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DIRAC'S EQUATION AND THE SPIN-SPIN INTERACTIONS OF TWO ELECTRONS

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Abstract

An improvement is made on a previous attempt to treat two particles by means of Dirac's equation. The approximate equation (1) below is considered in successive steps. The first step, following Oppenheimer, includes the electrostatic energy exactly, rather than to the first in power in e^2. This makes it possible to use it as a good starting point in the calculation of spectral terms. The second step brings in the energy due to the interaction of the electric currents. It is given by (9) below. Maxwell's equations and the conservation of energy (see (8.5)) demand the validity of the diagonal matrix elements of this expression as a first order perturbation energy, independently of theories of light quanta. The interactions of the particles with themselves give *additive constants* in the energy within the limits of the approximation used. Within the same limits the results are in agreement with experiment.

(I). Statement of Problem

In a previous paper[1] an attempt has been made to treat the spin-spin interactions of two electrons by means of Dirac's relativistic equation.[2] The conclusion reached at the time was that the equation

$$\{p_0+\boldsymbol{\alpha}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{I}}+\boldsymbol{\alpha}^{\mathrm{II}}\boldsymbol{p}^{\mathrm{II}}+(\alpha_4{}^{\mathrm{I}}+\alpha_4{}^{\mathrm{II}})mc + (e^2/2c)\left[\boldsymbol{\alpha}^{\mathrm{I}}\boldsymbol{\alpha}^{\mathrm{II}}r^{-1}+(\boldsymbol{\alpha}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\alpha}^{\mathrm{II}}\boldsymbol{r})r^{-3}\right]\}\psi=0 \qquad (1)$$

is not in agreement with experiment.[3]
Here

$$p_0=-\frac{h}{2\pi i}\frac{\partial}{c\partial t}+(e/c)(A_0{}^{\mathrm{I}}+A_0{}^{\mathrm{II}})-(e^2/cr)$$

$$\boldsymbol{p}^{\mathrm{I}}=(p_1{}^{\mathrm{I}},p_2{}^{\mathrm{I}},p_3{}^{\mathrm{I}}),\qquad \boldsymbol{p}^{\mathrm{II}}=(p_1{}^{\mathrm{II}},p_2{}^{\mathrm{II}},p_3{}^{\mathrm{II}})$$

$$p_k{}^{\mathrm{I}}=\frac{h}{2\pi i}\frac{\partial}{\partial x_k{}^{\mathrm{I}}}+(e/c)A_K{}^{\mathrm{I}},\quad p_k{}^{\mathrm{II}}=\frac{h}{2\pi i}\frac{\partial}{\partial x_k{}^{\mathrm{II}}}+(e/c)A_K{}^{\mathrm{II}},\quad (k=1,2,3)$$

(A_0, A_1, A_2, A_3) = electromagnetic four vector
$\boldsymbol{r}=+\left[(x_1{}^{\mathrm{I}}-x_1{}^{\mathrm{II}})^2+(x_2{}^{\mathrm{I}}-x_2{}^{\mathrm{II}})^2+(x_3{}^{\mathrm{I}}-x_3{}^{\mathrm{II}})^2\right]^{1/2}$ = distance between electrons I and II $\boldsymbol{\alpha}^{\mathrm{I}}=(\alpha_1{}^{\mathrm{I}},\alpha_2{}^{\mathrm{I}},\alpha_3{}^{\mathrm{I}})$, $\boldsymbol{\alpha}^{\mathrm{II}}=(\alpha_1{}^{\mathrm{II}},\alpha_2{}^{\mathrm{II}},\alpha_3{}^{\mathrm{II}})$
Matrices $(\alpha_1{}^{\mathrm{I}},\alpha_2{}^{\mathrm{I}},\alpha_3{}^{\mathrm{I}},\alpha_4{}^{\mathrm{I}})$, $(\alpha_1{}^{\mathrm{II}},\alpha_2{}^{\mathrm{II}},\alpha_3{}^{\mathrm{II}},\alpha_4{}^{\mathrm{II}})$, identical in form with Dirac's

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[1] Breit, Phys. Rev. **34**, 553 (1929).

[2] An attempt along the same lines has been also made by Gaunt (Phil. Trans. Roy. Soc. **A228**, 151 (1929). A discussion of this will be found in references I, 3.

