

9. Some Additional Remarks Concerning the Mathematical Foundation of the Theory[13]

If we consider the homogeneous equation

$$F(u_{ik}) = 0 \tag{9.1}$$

[13] This chapter, written in the beginning of December, contains some further improvements of the author in the subject presented above. The formulation of this chapter is influenced by some critical remarks of Professor H. P. Robertson, who examined the contents of the previous chapters with great care. He deduced also independently of the author the fundamental Eq. (9.11) for the vector potential, recognizing at the same time an error in the sign of a formula in a previous paper of the author. (See the next footnote). The last chapter 10, which establishes the connection with Maxwell's equations, was added January 1932.

of the field Eqs. (2.14) and we suppose a Euclidian metric we find a solution of the form

$$u_{ik} = \frac{\partial \phi_i}{\partial x_k} + \frac{\partial \phi_k}{\partial x_i} - \frac{\partial \phi^\alpha}{\partial x_\alpha} g_{ik} \tag{9.2}$$

with an arbitrary vector ϕ_i. For this reason we should expect that the quite similar appearance of the free vector ϕ_i in the integrated form (3.18) of the field equations (2.14) should be connected with the fact that the homogeneous equation has a solution with an arbitrary vector ϕ_i. That was originally my conjecture. As a consequence of that it would be necessary that there exist a vectorial identity for $F(u_{ik})$ and corresponding to that a vectorial condition for the right side of the equation. However, we cannot prove the existence of such an identity, and also, substituting our solution (3.18) for the special case $v_{ik}=0$ in our field equations we do not find that the homogeneous Eq. (9.1) is necessarily satisfied. The real situation is the following.

We introduced in our procedure the idea that a pure transformation of the coordinates cannot influence our variation principle which is an invariant. We used this fact to restrict the variation of g_{ik} by the condition of (3.2) which cannot change anything in our results. However, while this treatment is justified if our integral of variation is deduced by a scalar Hamiltonian function $H(R_{ik}, g_{ik})$, we do not have the same statement in the "non-holonomic" case, i.e. if u_{ik} and v_{ik} are considered as *general* quantities. In this case the exclusion of the pure transformation of coordinates is a *real* restriction and treating this restriction as an auxiliary condition in the variation of g_{ik} we obtain our field equations instead of (2.14) in the more general form

$$F(u_{ik}) - 2v_{ik} = \frac{\partial \Phi_i}{\partial x_k} + \frac{\partial \Phi_k}{\partial x_i} - \frac{\partial \Phi^\alpha}{\partial x_\alpha} g_{ik} \tag{9.3}$$

with the free vector Φ_i as a Lagrangian factor. It therefore follows that already in the *original* form of the field equations there appears a free function Φ_i, if we operate with the general quantities u_{ik} and v_{ik}. Assuming the existence of a Hamiltonian function, the vector Φ_i disappears automatically. We know namely in this case that the left side of (9.3) satisfies an identity, viz.,

$$\text{div}\,[F(u_{ik}) - 2v_{ik}] = 0. \tag{9.4}$$

That yields for the right side

$$\Delta\Phi_i - R_i^{\alpha}\Phi_\alpha = 0 \tag{9.5}$$

which is just our equation which has no solutions except zero.

Now we can integrate our Eqs. (9.3), reducing them to the self-adjoint differential expression $E(u_{ik})$, if we write (9.3) in the following form:

$$E(u_{ik}) = 2v_{ik} + \frac{\partial \psi_i}{\partial x_k} + \frac{\partial \psi_k}{\partial x_i} - \frac{\partial \psi^\alpha}{\partial x_\alpha} g_{ik} \tag{9.6}$$

where we have put

$$\psi_i = \sigma_i + \Phi_i \tag{9.7}$$

with

$$\sigma_i = \frac{\partial u_i^\alpha}{\partial x_\alpha} \cdot \tag{9.8}$$

We can treat ψ_i as well as Φ_i as a free vector. Comparing the Eq. (9.6) with the form of the solution (3.18) obtained by using Mach's principle we observe the following connection between the vector ϕ_i appearing in (3.16), and the vector Ψ_i appearing in (9.6):

$$\psi_i = \Delta\phi_i + R_i^\alpha\phi_\alpha. \tag{9.9}$$

This can be proved easily if we introduce the solution (3.16) in the equation (9.6)[14]

