WAVE MECHANICS OF RADIATION AND FREE PARTICLES

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Abstract

The quantum theory of radiation, formulated in wave-mechanical language but in terms of running waves, is shown to lead under suitable conditions to the classical theory of emission by a particle as a first approximation. Non-relativistic mechanics is used, and the limitations of the method are brought out in terms of classical theory. The characteristic quantum phenomenon of the Compton effect requires that the Compton recoil shall exceed the spread in momentum of the particle, and so that the wave-packet shall greatly exceed in diameter the wave-length of the incident radiation; this condition is realizable on any scale of magnitude. For quasi-classical motion just the reverse is required; furthermore, if the motion is to last for a number of wave-periods, the wave-length must greatly exceed h/mc (m = mass of the particle). The classical packet-motion arises from combinations between quantum-states of the particle-field system; this fact suggests that the photon is not a fundamental constituent of the field but only one possible aspect of its action. The infinite "zero-energy" associated apparently even with the lowest energy-state of the field implies no electromagnetic field intensities at any point. Whereas for problems on energy or momentum it is customary to represent an incident wave-train by a single typical quantum state, to obtain motional phenomena of classical type a more complete expression such as that given in the paper must be used. The physically empty field likewise corresponds to a mixture of quantum states but without correlation among their amplitudes.

THE quantum-mechanical theory of radiation initiated by Dirac, which yields a correct formula for the quantum phenomenon of the Compton effect, must likewise lead to the classical theory of radiation as an approximation valid under certain conditions. A certain theoretical interest seems to attach to the manner in which it does this. In particular, one would like to see how we can bring under the same theoretical roof the continuous oscillation, chiefly along the electric vector, of a free electron in a Hertzian wave-train, on the one hand, and the billiard-ball Compton recoil of an electron on the other, both occurring in radiation fields that appear to differ only in respect of wave-length.[1] Such a discussion forms the object of the present paper. A wave-mechanical form of the theory, new in some details, is employed, partly because wave-mechanics seems less repellently abstract to many than the more usual symbolic matrix formulation; and the treatment is non-relativistic throughout. The mathematical machinery remains, nevertheless, in spite of every simplification rather voluminous,[2] and is given only in skeleton; it is hoped that those who are interested chiefly in the ideas involved will find no difficulty in following the discussion while those who wish to follow the

[1] Cf. A. H. Compton, Phys. Rev. **31**, 59 (1928).
[2] Cf. W. Heisenberg, Ann. d. Physik **9**, 338 (1931).

mathematics in detail will have no trouble in following directions and filling in the gaps.

The classical theory of the field in Hamiltonian form is first developed as a prototype in greater detail than has been done elsewhere; then wave-mechanical formulas are deduced for the motion of the packet centroid in a radiation field, and from these the classical emission-formula for a point charge is obtained as an approximation. Letting this charge recede to infinity we have a typical quantum-mechanical expression for a physical plane wave-train. The paper closes with a brief discussion of the relation between the Compton effect and the classical law of force.

Classical Theory in Hamiltonian Form

A pure radiation field, characterized as such by the absence of electric charges in it, can be described completely in terms of a vector potential $\boldsymbol{A}$; and it can further be resolved into an infinite set of plane wave-trains running in all directions, that is, we can write:

$$\boldsymbol{A} = \int [\boldsymbol{a}'(\nu, \boldsymbol{n}) \cos 2\pi\nu(t - \gamma \boldsymbol{n}\cdot\mathbf{x}) + \boldsymbol{a}''(\nu, \boldsymbol{n}) \sin 2\pi\nu(t - \gamma\boldsymbol{n}\cdot\mathbf{x})]d\omega, \quad (1)$$

in which ν = frequency, $c = 1/\gamma$ = speed of light, $\mathbf{x}$ is the vector distance of the field-point from the origin, $\boldsymbol{n}$ is a unit vector in the direction of the element of solid angle $d\omega$ and $\boldsymbol{a}'$ and $\boldsymbol{a}''$ are vector functions of ν and $\boldsymbol{n}$ perpendicular to each other and to $\boldsymbol{n}$; the integral extends over all positive ν and all directions of $\boldsymbol{n}$. Associated with each pair of values of ν and $\boldsymbol{n}$ there are thus two wave-trains polarized at right angles. For the energy in the field one finds by means of Maxwell's equations, the formulas $\boldsymbol{E} = -\gamma\partial \boldsymbol{A}/\partial t$, $\boldsymbol{H} = \text{curl } \boldsymbol{A}$, and a simple application of the divergence theorem:

$$W_0 = \int (E^2 + H^2)/8\pi d\tau = (\gamma^2/8\pi) \int [(\partial \boldsymbol{A}/\partial t)^2 - \boldsymbol{A}\cdot\partial^2\boldsymbol{A}/\partial t^2]d\tau,$$

the field being assumed to vanish fast enough at infinity so that $\int \text{div } \boldsymbol{A}\times\boldsymbol{H}d\tau = 0$. We now substitute here for $\boldsymbol{A}$ from (1), replace ν and ω as variables in one integral by νn_x, νn_y, νn_z (so that $\nu^2 = (\nu n_x)^2 + (\nu n_y)^2 + (\nu n_z)^2$ and $\nu^2 d\nu d\omega$ = element of volume in polars $= d(\nu n_x)d(\nu n_y)d(\nu n_z)$), and evaluate the result by means of repeated applications of the Fourier integrals,