[3] Breit, Phys. Rev. **36**, 385 (1930).

s, are operators on two independent indices belonging respectively to I and II, each index taking four values 1, 2, 3, 4.

With a slight modification it has been shown however to fit the observed facts as well as could be expected. The modification required consists in the removal of terms in e^4 in the result of eliminating the "small" components of the wave-function. These terms do not involve h so that their physical significance is doubtful even for this reason alone.

Eq. (1) is not exact. It is wrong when the distance between the particles or their velocities are too great. It also was not intended to give account of radiation from the atom. It will now be shown that a modification of the first interpretation of (1) can be given in such a way that the difficulty with the e^4 terms disappears. From the present point of view the e^4 terms derived in reference 1 have no direct physical significance. Their appearance in the previous work was due to trying to attach more exactness to the last term in (1) than it possesses.

In reference 1 the derivation of the equation has been given from the point of view of the Heisenberg-Pauli theory of electrodynamics. It was supposed in this calculation that the interaction energy between the particles is small. The first order effect in the interaction energy was calculated and it was found that the values of the interaction integrals to be used in the calcalculation of the mutual energy are given by the formula (63) (using umbral notation)

$$\begin{aligned} A_{st,ts} &= \int [(u_\rho{}^{*s} u_\rho{}^t) \{ \overset{-1}{r_{PP'}} \} (u_\sigma{}^{*t} u_\sigma{}^s)' \\ &\qquad - (u_\rho{}^{*s} \boldsymbol{\alpha}_{\rho\sigma} u_\sigma{}^t) \{ 1/r_{PP'} \} (u_\mu{}^{*t} \boldsymbol{\alpha}_{\mu\nu} u_\nu{}^s)' dV dV' \qquad (2) \\ \{1/r_{PP'}\} &= (1/r_{PP'}) \cos (2\pi r_{PP'}/\lambda_{st}) . \end{aligned}$$

Here s, t are unperturbed electron levels. The wave-length of the transition $s \rightarrow t$ is denoted by λ_{st}. Recently, C. Møller[4] and L. Rosenfeld[5] have given simpler derivations of the same result also to the first order in e^2 without using the formalism of the quantum electrodynamics.[6] There is hardly any doubt that the mutual energy of two electrons is given correctly to the first order in e^2 by the interaction integrals $A_{st,ts}$, as long as one is justified in speaking of a constant energy.

This derivation is not sufficient to make practical applications to the fine structure of He possible. The Shroedinger non-relativistic wave equation gives, according to Kellner and Hylleraas as well as others, term value energies for the two electron problem which are in good agreement with experiment. In these calculations the electrostatic energy e^2/r is used much more accurately than to the first order. It is desirable to have such a theory that Schroedinger's non-relativistic wave equation with the e^2/r term for the mu-

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[4] C. Møller, Zeits. f. Physik **70**, 786 (1931).

[5] L. Rosenfeld, Zeits. f. Physik **73**, 253 (1931).

[6] The fact that the easiest interpretation of $A_{st,ts}$ is in terms of the retarded potentials has been stated in the writer's paper. (p. 572).

tual energy could be taken as a starting point for the calculations. In order to have such a theory it is necessary to explain the validity of the e^2/r term in an exact sense. We must strive therefore for a theory in which the Schroedinger non-relativistic value of the energy is modified by the presence of small terms.

II. ELECTROSTATIC ENERGY

The proper point of view for such a treatment is given by Oppenheimer[7] making use of the second form of the Heisenberg-Pauli theory. It is shown in Oppenheimer's paper that the variables describing the longitudinal components of the electromagnetic field can be eliminated from the wave equations. (Formula (8) of J. R. Oppenheimer.[7]) The process of elimination brings in infinite terms into the energy expression. The infinite terms represent the infinite electrostatic energy of point charges. It is stated in Oppenheimer's paper that these terms may be considered as constants and do not interfere with the application of the theory. This point is of special interest to us and will therefore be discussed somewhat more in detail.