The vanishing of Φ_i results in the following connection between the vector potential and the metrical quantities

$$\Delta\phi_i + R_i^\alpha\phi_\alpha = \frac{\partial u_i^\alpha}{\partial x_\alpha} \cdot \tag{9.10}$$

In the case of our Hamiltonian we see that the right side *vanishes* for the expression (4.4) as well as for (4.6) if we employ our first integral $R=\text{const}$.

We obtain therefore the following fundamental equation as the determining equation of ϕ_i:

$$\Delta\phi_i + R_i^\alpha\phi_\alpha = 0. \tag{9.11}$$

This equation is the *exact equation for the vector potential.* We observe the familiar potential equation as a first approximation and we can also deduce immediately the conservation law, building

$$\frac{\partial}{\partial x_\alpha}(\Delta\phi^\alpha + R^{\alpha\sigma}\phi_\sigma) = 0 \tag{9.12}$$

that yields

$$\Delta \frac{\partial\phi^\alpha}{\partial x_\alpha} = 0 \tag{9.13}$$

and leads to the results (6.3) and (6.4).

We can consider the Eq. (9.11) also as a consequence of the conservation

[14] Unfortunately in the corresponding treatment of the previous paper (Zeits. f. Physik **32,** 169 (1925)) the author made a mistake, which was also recognized by Professor H. P. Robertson. In the equation of (31) and (32) of this paper there appears a minus sign instead of the correct formula:

$$\Phi_i = \tfrac{1}{2}(\Delta\Psi_i + R_i^s\Psi_s). \tag{32}$$

Having discovered this error already a long time ago, I could never understand the real significance of this change in the sign on comparing (32) in the rectified form with the Eq. (35) in which the minus sign is valid. In the present theory this change of the signs has a fundamental importance. While the equation with the minus sign is supposed to have no proper solutions besides zero, the equation with the plus sign is the fundamental equation for the vector potential and the existence of the material particles must be connected with the proper solutions of this equation. In fact just this change of signs causes Maxwell's equations to appear in *exact form* as shown in the next chapter.

law of R_{ik}, because the vanishing of the right side of (9.10), using for u_{ik} the form (4.4), is just a consequence of this law. We had originally 10+4 equations: the 10 field equations for R_{ik} and the vectorial conservation law. The latter conditions caused a Lagrangian factor ϕ_i which must be determined by the surplus equations. Indeed, we can consider the determining equation (9.11) as a representation for the conservation law. We have at last 10+4 equations for 10+4 quantities, the R_{ik} and the ϕ_i. In the first place there exist the 10 field equations in the form (7.1), which are to be completed only by the gravitational terms given by the second Hamiltonian H_2 and by the cosmological term. Secondly, there are the four Eqs. (9.11) for the vector potential.

We expect the possibility of proper solutions of (9.11) which must represent physically the material particles electron and proton. In the perception of this theory both electricity and gravitation are manifestation of a certain geometrical structure of the Universe. This structure is of the Riemannian type and governed by a Hamiltonian function which has the simplest form if we require the gauge-invariance.

Any completion of these equations by new "material" terms like the completion of Einstein's gravitational equations by a phenomenological matter-energy-tensor would be impossible. Such a completion means in fact the *denial* of the field equations in certain regions and is practically equivalent to the permission of singularities. The fundamental idea of this theory, however, is the consideration of Einstein's curvature-tensor R_{ik} as the fundamental characteristic quantity of Riemann's geometry. That is only possible if we exclude any singularities of the metric.