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f(\mu') \cos 2\pi x(\mu' - \mu)d\mu' = f(\mu),$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f(\mu') \sin 2\pi x(\mu' - \mu)d\mu' = 0. \quad 2)$$

The final result is an expression for the energy of the field in terms of the amplitudes a' and a'' of the wave-trains, viz.,

$$W_0 = (\pi c/2) \int (a'^2 + a''^2)d\nu d\omega. \quad (3)$$

In order to apply quantum theory we must now substitute for Maxwell's equations a statement of the laws of the field in Hamiltonian form, that is, we must describe the field in terms of variables obeying the Hamiltonian equations of motion. For simplicity we shall make these finite in number by replacing our continuous set of wave-trains by a discrete set, associated in pairs with equal intervals $\Delta\nu$ on the frequency axis and equal elements of solid angle $\Delta\omega$ in direction; the integrals above are then to be replaced by sums. Then at the end of our calculations we let $\Delta\nu$ and $\Delta\omega$ approach zero. There is a considerable range of choice for the Hamiltonian variables, but all must lead to the same result. The simplest choice for wave-mechanical purposes[3] seems to be to take as variables describing the wave-trains associated with given ν and $\boldsymbol{n}$ the two pairs:

$$\begin{aligned} Q' &= Ba' \cos 2\pi\nu t & Q'' &= Ba'' \sin 2\pi\nu t \\ P' &= -Ba' \sin 2\pi\nu t & P'' &= Ba'' \cos 2\pi\nu t \end{aligned} \tag{4}$$

where $B=(c\Delta\nu\Delta\omega/2\nu)^{1/2}$. It will be more convenient, however, to number these pairs of variables in a single sequence, merely remembering that two members of the sequence belong to each ν and $\boldsymbol{n}$. Eqs. (1) and (3) then become, in terms of these coordinates:

$$A = \sum(\sigma)\boldsymbol{b}_\sigma(Q_\sigma \cos \gamma\mu_\sigma x_\sigma - P_\sigma \sin \gamma\mu_\sigma x_\sigma), \tag{5}$$

$$W_0 = \sum(\sigma)\pi\nu_\sigma(Q_\sigma{}^2 + P_\sigma{}^2), \tag{6}$$

where $\mu_\sigma = 2\pi\nu_\sigma$, $\boldsymbol{n}_\sigma$ is the unit ray-vector for wave-train number σ, $\mathbf{x}_\sigma = \boldsymbol{n}_\sigma \cdot \mathbf{x}$, $\boldsymbol{b}_\sigma = (2\nu_\sigma\Delta\nu\Delta\omega/c)^{1/2}\boldsymbol{l}_\sigma$, $\boldsymbol{l}_\sigma$ being a unit vector in the direction of the electric vector (i.e., of the original $\boldsymbol{a}'$ or $\boldsymbol{a}''$). Obviously with W_0 as Hamiltonian the new variables obey the Hamiltonian equations: $\dot{Q}_\sigma = \partial W_0/\partial P_\sigma$, $\dot{P}_\sigma = -\partial W_0/\partial Q_\sigma$.

For the treatment of physical problems it is necessary to introduce electricity into the field, contrary to our assumption that there is none there. It was tacitly assumed by Dirac[4] that this inconsistency would not invalidate the results so far as concerns pure radiation effects; we shall see presently to what extent this assumption is justified. Accordingly we add the usual (non-relativistic) expression for one or more independent particles having charges e_j and masses m_j and obtain as the classical Hamiltonian function for a number of particles interacting only by way of the radiation field:

$$W = \sum(\sigma)\pi\nu_\sigma(Q_\sigma{}^2 + P_\sigma{}^2) + \sum(j)(1/2m_j)(\boldsymbol{p}_j - \gamma e_j \boldsymbol{A}_j)^2, \tag{7}$$

where $\boldsymbol{A}_j$ stands for $\boldsymbol{A}$ with the coordinates of particle no. j inserted for $\mathbf{x}$, $\boldsymbol{p}_j$ is the vector generalized momentum for the jth particle, and $\Sigma\,(j)$ extends over all particles.

It is well known that the second term in W as given by (7) yields correct values for the motion of the particles themselves in a given field with vector

[3] Similar variables are employed by Born and Jordan in their book, Quantenmechanik, but they employ standing waves inside a box in place of running waves.