The mathematical apparatus of Heisenberg-Pauli II can be applied to classical electrodynamics. Its direct application also leads to infinities as would be expected of any theory with point charges. Nevertheless, in classical theory, there is a way of escaping the difficulty. In all of the expressions for the interaction energy we can replace $\boldsymbol{A}(\boldsymbol{r})\dot{\boldsymbol{r}}$ by $\int \boldsymbol{A}(\boldsymbol{r}')D(\boldsymbol{r}-\boldsymbol{r}')\dot{\boldsymbol{r}}dV'$ where $\boldsymbol{A}$ is the vector potential, $\dot{\boldsymbol{r}}$ the velocity, $\boldsymbol{r}$ is the position of the particle and $\boldsymbol{r}'$ is a variable point in space. The function D is a concentrated function satisfying

$$\int D(\boldsymbol{r}-\boldsymbol{r}')dV' = 1$$

resembling a δ function before the latter passes to the limit of infinite concentration. Such a scheme can be carried through only non-relativistically in a classical particle theory. Nevertheless the result is of interest because of its simplicity. The longitudinal components of the field can be eliminated by using the equation div $\mathcal{E}=4\pi\rho$. The elimination introduces additional terms into the Hamiltonian. Their sum is simply the energy of the electrostatic field (defined by div $\mathcal{E}=4\pi\rho$, curl $\mathcal{E}=0$) of the charge distributions described by $D(\boldsymbol{r}-\boldsymbol{r}')$ for each particle. It may be decomposed into a sum of the electrostatic energy of the field which would exist if the particles were infinitely far from each other and the mutual energy of the particles $(\Sigma_{i,j}e_ie_j/r_{ij})$. The self energy is constant and may be removed. Similarly in quantum theory we can replace all integrals

$$\int \boldsymbol{A}(\boldsymbol{r})(\psi_\mu{}^*\boldsymbol{\alpha}_{\mu\nu}\psi_\mu)dV \text{ by } \int\int \boldsymbol{A}(\boldsymbol{r}')D(\boldsymbol{r}-\boldsymbol{r}')(\psi_\mu{}^*\boldsymbol{\alpha}_{\mu\nu}\psi_\nu)_{\boldsymbol{r}}dVdV'.$$

We then obtain large additive constants *identical with those of the classical theory*. The additive constants can be removed from the wave equation with-

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[7] J. R. Oppenheimer, Phys. Rev. **35**, 461 (1930).

out affecting the physically interesting conclusions about numbers of light quanta, distribution of particles, etc. The fact that a hydrogenic atom may be treated by means of Dirac's equation shows that this removal of the additive constants is a safe procedure. From our present point of view there is no essential difference between the mutual electrostatic energy of a proton and an electron and the mutual electrostatic energy of a collection of several protons and several electrons.

There is an apparent objection. As long as the D function is not the δ function the equations are not relativistically invariant. The results however should approach relativistically invariant results as D approaches δ.

On elimination we end up therefore with Oppenheimer's (11) slightly modified by the removal of the large additive constants and by the replacement of every e_ie_j/r_{ij} by the mutual electrostatic energy of two charge distributions $e_iD_i(\boldsymbol{r}_i-\boldsymbol{r}')$ $e_jD_j(\boldsymbol{r}_j-\boldsymbol{r}')$, $r_{ij}=|\boldsymbol{r}_i-\boldsymbol{r}_j|$. This is then identical in content with Dirac's theory of light quanta.[8]

III. INTERACTOIN WITH TRANSVERSE WAVES

In Dirac's theory of light quanta part of the electromagnetic field is described by means of a vector potential A, satisfying div $A=0$. In order to describe the whole field we need in addition an electrostatic field. We call it $\mathcal{E}'$ and the total electric field we call $\mathcal{E}$. The defining equations are then

$$\mathcal{E} = -\frac{\partial A}{c\partial t} + \mathcal{E}'; \quad \mathcal{H} = \operatorname{rot} A; \quad \operatorname{div} \mathcal{E}' = 4\pi\rho; \quad \operatorname{rot} \mathcal{E}' = 0 \tag{3}$$

$$\operatorname{div} A = 0.$$

Here ρ is the charge density supposed to satisfy the continuity equation

$$\frac{\partial\rho}{\partial t} + \operatorname{div} j = 0 \tag{4}$$

where j is the current density. It follows from Maxwell's equations that

$$\left(\Delta - \frac{\partial^2}{c^2\partial t^2}\right)A = -\frac{4\pi j}{c} - \frac{\partial \mathcal{E}}{c\partial t}\cdot \tag{5}$$

