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LOCAL MOMENTUM IN WAVE MECHANICS. II

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ABSTRACT

It is shown that in the limit $h\rightarrow 0$ the quantity local momentum becomes equal to the classical momentum and the equations involving local momentum become equations of classical mechanics. Operators are found for this quantity corresponding to the operator $(h/2\pi i)$ (d/dx).

1.

THE most usual statement concerning the relation of quantum to classical mechanics is that in the limit $h\rightarrow 0$ quantum mechanical relations connecting the observables of this theory pass over into classical equations. One of the most important of such relations is the commutator of matrix mechanics

$$px - xp = \frac{h}{2\pi i} 1$$

connecting the dynamical variables p and x. In the limit $h=0$ we obtain the following results for p and x: (1) they become commuting quantities of the classical theory (2) they are simultaneously observable to any degree of accuracy. The above results along with the behavior of the quantum mechanical equations of motion are sufficient to show that, if we represent the motion of a system by a wave packet, in the limit $h\rightarrow 0$ (1) the initial size of the packet may be made as small as we please, (2) the wave packet does not spread, and (3) it moves in accordance with the laws of classical mechanics. The above is not intended as a proof that

$$\lim_{h\rightarrow 0} \text{(quantum mechanics)} = \text{classical mechanics}$$

but is simply an outline of the steps which might be taken in such a proof.

In a recent issue of the Physical Review the author published a paper ("A Note on Local Momentum in Wave Mechanics")[1] in which it was shown that the wave functions for one-dimentional problems could be written in the forms

$$\begin{aligned} \psi^{+} &= P(x)^{-1/2} e^{2\pi i/h \int P dx} \\ \psi^{-} &= P(x)^{-1/2} e^{-2\pi i/h \int P dx} \end{aligned} \qquad (1)$$

or for discrete states[2]

$$\psi = P(x)^{-1/2} \cos \frac{2\pi}{h} \int P dx \qquad (2)$$

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[1] Young, Phys. Rev. **38**, 1612 (1931).

[2] For discrete states we require $(2\pi/h)\int P dx = n\pi$ the range of integration being taken over the entire range of the coordinate. Note that twice the integral makes the analogy with the Bohr-Sommerfeld quantum condition $\oint p dx = nh$ complete.

where $P(x)$ (called local momentum) satisfies the differential equation

$$P^2 + 2m(V - E) = \frac{h^2}{4\pi^2} P^{1/2} \frac{d^2}{dx^2} P^{-1/2}. \tag{3}$$

The Bohr-Sommerfeld quantum condition was there revived in terms of the local momentum and it was stated that in the limit, $P(x)$ becomes equal to $p(x)$, the classical momentum. This may be demonstrated easily from Eq. (3) for if we go to the limit $h = 0$ we obtain

$$P^2(x) + 2m(V - E) = 0 \tag{4}$$

but classically we have

$$p^2(x) + 2m(V - E) = 0$$

Therefore in the limit $h \rightarrow 0$, $P(x) \rightarrow p(x)$. If in Eq. (3) we introduce $S_h = \int P dx$ we obtain as $h \rightarrow 0$

$$\left(\frac{dS_0}{dx}\right)^2 + 2m(V - E) = 0. \tag{5}$$

This is the one-dimensional Hamilton-Jacobi equation, S_0 is the classical action function and S_h its wave mechanical analogue.

It is clear that $P(x)$ is not a dynamical variable but it is interesting to try to find an operator $\boldsymbol{P}$ corresponding to the operator $\boldsymbol{p} = (h/2\pi i)(d/dx)$. To do this let us consider a wave function of the form ψ^+. Let us answer the question as to what function $Q(x)$ satisfies the relation

$$\frac{h}{2\pi i} \frac{d}{dQ(x)} \psi^+ = P(x)\psi^+ \tag{6}$$

We find

$$\frac{dQ}{dx} = \frac{h}{2\pi i} P^{-1} \frac{d}{dx} \log \psi^+ \tag{7}$$

introduction for ψ^+ its value in terms of $P(x)$ and integrating yields

$$Q(x) = x + \frac{1}{4\pi i} \cdot \frac{h}{P(x)} \cdot \tag{8}$$

Note that for finite h the last term in this equation is just $1/4\pi i$ times a "de Broglie wave-length" and that in the limit $Q(x) \rightarrow x$ the dynamical variable conjugate to p. $Q(x)$ also satisfies

$$\frac{h}{2\pi i} \frac{d}{dQ} \psi^- = - P(x)\psi^- \tag{9}$$

Moreover we may write

$$\psi^+ = e^{2\pi i/h \int P dQ}$$
$$\psi^- = e^{-2\pi i/h \int P dQ} \tag{10}$$

and the real wave function

$$\psi = \cos (2\pi/h) \int P dQ. \tag{11}$$

So far the author has discovered no real practical advantage in writing the wave functions in this form but their simplicity has a definite elegrance and aesthetic value. If we define an operator $\boldsymbol{P}$ by the relation

$$\boldsymbol{P} = \frac{h}{2\pi i}\,\frac{d}{dQ}$$

we can easily show that

$$\boldsymbol{P} = \left[1 + \frac{1}{4\pi i}\,\frac{d}{dx}\left(\frac{h}{P(x)}\right)\right]\boldsymbol{p}. \tag{12}$$

It is clear from our definition of the operator $\boldsymbol{P}$ that

$$\boldsymbol{PQ} - \boldsymbol{QP} = \frac{h}{2\pi i}\,\boldsymbol{1}$$

but a more significant relation may be derived, namely

$$\boldsymbol{PQ} - \boldsymbol{QP} \equiv \boldsymbol{px} - \boldsymbol{xp}. \tag{13}$$

These results are interesting and suggestive and may be extended easily. We consider that a close and fundamental relationship has been established between the local momentum $P(x)$ and the classical momentum.

2.

The differential equation satisfied by $P(x)$, Eq. (3), is not in a form which allows easy analytic calculation, but, by making use of the relationship existing between $P(x)$ and the fundamental solutions of the Schrödinger equation for the same potential, analytic expressions may be obtained for most of the well-known equations of wave mechanics whose solutions are known exactly. By taking $P^{-\frac{1}{2}}$ as the dependent variable Eq. (3) may be put into a form admirably suited to numerical methods. As might be expected results of problems depending essentially on wave interference may be expressed simply in terms of the phase integral $\int P dx$. For example, consider the elastic scattering of electrons by atoms represented by a central field of potential $V(r)$ (Ramsauer effect). We obtain the well-known result for the cross section in units equal to the square of the Bohr radius for hydrogen

$$Q = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l \tag{14}$$

where

$$\delta_l = \lim_{r\to\infty}\left[\int_0^r P_{1,l}(r)dr - \int_0^r P_{0,l}(r)dr\right] \tag{15}$$

where the $P_{1,l}$ and $P_{0,l}$ are the local momenta corresponding to the wave equations

$$R''_{1,l} + \left[k^2 + 2V(r) - \frac{l(l+1)}{r^2}\right]R_{1,l} = 0 \tag{16}$$

and

$$R''_{0,l} + \left[k^2 - \frac{l(l+1)}{r^2}\right]R_{0,l} = 0 \tag{17}$$

In these equations k^2 represents the energy of the electron beam in hydrogenic units.