[4] P. A. M. Dirac, Roy. Soc. Proc. **114**, 243 (1927).

potential $\boldsymbol{A}_j$, the velocity of the jth particle being $\boldsymbol{v}_j=(\boldsymbol{p}_j-\gamma e_j\boldsymbol{A}_j)/m_j$. We shall treat here the converse case, the reaction of the particle upon the field according to classical theory, doing this partly because the calculations will serve as a valuable guide in the quantum work and partly to reveal the nature of the limitations upon Dirac's hypothesis. For simplicity we shall assume but one particle present; the extension to many is easy. We have then:

$$\dot{Q}_\sigma = \frac{\partial W}{\partial P_\sigma} = 2\pi\nu_\sigma P_\sigma + \frac{e}{cm}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)\cdot\boldsymbol{b}_\sigma \sin\gamma\mu_\sigma x_\sigma,$$
$$\dot{P}_\sigma = -\frac{\partial W}{\partial Q_\sigma} = -2\pi\nu_\sigma Q_\sigma + \frac{e}{cm}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)\cdot\boldsymbol{b}_\sigma \cos\gamma\mu_\sigma x_\sigma. \tag{8}$$

Let us suppose that the field vanishes until $t=0$, when the charge begins to move. The appropriate solution of (8) is,

$$Q_\sigma + iP_\sigma = \gamma e\int_0^t \boldsymbol{b}_\sigma\cdot\boldsymbol{v}[\sin\mu_\sigma(t-t'+\gamma x_{\sigma e}) + i\cos\mu_\sigma(t-t'+\gamma x_{\sigma e})]dt', \tag{8a}$$

where $\boldsymbol{v}=(\boldsymbol{p}-e\boldsymbol{A}/c)/m$, and by (5) the resulting potential is:

$$\boldsymbol{A} = 2e\gamma^2\sum(\sigma)\nu_\sigma\Delta\nu\Delta\omega\boldsymbol{l}_\sigma\int_0^t \boldsymbol{l}_\sigma\cdot\boldsymbol{v}\sin\mu_\sigma(t-t'+\gamma x_{\sigma e}-\gamma x_\sigma)dt', \tag{9}$$

in which $\boldsymbol{v}$ and $x_{\sigma e}$ stand for values at the particle and at time t' while x_σ refers to the field-point at which $\boldsymbol{A}$ is given by the equation. Now if $\boldsymbol{B}$ is any vector and if we denote the two perpendicular $\boldsymbol{l}$'s that belong to given ν_σ and $\boldsymbol{n}_\sigma$ by $\boldsymbol{l}_{\sigma1}$, $\boldsymbol{l}_{\sigma2}$, we have

$$\boldsymbol{l}_{\sigma1}\cdot\boldsymbol{B}\,\boldsymbol{l}_{\sigma1} + \boldsymbol{l}_{\sigma2}\cdot\boldsymbol{B}\,\boldsymbol{l}_{\sigma2} = \boldsymbol{B}_{\perp\sigma} \tag{10}$$

where $\boldsymbol{B}_{\perp\sigma}$ denotes the vector component of $\boldsymbol{B}$ perpendicular to $\boldsymbol{n}_\sigma$. Hence

$$\boldsymbol{A} = 2e\gamma^2\int_0^\infty \nu_\sigma d\nu_\sigma\int d\omega\int_0^t \boldsymbol{v}_{\perp\sigma}\sin\mu_\sigma(t-t'+\gamma x_{\sigma e}-\gamma x_\sigma)dt'.$$

The integration over $d\omega$ is easily carried out first. We note that $\int\boldsymbol{v}_{\perp\sigma}\sin\mu_\sigma\gamma(x_{\sigma e}-x_\sigma)d\omega$ vanishes by symmetry, $\boldsymbol{v}_{\perp\sigma}$ being the same and $x_{\sigma e}-x_\sigma$ reversed in sign for trains moving oppositely. Hence

$$\boldsymbol{A} = 2e\gamma^2\int_0^\infty \nu_\sigma d\nu_\sigma\int_0^t dt'\int \boldsymbol{v}_{\perp\sigma}\sin\mu_\sigma(t-t')\cos\mu_\sigma\gamma(x_{\sigma e}-x_\sigma)d\omega.$$

Now let $\boldsymbol{r}$ denote distance from the particle to the field-point and take this line as the axis of polars. Then $\cos\mu_\sigma\gamma(x_{\sigma e}-x_\sigma) = \cos(\mu_\sigma\gamma r\cos\theta)$, θ referring to the direction of $\boldsymbol{n}_\sigma$, $\boldsymbol{v}_{\perp\sigma}=\boldsymbol{v}-\boldsymbol{v}\cdot\boldsymbol{n}_\sigma\boldsymbol{n}_\sigma=\boldsymbol{v}-(v_r\cos\theta-v_\perp\sin\theta\cos\phi)\boldsymbol{n}_\sigma$, $\boldsymbol{v}_r$ and $\boldsymbol{v}_\perp$ being vector components respectively parallel and perpendicular to $\boldsymbol{r}$, $d\omega=\sin\theta d\theta d\phi$; and after several partial integrations over θ, and then one in the $1/r^3$ terms with respect to ν_σ, one finds that several terms cancel and

$$A = 2e\gamma^2 \int_0^\infty d\nu_\sigma \int_0^t \left[\frac{2v_\perp}{\gamma r}\sin \mu_\sigma(t-t') + \frac{2v_r - v_\perp}{\pi\nu_\sigma\gamma^3 r^3}(t-t')\cos\mu_\sigma(t-t')\right]\sin \mu_\sigma \gamma r dt'$$