By (3) and (4)

$$\left(\frac{\partial \mathcal{E}'}{\partial t}\right)_P = \nabla_P \int \frac{(\operatorname{div}_{P'} j_{P'})}{r_{PP'}} dV_{P'} \tag{6}$$

where $r_{PP'}$ is the distance between points P and P'. Hence substituting (6) into (5) using the retarded expression for A we have

$$A_{P''} = \frac{1}{c}\int \frac{\{j_{P'}\}}{r_{P'P''}} dV_{P'} + \frac{1}{4\pi c}\int \frac{1}{r_{P''P}}\left\{\nabla_P \int j_{P'}\nabla_P\left(\frac{1}{r_{PP'}}\right)dV_{P'}\right\}dV_P \tag{7}$$

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[8] The obvious changes of using progressive waves in the Heisenberg-Pauli theory and of using Dirac's relativistic equation in Dirac's presentation of the theory of light quanta must be made in order to obtain also identity in form.

where $\{\}$ denotes retarded values. Neglecting retardation in (7) we obtain after some easy transformations

$$\int j_{P''}A_{P''}dV_{P''} = \frac{1}{2c}\int\left(\frac{j_{P'}j_{P''}}{r} + \frac{(j_{P'}\boldsymbol{r})(j_{P''}\boldsymbol{r})}{r^3}\right)dV_{P'}dV_{P''} \qquad (8)$$

where r is the distance between points P', P''. The above deduction of (8) applies in the classical as well as in the quantum theory as long as the current operators commute with the coordinates. The operator which represents the charge density at a point $\boldsymbol{r}'$ is $-\Sigma e_i D_i(\boldsymbol{r}_i - \boldsymbol{r}')$ while the current density operator is given by $\Sigma e_i c\,\boldsymbol{\alpha}^i D_i(\boldsymbol{r}_i - \boldsymbol{r}') = -\Sigma e_i \dot{\boldsymbol{r}}_i D_i(\boldsymbol{r}_i - \boldsymbol{r}')$.

We now consider the wave equation which results on eliminating the electrostatic field:

$$\left(p_0 + \sum_i \boldsymbol{\alpha}^i\left(\boldsymbol{p}^i + \frac{e_i}{c}\int A(\boldsymbol{r}')D_i(\boldsymbol{r}_i - \boldsymbol{r}')dV'\right)\right.$$
$$\left. + \sum_i \alpha_4{}^i m_i c - \frac{1}{8\pi c}\int\left[\left(\frac{\partial A}{c\partial t}\right)^2 + \mathcal{H}^2\right]dV\right)\phi = 0 \qquad (58).$$
$$\left(p_0 = -\frac{h\partial}{2\pi i c\partial t}, \quad p_k{}^i = \frac{h\partial}{2\pi i\partial x_k{}^i}\right) \qquad (k = 1, 2, 3).$$

The wave function contains the coordinates of the particles as well as those describing the radiation (e.g., numbers of light quanta). We shall not use explicitly the way in which ϕ depends on the radiation variables. The term due to $(\partial A/c\partial t)^2$ will be neglected since it has to do with the acceleration. It cannot be taken into account consistently without considering the emission of radiation. We may now use expressions (8), (7) in the above wave equation. The energy operator is then expressed in terms of the particle variables without the aid of the radiation variables. The resultant contribution to the energy operator is

$$E = -\sum_i e_i\boldsymbol{\alpha}^i\int A(\boldsymbol{r}')D_i(\boldsymbol{r}_i - \boldsymbol{r}')dV' + (1/8\pi)\int\mathcal{H}^2 dV.$$

We have

$$\int\mathcal{H}^2 dV = \int\mathcal{H}\,\mathrm{rot}\,A dV = \int(\mathrm{rot}\,\mathcal{H})A dV = \int\left(\frac{4\pi j}{c} - \frac{\partial^2 A}{c^2\partial t^2} + \frac{\partial\mathcal{E}'}{c\partial t}\right)A dV$$