To determine the vector potential ϕ_i as a proper solution of an equation like (9.11) without further restrictions does not seem to be possible if we consider for example the arbitrariness of a factor in such a solution which does not correspond to the real physical statement. We must realize, however, that we should not conceive these equations as equations with definitely determined coefficients. There is a strict interconnection between vector potential and metric; on the one hand the metric is determined by the vector potential, on the other hand the constitutive equation of the vector potential is essentially influenced by the metric which produces the possibility of a proper solution. Thus the fundamental differential Eq. (9.11) is in reality a kind of complicated non-linear equation and the arbitrariness of a linear proper solution does not appear in the solution of such an equation. To determine the ultimate material particles and their inter-action as the proper solution of such a system of non-linear equations seems to me very satisfactory, especially in regard to such problems as the equality of the charge of all the electrons. The limitation attained by such a requirement without further restrictions seems to correspond to the real degree of limitation to be found in nature and corresponds also to a natural mathematical point of view according to which the solution of a variation problem is given by some differential equations completed by corresponding natural boundary conditions which can also become superfluous if the manifold is closed.

10. Connection with Maxwell's Equations and with the Completed form of Maxwell's Equations.

If we compare the left sides of (9.5) and (9.11) the change in the sign of the second term has a striking significance with respect to Maxwell's equations. The left side of (9.5) arises if we build the divergence of the *symmetric* combination

$$\frac{\partial \phi_i}{\partial x_k} + \frac{\partial \phi_k}{\partial x_i} \tag{10.1}$$

noticing the fact that we can leave out the additional term $-(\partial\phi^\alpha/\partial x_\alpha)g_{ik}$, which vanishes as a consequence of the conservation law of the vector potential. The left side of (9.11) arises if we build the divergence of the *antisymmetric* combination

$$F_{ik} = \frac{\partial \phi_i}{\partial x_k} - \frac{\partial \phi_k}{\partial x_i}. \tag{10.2}$$

The antisymmetric tensor F_{ik} must be considered as the electro-magnetic field strength. The equation

$$\operatorname{div} F_{ik} = 0 \tag{10.3}$$

is the one system of Maxwell's equations. The other system

$$\operatorname{div} F_{ik}^* = 0 \tag{10.4}$$

belonging to the "dual field-strength" F_{ik}^*, is an immediate consequence of (9.15). Thus *the whole system of Maxwell's equations is fulfilled without any change*. We may also build the symmetric Maxwellian stress-energy-tensor

$$S_{ik} = F_i^{\alpha} F_{k\alpha} - \tfrac{1}{4} F_{\alpha\rho} F^{\alpha\rho} g_{ik}. \tag{10.5}$$

The divergence of this tensor vanishes also exactly

$$\operatorname{div} S_{ik} = 0. \tag{10.6}$$

We obtain therefore, the surprising result that the classical form of electromagnetism appears in a very strict connection with Riemann's geometry. The difference compared with the former situation is that the electromagnetic quantities appear merely as *auxiliary quantities* in building up a metric. And the interaction between electromagnetic and metrical quantities is quite different from that assumed in any earlier attempt to combine electromagnetism with Einstein's theory of gravitation. As a consequence of this new interaction of metrical and electromagnetic quantities we may expect quite new insights in the problems connected with the constitution of matter.

However, a closer examination of the proper solutions of Maxwell's equations contradicts strongly what we expect. The equation

$$\frac{\partial (g)^{1/2} F^{91}}{\partial x_1} + \frac{\partial (g)^{1/2} F^{92}}{\partial x_2} + \frac{\partial (g)^{1/2} F^{93}}{\partial x_3} = 0 \tag{10.7}$$

in which the operation $\partial/\partial x_i$ now means an ordinary differentiation, permits the application of Gauss' theorem and leads generally to the *vanishing* of the electric charge.[15] We see, therefore, that the introduction of a curved manifold is not sufficient for producing useful proper solutions of the Maxwellian equations.