in which v_r, $v_\perp$ and r are all functions of t'. We next replace the product of sines or cosines by functions of $t-t'\pm\gamma r$, change the variable of integration by means of the formula, $d(t'\mp\gamma r)=(1\pm\gamma v_r)dt'$, and then apply the formulas:

$$\int_0^\infty d\mu_\sigma \int_0^T f(\tau)\cos\mu_\sigma(\tau-k)d\tau = \lim_{\mu\to\infty}\int_0^T f(\tau)\frac{\sin\mu(\tau-k)}{\tau-k}d\tau = \pi f(k) \text{ or } 0$$

according as $0<k<T$ or not, k and T being constants; and

$$\int_0^\infty \frac{d\nu_\sigma}{\nu_\sigma}\sin M\nu_\sigma = \frac{\pi}{2} \text{ if } M>0 \text{ and } = -\frac{\pi}{2} \text{ if } M<0.$$

We thus find that for $t>\gamma r$

$$A = \frac{e}{c}\left(\frac{v_\perp}{r(1-\gamma v_r)}\right)_{t-\gamma r} + ec\int_{t-\gamma r}^t \frac{2v_r - v_\perp}{r^3}(t-t')dt', \tag{11}$$

while for $0<t<\gamma r$

$$A = ec\int_0^t \frac{2v_r - v_\perp}{r^3}(t-t')dt'.$$

A as given by (11) should represent at least the radiative component of the field around a point-charge moving with variable velocity v. It is immediately obvious, that the result cannot be entirely correct, for the second term is not properly retarded. Even the first term has an unfamiliar look; but this is merely a consequence of the fact that our wave-trains must necessarily yield a solenoidal potential whereas in the usual theory div A does not always vanish; upon differentiating the first term one nevertheless obtains correct values for the electric and magnetic fields in so far as they are of order $1/r$. The true magnitude of the error is best appreciated by passing to the case of simple harmonic motion. If we put $v=v_0 \sin 2\pi\nu_0 t$, where v_0 is small, the integral in (11) is easily evaluated (r being treated as sensibly constant). Omitting details we shall merely state here that the result represents the correct Hertzian electric and magnetic fields around a vibrating doublet of instantaneous moment $-ev_0 \cos 2\pi\nu_0 t/2\pi\nu_0$, starting from rest at $t=0$, *except* that there is superposed upon this field for $t>0$ the static electric field of a doublet of instantaneously equal and opposite moment, and also out to $r=ct$ the same for a doublet of moment $-ev_0/2\pi\nu_0$. The error is thus of order $(r/\lambda_0)^3$, $\lambda_0=c/\nu_0$. (The complete classical field would likewise contain, of course, the simple electrostatic field of the particle.)

In general, then, in classical theory the application of the Hamiltonian equations to the running-wave formulation gives correctly the radiation field around a moving point-charge at distances rather large relative to all the principal wave-lengths involved in the Fourier representation of the motion of the charge. This restriction excludes the simple case of a charge in uniform relative motion, and it also prevents, unfortunately, any application to the problem of radiation reaction.

Wave Mechanics of the Radiation Field

To apply wave mechanics we now introduce a wave-scalar ψ, which is a function of the Q's, the coordinates of the particles and the time, and form a Schrödinger equation for it in the usual way, assuming that W as given by (7) is the correct Hamiltonian so long as the particles interact only indirectly by way of the radiation field. The equation reads:

$$-\epsilon\frac{\partial\psi}{\partial t} = \sum(\sigma)\pi\nu_\sigma\left(\epsilon^2\frac{\partial^2}{\partial Q_\sigma^2}+Q_\sigma^2\right)\psi + \sum(j)\frac{1}{2m_j}\left(\epsilon\nabla_j - \frac{e_j}{c}A_j\right)^2\psi, \quad (12)$$

where $\epsilon = h/2\pi i$, the subscript j on ∇ and A specifies that the space coordinates occurring therein are those of the jth particle, and A is the operator vector-potential:

$$A = \sum(\sigma)\boldsymbol{b}_\sigma R_\sigma, \qquad R_\sigma = Q_\sigma\cos\gamma\mu_\sigma x_\sigma - \epsilon\frac{\partial}{\partial Q_\sigma}\sin\gamma\mu_\sigma x_\sigma. \quad (13)$$

We shall presently deduce formulas for the motion of packets directly from this equation, but it seems impossible to make use of them without resorting to the usual cumbrous resolution into characteristic functions. The latter may be summarized as follows.