We neglect $\partial^2 A/c^2\partial t^2$ and $\partial\mathcal{E}'/\partial t$ being a gradient contributes nothing by Green's theorem, div A being zero. Thus

$$\Delta E = -\frac{1}{2c^2}\int jAdV = -\frac{1}{4c^2}\int\left(\frac{(j_{P'}j_{P''})}{r} + \frac{(j_{P'}\boldsymbol{r})(j_{P''}\boldsymbol{r})}{r^3}\right)dV_{P'}dV_{P''}. \qquad (9)$$

This is equivalent to Oppenheimer's (28).[7] In the derivation of (9) we have made a number of approximations. These have been suggested to us by the belief that acceleration and radiation have little to do with the energy of stationary states. It is not possible to justify this belief by present mathematical theories. There is thus an element of speculation in the formula (9). We believe that it is correct to start with the wave equation omitting A, to use the energy eigenwerte of this equation and to add to them the diagonal matrix elements of (9). This should give the first order perturbation in the energy due to the interaction of the particles with the transverse electromagnetic waves. Again it is not possible to give a mathematical justification because there exists no proper quantum theory of electrodynamics. However, simple considerations indicate that the diagonal matrix elements of (9) are good approximations to the desired correction for the interaction with the field.

The point is that we may multiply (8.5) by ϕ^* on the left and integrate over the configuration space including that of light quanta. We then have an equation connecting the "expectations" (Erwartungwerte) of the terms in the operator of (8.5). This equation simply means that the "expectation" of the energy is the sum of the kinetic energy of the material particles and of the energy of the electromagnetic field. There is every reason to believe that this is correct even though present theories of electrodynamics based on (8.5) and special assumptions about the nature of light quanta lead to some physically impossible results. The coupling of the motions of particles due to their interactions with transverse waves is small compared to that due to electrostatic forces. We may therefore compute the effect on the "expectation" of the energy by using the unperturbed proper functions of (8.5) with A omitted. Thus the correctness of the diagonal elements of ΔE in energy calculations follows from the principle of conservation of energy and the form of (8.5) independently of the assumptions made about the nature of light quanta.

(IV) CONSTANCY OF SELF-ENERGY TERMS

Replacing the current operators by $\Sigma e_i c \boldsymbol{\alpha}^i D_i(\boldsymbol{r}_i - \boldsymbol{r}')$ we have for the addition to the energy characterised by quantum numbers $n_1, n_2, n_3 \ldots$ in the wave equation

$$\left(p_0 + \sum_i \boldsymbol{\alpha}^i \boldsymbol{p}^i + \sum_i \alpha_4{}^i m_i c\right)\psi = 0 \tag{10}$$

the diagonal matrix element of the operator (9):

$$(\Delta E)_{n,n} = -\left(\tfrac{1}{4}\right) \sum_{i,j} e_i e_j \int\!\!\int \left[\left(\frac{\boldsymbol{\alpha}^i \boldsymbol{\alpha}^j}{r_{P'P''}} + \frac{(\boldsymbol{\alpha}^i \boldsymbol{r}_{P'P''})(\boldsymbol{\alpha}^j \boldsymbol{r}_{P'P''})}{r_{P'P''}}\right) D_i(\boldsymbol{r}_i - \boldsymbol{r}') \cdot \right. \\ \left. \cdot D_j(\boldsymbol{r}_j - \boldsymbol{r}'')\right]_{n,n} dV_{P'} dV_{P''}. \tag{11}$$

According to the above reasoning this formula is independent of the detailed assumptions about the nature of light quanta which are made in Dirac's theory. The essential requirements for its validity are the correctness of Maxwell's equations, the legitimacy of neglecting retardation in (7) and of $(\partial^2 A/c^2\partial t^2)$, $1/c^2(\partial A/\partial t)^2$ in comparison with $4\pi j/c$, $\mathcal{H}^2$ respectively. It may be also derived by means of Dirac's theory of light quanta or its equivalent

Heisenberg-Pauli theory as has been done by Oppenheimer.[7] Carrying through the calculation by the method of successive approximations of wave functions used in Heisenberg-Pauli I for the evaluation of their E^2 there is no difficulty in deriving $(\Delta E)n,\, n$ in agreement with (11). The treatment of resonance denominators is then exactly similar to that in HPI for e^2/r. This adds to our confidence in (9) to the first order in e^2.