If we review our method of building up the fundamental equations, we observe the use of some restrictional assumptions which are in reality not necessary for the fulfilling of the field equations. We introduced the vector potential ϕ, by the use of a process which we called "Mach's principle." During this process we made the assumption that the equation $E(u_{ik})=0$ as well as the equation $\Delta\Phi=0$ should not have nontrivial proper solutions. The latter restriction led to the first integrals $R=$const. and div $\phi_1=0$.

In Chapter 9 we proved the validity of the integrated form of the field equations without the help of Mach's principle. It turned out that we have to add only the equation (9.10) as a condition for the vector potential. Any other condition is unnecessary. Since we have seen now that the strict validity of Maxwell's equations leads to a contradiction, we have to abandon the unnecessary supposition that (5.4) shall be the only solution of (5.3). And we have therefore to abandon our first integrals (5.4) and (6.4). These integrals are responsible for the strict appearance of Maxwell's equations. If we do not make the above-mentioned restriction, we get by (9.10) the following constitutive equation for the vector potential ϕ_1:

$$\Delta\phi_i + R_i{}^\alpha\phi_\alpha = (1 + 2C)\frac{\partial R}{\partial x_i} \tag{10.8}$$

and Maxwell's equations for the antisymmetric field strngth F_{ik}, defined again by (10.2), appear in the following form:

$$\frac{\partial F_i{}^\alpha}{\partial x_\alpha} + \frac{\partial \phi}{\partial x_i} = 0 \tag{10.9}$$

where we introduced the scalar ϕ by putting

$$\phi = \frac{\partial \phi^\alpha}{\partial x_\alpha} - (1 + 2c)R. \tag{10.10}$$

The second set of Maxwell's equations remains unchanged in the previous form (10.4).

If we use the familiar notations of the 3-dimensional vector-analysis, we can write our equations in the case of a Euclidian metric in the following form

[15] The equation (10.7) being valid without exception also in the central part of the particle, we can build a surface integral over a closed surface surrounding the particle at such a distance that the metric is there practically Euclidian. That gives

$$\int F_{\Phi\alpha} v_\alpha dS = 0$$

The left integral, built in the rest system of the particle, has the significance of the charge.

$$\frac{1}{ic}\frac{\partial F}{\partial t} - \text{curl } F - \text{grad } \Phi = 0$$
$$\text{div } F + \frac{\partial \Phi}{ic\partial t} = 0. \qquad (10.11)$$

We introduced there the complex vector F with the significance

$$F = H + iE \qquad (10.12)$$

where H is the magnetic, E the electric field-strength. Compared with Maxwell's equations this system is enlared with the terms containing Φ. This self-adjoint system of equations has just the right degree of determination, constituting 4 equations for the 4 quantities F and ϕ. Generally we have to consider F and Φ as complex quantities. The system is then equivalent—as I have proved in a former investigation[16]— to Dirac's equation for the electron, if we leave out the term with the mass. The only difference in our case is that Φ must be real.

The stress-energy-tensor of Maxwell must be enlarged in the following form:[17]

$$S_{ik} = F_i^{\alpha} F_{k\alpha} - \tfrac{1}{4}(F_{\alpha\beta}F^{\alpha\beta} + 2\Phi^2)g_{ik} + \Phi F_{ik} \qquad (10.13)$$

in order that the divergence vanish. This enlarged tensor is no longer symmetric, due to the last term which is antisymmetric.

The difficulty of the vanishing of the charge is now overcome. We see, however, that the equation (10.7) appears also now if we require a *static* solution. Because the equation $\Delta\Phi = 0$ has certainly no *statical* proper solutions. It seems that there is a general *pulsation* in the metrical quantities which gives a "guiding field" for the pulsation of all matter. It seems that we must understand the electrostatic field as a *quadratic effect* of these pulsating fields, taking into consideration the quadratic character of our action-principle. This character appears also in the expression of the ponderomotive force as a product of charge and field-strength.

[16] Zeits. f. Physik **57**, 447 (1929).
[17] Zeits. f. Physik **57**, (1929); Eq. (15′).