The Hamiltonian for the field alone, $W_0 = \Sigma(\sigma)\pi\nu_\sigma(Q_\sigma^2+P_\sigma^2)$, is that for a collection of independent harmonic oscillators, expressed in what might be called "canonical form"; accordingly we have, as usual, as characteristic functions for the field alone

$$U_s(Q, N) = \eta(Q_1, N_{1s})\eta(Q_2, N_{2s})\cdots,$$

where $N_{\sigma s}$ denotes a particular choice of the N_σ's and $\eta(Q_\sigma, N_\sigma)$ stands for a normalized characteristic solution of the harmonic-oscillator equation:

$$\pi\nu_\sigma(Q_\sigma^2 - h^2/4\pi^2\partial^2/\partial Q_\sigma^2)\eta(Q_\sigma, N_\sigma) = (N_\sigma + \tfrac{1}{2})h\nu_\sigma\eta(Q_\sigma, N_\sigma).$$

Inserting then M particles with vector coordinates $\mathbf{x}_j$ and momenta $\boldsymbol{p}_j$, we can write:

$$\psi = h^{-3M/2}\sum(s)U_s\int a(s\boldsymbol{p})e^{\zeta(\Sigma p\cdot \mathbf{x} - W_{sp}t)}$$
$$= h^{-3M/2}\sum(s)U_s e^{-\zeta W_s t}\int \alpha(s\boldsymbol{p})e^{\zeta\Sigma p\cdot\mathbf{x}}dp. \quad (14)$$

where $\zeta = 1/\epsilon = 2\pi i/h$, $W_s = \Sigma(\sigma)(N_{\sigma s}+1/2)h\nu_\sigma$, $W_{sp} = W_s + \Sigma(j)p_j^2/2m_j$, $\Sigma\boldsymbol{p}\cdot\mathbf{x}$ stands for $\Sigma(j)\boldsymbol{p}_j\cdot\mathbf{x}_j$, and the integration extends over all components of momentum of all particles; $\alpha(s\boldsymbol{p}) = a(s\boldsymbol{p}) \exp(-\zeta\Sigma(j)p_j^2t/2m_j)$, both α and a being functions of s and the momenta of all particles and normalized so that

$$\sum(s) \int a^*(s\boldsymbol{p})a(s\boldsymbol{p})dp = \sum(s) \int \alpha^*(s\boldsymbol{p})\alpha(s\boldsymbol{p})dp = 1. \tag{15}$$

To find the time-variation of the coefficient α due to the interaction terms in (12) we substitute for ψ from (14) in (12) and select a particular coefficient in the usual way, by multiplying by a particular $U_s e^{-\zeta\Sigma \boldsymbol{p}.\boldsymbol{x}}$ and integrating over all Q's and x's; the orthogonality relations for the x's are expressed by the Fourier formulas (2). For the operator R we note that according to well-known recurrence relations

$$Q\eta(N) = (h/4\pi)^{1/2}[N^{1/2}\eta(N-1) + (N+1)^{1/2}\eta(N+1)],$$
$$d\eta(N)/dQ = (\pi/h)^{1/2}[N^{1/2}\eta(N-1) - (N+1)^{1/2}\eta(N+1)],$$

with $\eta(-1)=0$, whence

$$R_\sigma U_s = (h/4\pi)^{1/2}[N_\sigma^{1/2}U(\overset{-}{s_\sigma})e^{i\mu_\sigma\gamma x_\sigma} + (N_\sigma+1)^{1/2}U(\overset{+}{s_\sigma})e^{-i\mu_\sigma\gamma x_\sigma}, \tag{16}$$

where $U(s_\sigma{}^\pm)$ stands for U_s with $\eta(Q_\sigma, N_\sigma \pm 1)$ substituted in it for $\eta(Q_\sigma, N_\sigma)$; the extension of this formula to $R_\tau R_\sigma U_s$ is easy. The exponentials in x_σ introduce a "Compton shift," expressed by (18) below, in the $\boldsymbol{p}$ occurring in $\boldsymbol{b}_\sigma \cdot \boldsymbol{p}$, but this has no effect because $\boldsymbol{b}_\sigma \cdot \boldsymbol{n}_\sigma = 0$. We thus find eventually for the rate of change of α

$$\begin{aligned}(\dot{\alpha}sp) = &- \sum(j)\zeta p_j^2/2m_j \\ &+ \sum(j)\zeta e_j/cm_j(h/4\pi)^{1/2}\sum(\sigma)\boldsymbol{b}_\sigma\cdot\boldsymbol{p}[(N_{\sigma s}+1)^{1/2}\alpha(\overset{+}{s_\sigma}, \overset{-}{\boldsymbol{p}_{j\sigma}})e^{-i\mu_\sigma t} \\ &+ N_{\sigma s}{}^{1/2}\alpha(\overset{-}{s_\sigma}\overset{+}{\boldsymbol{p}_{j\sigma}})e^{i\mu_\sigma t}] - \sum(j)ie_j^2/4c^2m_j\sum(\sigma,\tau)\boldsymbol{b}_\sigma\cdot\boldsymbol{b}_\tau \\ &\quad[(N_{\sigma s}+1)^{1/2}(N_{\tau s}+1+\delta_\sigma^\tau)^{1/2}\alpha(\overset{++}{s_{\sigma\tau}}\overset{--}{\boldsymbol{p}_{j\sigma\tau}})e^{-i(\mu_\sigma+\mu_\tau)t} \\ &+ (N_{\sigma s}+1)^{1/2}(N_{\tau s}+\delta_\sigma^\tau)^{1/2}\alpha(\overset{+-}{s_{\sigma\tau}}\overset{-+}{\boldsymbol{p}_{j\sigma\tau}})e^{-i(\mu_\sigma-\mu_\tau)t} \\ &+ N_{\sigma s}{}^{1/2}(N_{\tau s}+1-\delta_\sigma^\tau)^{1/2}\alpha(\overset{-+}{s_{\sigma\tau}}\overset{+-}{\boldsymbol{p}_{j\sigma\tau}})e^{i(\mu_\sigma-\mu_\tau)t} \\ &+ N_{\sigma s}{}^{1/2}(N_{\tau s}-\delta_\sigma^\tau)^{1/2}\alpha(\overset{--}{s_{\sigma\tau}}\overset{++}{\boldsymbol{p}_{j\sigma\tau}})e^{i(\mu_\sigma+\mu_\tau)t}],\end{aligned} \tag{17}$$