The summation over i, j can be broken up into two parts one part containing terms with $i \neq j$ and another with $i = j$.

$$-\tfrac{1}{4}\sum_{i,j} = -\tfrac{1}{2}\sum_{i>j} - \tfrac{1}{4}\sum_{i=j}.$$

The first part is in agreement with (1) provided the last term in (1) is used for the calculation of the energy by the usual method of the perturbation theory. The second part contributes a constant. In fact

$$\frac{\boldsymbol{\alpha}^i\boldsymbol{\alpha}^i}{r_{P'P''}} + \frac{(\boldsymbol{\alpha}^i\boldsymbol{r}_{P'P''})(\boldsymbol{\alpha}^i\boldsymbol{r}_{P'P''})}{r_{P'P''}} = \frac{4}{r_{P'P''}}.$$

The contribution to (11) due to these terms is

$$-\sum_i e_i^2\left(\iint \frac{D_i(\boldsymbol{r}_i - \boldsymbol{r}_{P'})D_i(\boldsymbol{r}_i - \boldsymbol{r}_{P''})}{r_{P'P''}}\, dV_{P'}dV_{P''}\right)_{n,n}.$$

The double integral is a constant and therefore the diagonal matrix element is simply this constant. We have thus a contribution

$$-\sum_i 2W_0^i\left(W_0^i = (e_i^2/2)\iint \frac{D_i(\boldsymbol{r}_i - \boldsymbol{r}_{P'})D(\boldsymbol{r}_i - \boldsymbol{r}_{P''})}{r_{P'P''}}\, dV_{p'}dV_{p''}\right)$$

to the energy where W_0^i is the energy of the electrostatic field due to the particle i. It is not very satisfactory to have this energy thrust upon us. However it must be remembered that only energy differences can be observed. For these the additive constant energy is not of interest. As long as $\Sigma_{i>j}$ is small the perturbation in the energy differences between various spectral terms due to (11) is also small.

We see therefore that Eq. (1) may be interpreted consistently in the sense that the interaction of the particles with the transverse waves brings about a perturbation in the energy of the amount

$$(\Delta E')_{n,n} = -\tfrac{1}{2}\sum_{i>j} e_i e_j\left(\frac{\boldsymbol{\alpha}^i\boldsymbol{\alpha}^j}{r_{ij}} + \frac{(\boldsymbol{\alpha}^i\boldsymbol{r}_{ij})(\boldsymbol{\alpha}^j\boldsymbol{r}_{ij})}{r_{ij}^3}\right)_{n,n}. \tag{12}$$

The unperturbed energy values are the eigenwerte of (10).

This conclusion is somewhat more restricted than the supposition that (1) is correct. We should substitute for the last term in (1) its diagonal matrix element referred to the proper energy values of (10). Another way of expressing the same result is to say that (1) may be used for energy calculations if the last term is taken into account to the first order in the usual way by perturbation theory working with 16 Ψ components.

(V) COMPARISON WITH EXPERIMENT

It remains to compare the results with experiment. It is convenient to use Dirac's original form of the α matrices. As in reference (1) we let $\boldsymbol{\sigma}$ be the negative of Pauli's matrices. We introduce a wave function Ψ having four components identical with the four "large" components of ψ. The σ's operate on Ψ. Thus

$$(\sigma^{\mathrm{I}}\Psi)_{\alpha,\beta} = \sum_{\alpha'}\sigma_{\alpha\alpha'}\Psi_{\alpha'\beta} \qquad (\alpha, \beta = 3 \text{ or } 4).$$

We let

$$\begin{aligned} (X^{\mathrm{I}}\Psi)_{3,\alpha} &= \psi_{1,\alpha}(X^{\mathrm{I}}\Psi)_{4,\alpha} = \psi_{2,\alpha} \\ (X^{\mathrm{II}}\Psi)_{\alpha,3} &= \psi_{\alpha,1};\ (X^{\mathrm{II}}\Psi)_{\alpha,4} = \psi_{\alpha,2} \\ (Y\Psi)_{3,3} &= \psi_{1,1};\ (Y\Psi)_{3,4} = \psi_{1,2};\ (Y\Psi)_{4,3} = \psi_{2,1};\ (Y\Psi)_{4,4} = \psi_{2,2}. \end{aligned} \tag{13}$$