in which such symbols as $s_\sigma{}^\pm$, $s_{\sigma\,\tau}^{\pm\pm}$ refer to states the same as s except that N_σ and N_τ have been increased or decreased by unity as directed by the signs (or increased or decreased by 2 when $\sigma=\tau$), α being put equal to 0 when it thus comes to contain a negative N, $\delta_\sigma{}^\tau = 0$ except that $\delta_\sigma{}^\sigma = 1$, and

$$\overset{\pm}{\boldsymbol{p}}_{j\sigma} = \boldsymbol{p}_j \pm \gamma h\nu_\sigma \boldsymbol{n}_\sigma,\ \overset{\pm\pm}{\boldsymbol{p}}_{j\sigma\tau} = \boldsymbol{p}_j \pm \gamma h\nu_\sigma\boldsymbol{n}_\sigma \pm \gamma h\nu_\tau\boldsymbol{n}_\tau. \tag{18}$$

It is to be understood that in a and α the $\boldsymbol{p}$'s of any particle not specifically indicated in the notation are the same throughout any given equation. In sums such as $\Sigma(\sigma, \tau)$ in (17) the terms $\sigma=\tau$ are a nuisance but they can fortunately usually be omitted in comparison with the other terms (i.e., their contribution comes out finally to be proportional to $\Delta\nu\Delta\omega$ and so vanishes in the limit).[5]

[5] A serious question as to the convergence of the sum in (17) arises if one really allows the range for ν_σ to extend to infinity, but all mathematical questions of this sort will be passed over in silence in this paper. Cf. L. Rosenfeld, Zeits. f. Physik 70, 454 (1931).

Motion of Packet Centroids

A fundamental condition for the approximate validity of classical laws seems to be that we must be able to represent the particles by wave-packets which remain of negligible size over a considerable interval of time. The packets then move about practically like particles. Strictly speaking, only a single packet exists for the whole system, and it lies in a space whose coordinates include those of all particles and also all of the Q's; but we can fix our attention upon its location in the space of any single particle and regard it, seen from this angle, as a packet for that particle, and under classical conditions we can even imagine the virtual packets so obtained for the various particles all to move about in the same space. When the packets are small it suffices to consider the location and motion of their centroids; the laws of nature then take on their classical simplicity.

The vector coordinate of the jth particle being $\mathbf{x}_j$, the coordinate of its centroid is defined by

$$\bar{\mathbf{x}}_j = \int \mathbf{x}_j \psi^* \psi dq, \tag{19}$$

in which dq covers the Q's as well as the coordinates of all particles. The time derivatives of $\mathbf{x}_j$ are most easily obtained by the symbolic method,[6] which is easily extended to cover the present case by first deducing from the wave equation, Eq. (12), the "Hamiltonian equation of motion":

$$\begin{aligned} \frac{d}{dt}\int \psi^* f(x, \boldsymbol{p}, Q, P)\psi dq = \int \psi^* \Big\{ \sum(i)\Big[\Big(\frac{\partial f}{\partial x_i}\frac{\partial}{\partial p_i}\Big)W - \Big(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial x_i}\Big)W\Big] \\ + \sum(\sigma)\Big[\Big(\frac{\partial f}{\partial Q_\sigma}\frac{\partial}{\partial P_\sigma}\Big)W - \Big(\frac{\partial f}{\partial P_\sigma}\frac{\partial}{\partial Q_\sigma}\Big)W\Big]\Big\}\psi dq. \end{aligned} \tag{20}$$

Here Σ (i) extends over the coordinates of all particles and W is the Hamiltonian given by (7); a "compound" derivative such as $(\partial f/\partial x_i\, \partial/\partial p_i)W$ is to be calculated as in matrix theory by replacing in W each unit factor p_i^1 in turn by $\partial f/\partial p_i$ and adding the results, and, in both f and W, p_i and P_σ are to be treated as algebraic quantities except for the familiar interchange rules, $p_k x_l - x_l p_k = \epsilon\delta_k^l$, $P_\sigma Q_\tau - Q_\tau P_\sigma = \epsilon\delta_\sigma^\tau$, and are finally to be replaced by $\epsilon\partial/\partial x_i$, $\epsilon\partial/\partial Q_\sigma$ respectively just before carrying out the indicated integrations.