Then (1) may be rewritten as

$$\begin{aligned} (p_0 - 2mc)\Psi - (\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{I}})(X^{\mathrm{I}}\Psi) - (\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{p}^{\mathrm{II}})(X^{\mathrm{II}}\Psi) + M(Y\Psi) &= 0 \\ - (\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{I}})\Psi + p_0(X^{\mathrm{I}}\Psi) + M(X^{\mathrm{II}}\Psi) - (\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{p}^{\mathrm{II}})(Y\Psi) &= 0 \\ - (\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{p}^{\mathrm{II}})\Psi + M(X^{\mathrm{I}}\Psi) + p_0(X^{\mathrm{II}}\Psi) - (\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{I}})(Y\Psi) &= 0 \\ M\Psi - (\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{p}^{\mathrm{II}})(X^{\mathrm{I}}\Psi) - (\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{I}})(X^{\mathrm{II}}\Psi) + (p_0 + 2mc)(Y\Psi) &= 0 \end{aligned} \tag{14}$$

where

$$M = \frac{e^2}{2c}\left(\frac{\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{\sigma}^{\mathrm{II}}}{r} + \frac{(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{r})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{r})}{r}\right).$$

If we omit M in (14) we obtain the result of rewriting (10). The perturbation due to the last term in (1) is thus the same as the perturbation due to M in (14) i.e.,

$$(\Delta E)_{n,n} = -c\int\{\Psi^*(Y\Psi) + (X^{\mathrm{I}}\Psi)^*\cdot(X^{\mathrm{II}}\Psi) + (X^{\mathrm{II}}\Psi)^*(X^{\mathrm{I}}\Psi) + (Y\Psi)^*\Psi\}d\tau$$

the integration being extended over the configuration space. Omitting M we obtain sufficiently exact expressions for $(X^{\mathrm{I}}\Psi)$, $(X^{\mathrm{II}}\Psi)$, $(Y\Psi)$ to substitute into this integral

$$(X^{\mathrm{I}}\Psi) \cong \frac{1}{2mc}(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{I}})\Psi,\ (X^{\mathrm{II}}\Psi) \cong \frac{1}{2mc}(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{p}^{\mathrm{II}})\Psi,$$

$$(Y\Psi) \cong (\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{I}})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{p}^{\mathrm{II}})/4m^2c^2.$$

Substituting we obtain

$$\begin{aligned} (\Delta E)_{n,n} = \frac{1}{4m^2c}\int \Psi^*[&(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{I}})M(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{p}^{\mathrm{II}}) + (\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{p}^{\mathrm{II}})M(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{I}}) + M(\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{I}})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{p}^{\mathrm{II}}) \\ &+ (\boldsymbol{\sigma}^{\mathrm{I}}\boldsymbol{p}^{\mathrm{I}})(\boldsymbol{\sigma}^{\mathrm{II}}\boldsymbol{p}^{\mathrm{II}})M]\Psi d\tau. \end{aligned}$$

This agrees exactly with the first order perturbation energy of the term in $(e^2/8m^2c^3)$ of Eq. (44) of reference (1). In the approximation of that reference the elimination of $(X^{\mathrm{I}}\Psi)$, $(X^{\mathrm{II}}\Psi)$, $(\mathrm{Y}\Psi)$ from (14) with M omitted leads to all the remaining terms of that Eq. (44) with the exception of the term — $-(e^4/16mc^3)X^2$. In the calculation of the energies of He we may use therefore Eq. (48) provided the e^4 are omitted. According to the calculations previously made[3] the results are in as good agreement with experiment as could be expected.

According to the view expressed above we may expect that the approximation of equation (10) corrected by (12) may have a wider range of application than the He fine structure. Two K electrons around a bare nucleus would be expected to obey this equation. The experimentally accessible K lines are unfortunately complicated by the screening due to other electrons.

In comparison with the calculation of reference 2 the present treatment has the advantage of being guided more closely by physical considerations in the choice of approximations. These are seen clearly only in the 16 component form of Eq. (1). The reduction to four "large" components[1] obliterated the direct significance of terms as "expectations" (Erwartungnwerte). The use of the form (14) makes it possible to carry through the applications without losing sight of the direct physical significance of the various terms.