To find the centroid velocity for the jth particle we put each of its coordinates x_{jk} in turn for f in (20) and find, in vector notation,

$$\bar{\mathbf{v}}_j = \frac{d}{dt}\bar{\mathbf{x}}_j = \int \psi^* \Big(\frac{\epsilon}{m_j}\nabla_j - \frac{e_j}{cm_j}\sum(\sigma)\boldsymbol{b}_\sigma R_{\sigma j}\Big)\psi dq, \tag{21}$$

$R_{\sigma j}$ denoting R_σ with $x_{\sigma j}$ or $\boldsymbol{n}_\sigma \cdot \mathbf{x}_j$ substituted for x_σ. For the next differentiation we put for f, first, p_{jk}, then $R_{\sigma j} = R_{\sigma j}(x_j, Q, P)$, and obtain easily:

[6] Cf. E. H. Kennard, Nat. Acad. Sci. Proc. **17**, 58, (1931).

$$\epsilon d/dt \int \psi^* \nabla_j \psi dq = d/dt \int \psi^* \boldsymbol{p}_j \psi dq = (e_j/2cm_j)$$

$$\int \psi^* \Big\{ [\boldsymbol{p}_j - \gamma e_j \sum(\sigma) \boldsymbol{b}_\sigma R_{\sigma j}] \cdot \sum(\tau) \boldsymbol{b}_\tau (\nabla_j R_{\tau j}) \tag{22}$$

$$+ \sum(\tau)(\nabla_j R_{\tau j}) \boldsymbol{b}_\tau \cdot [\boldsymbol{p}_j - \gamma e_j \sum(\sigma) \boldsymbol{b}_\sigma R_{\sigma j}] \Big\} \psi,$$

the dots denoting as usual the scalar vector product, and, after replacing $Q_\sigma \sin \gamma\mu_\sigma x_\sigma + P_\sigma \cos \gamma\mu_\sigma x_\sigma$, when met, by $-(\partial R_\sigma/\partial x_\sigma)/\gamma\mu_\sigma$,

$$d/dt \int \psi^* R_{\tau j} \psi dq = - c \int \psi^* (\partial R_{\tau j}/\partial x_{\tau j}) \psi dq$$

$$+ (1/2m_j) \int \psi^* \{ [\boldsymbol{p}_j - \gamma e_j \sum(\sigma) \boldsymbol{b}_\sigma R_{\sigma j}] \cdot (\nabla_j R_{\tau j})$$

$$+ (\nabla_j R_{\tau j}) \cdot [\boldsymbol{p}_j - \gamma e_j \sum(\sigma) \boldsymbol{b}_\sigma R_{\tau j}] \} \psi dq.$$

In both of these equations $(\nabla_j R_\tau)$ stands for $\boldsymbol{n}_\tau \, \partial R_\tau/\partial x_{\tau j}$, this ∇ not operating, as does the one below, on the following ψ. Upon substituting in $d\bar{\boldsymbol{v}}_j/dt$ as obtained by differentiating (21), the long integrals combine according to the vector formula for any vector $\boldsymbol{B}$, $\boldsymbol{B} \cdot \boldsymbol{b}_\tau (\nabla R) - \boldsymbol{B} \cdot (\nabla R_\tau) \boldsymbol{b}_\tau = \boldsymbol{B} \times [\nabla R_\tau \times \boldsymbol{b}_\tau] = \boldsymbol{B} \times (\boldsymbol{n}_\tau \times \boldsymbol{b}_\tau) \, \partial R_\tau/\partial x_\tau$. We may also consolidate the two terms occurring in each integral by means of the easily established interchange relations,

$$\boldsymbol{p}_j \frac{\partial R_{\tau j}}{\partial x_{\tau j}} - \frac{\partial R_{\tau j}}{\partial x_{\tau j}} \boldsymbol{p}_j = \epsilon \boldsymbol{n}_\tau \frac{\partial^2 R_{\tau j}}{\partial x_{\tau j}^2}, \quad R_{\sigma j} \frac{\partial R_{\tau j}}{\partial x_{\tau j}} - \frac{\partial R_{\tau j}}{\partial x_{\tau j}} R_{\tau j} = \epsilon \gamma \mu_\tau \delta_\sigma^\tau. \tag{23}$$

The final result thus obtained can be written:

$$\frac{d^2}{dt^2} \bar{\mathbf{x}}_j = \frac{e_j}{m_j} \int \psi^* \sum(\sigma) \boldsymbol{b}_\sigma \frac{\partial R_{\sigma j}}{\partial x_{\sigma j}} \psi dq$$

$$+ \frac{e_j}{cm_j} (Rl) \int \psi^* \Big(\frac{\epsilon}{m_j} \nabla_j - \frac{e_j}{cm_j} \sum(\sigma) \boldsymbol{b}_\sigma R_{\sigma j} \Big) \times \sum(\tau) \boldsymbol{n}_\tau \times \boldsymbol{b}_\tau \frac{\partial R_{\tau j}}{\partial x_{\tau j}} \psi dq, \tag{24}$$

where (Rl) means "real part only of what follows". The added terms arising from (23) have here been dropped because they are pure-imaginary, whereas $d^2\bar{\mathbf{x}}_j/dt^2$ must be real; the integrals multiplying ϵ in these dropped terms, like the first integral in (24), contain only real operators linear in P and Q and so are real (cf., e.g.,

$$I = \epsilon \int \psi^* \partial\psi/\partial Q \;\, dQ = \epsilon \psi^* \psi / - \epsilon \int \psi \partial \psi^* / \partial Q dQ = I^* \Big).$$

Formulas (21) and (24) enable us to calculate the motion of all the packet centroids when ψ is known. They are strikingly similar in form to the formulas for a particle in an electromagnetic field expressed in the ordinary (but of course only approximately correct) manner; in fact the only difference is that here the field vectors are replaced by the three operators, $\boldsymbol{A}=\Sigma(\sigma)\boldsymbol{b}_\sigma R_\sigma$, $\boldsymbol{E}=\Sigma(\sigma)\boldsymbol{b}_\sigma \partial R_\sigma/\partial x_\sigma$, $\boldsymbol{H}=\text{curl } \boldsymbol{A}=\Sigma(\sigma)\boldsymbol{n}_\sigma\times\boldsymbol{b}_\sigma\ \partial R_\sigma/\partial x_\sigma$, in which the coordinates of the particle under study are to be inserted. Each particle has, of course, its own electromagnetic field in its own space. One is tempted to form such averages as $\int\psi^*\Sigma(\sigma)\boldsymbol{b}_\sigma R_{\sigma j}\psi dq_{[j]}$, $dq_{[j]}$ covering all coordinates except those of one particle, and then to treat these quantities as field vectors in the space of that particle; if we leave the particle out of consideration altogether these vectors are indeed easily shown to satisfy the Maxwell equations, but with the particle present one is led to very complicated expressions.

For some purposes, on the other hand, we need an alternative form of these expressions in terms of the harmonic constituents of the wave-function. Such forms are readily obtained by inserting ψ from (14) into (21) and (24) and evaluating the integrals in the same way as in arriving at (17). One thus finds for the jth particle:

$$\overline{A}_j = (h/4\pi)^{1/2}\sum(s)\int dp\sum(\sigma)\boldsymbol{b}_\sigma\Big[(N_{\sigma s}+1)^{1/2}\alpha^*(\overset{+}{s_\sigma}\overset{-}{\boldsymbol{p}_{j\sigma}})e^{i\mu_\sigma t} + N_{\sigma s}{}^{1/2}\alpha^*(\overset{-}{s_\sigma}\overset{+}{\boldsymbol{p}_{j\sigma}})e^{-i\mu_\sigma t}\Big]\alpha(s\boldsymbol{p}), \tag{25}$$

$$d\boldsymbol{x}_j/dt = (1/m_j)\ \sum(s)\int \boldsymbol{p}_j\alpha^*(s\boldsymbol{p})\alpha(s\boldsymbol{p})dp - e_j\overline{A}_j/cm_j, \tag{26}$$

$$\begin{aligned} d^2\overline{\mathbf{x}}_j/dt = &-(ie_j/cm_j)(\pi h)^{1/2}\sum(s)\int dp\sum(\sigma)\nu_\sigma[\boldsymbol{b}_\sigma+(1/cm_j)\boldsymbol{p}_j\times(\boldsymbol{n}_\sigma\times\boldsymbol{b}_\sigma)] \\ &\quad [(N_{\sigma s}+1)^{1/2}\alpha^*(\overset{+}{s_\sigma}\overset{-}{\boldsymbol{p}_{j\sigma}})e^{i\mu_\sigma t} - N_{\sigma s}{}^{1/2}\alpha^*(\overset{-}{s_\sigma}\overset{+}{\boldsymbol{p}_{\sigma j}})e^{-i\mu_\sigma t}]\alpha(s\boldsymbol{p}) \\ &+(ie_j{}^2h/2c^3m_j{}^2)\sum(s)\int dp\sum(\sigma\tau)\nu_\tau\boldsymbol{b}_\sigma\times(\boldsymbol{n}_\tau\times\boldsymbol{b}_\tau) \\ &\quad\{[(N_{\sigma s}+1)(N_{\tau s}+1+\delta_\sigma{}^\tau)]^{1/2}\alpha^*(\overset{++}{s_{\sigma\tau}}\overset{--}{\boldsymbol{p}_{j\sigma\tau}})e^{i(\mu_\sigma+\mu_\tau)t} \\ &+[(N_{\sigma s}+1)(N_{\tau s}+\delta_\sigma{}^\tau)]^{1/2}\alpha^*(\overset{+-}{s_{\sigma\tau}}\overset{-+}{\boldsymbol{p}_{j\sigma\tau}})e^{i(\mu_\sigma-\mu_\tau)t} \\ &-[N_{\sigma s}(N_{\tau s}+1-\delta_\sigma{}^\tau)]^{1/2}\alpha^*(\overset{-+}{s_{\sigma\tau}}\overset{+-}{\boldsymbol{p}_{j\sigma\tau}})e^{-i(\mu_\sigma-\mu_\tau)t} \\ &-[N_{\sigma s}(N_{\tau s}-\delta_\sigma{}^\tau)]^{1/2}\alpha^*(\overset{--}{s_{\sigma\tau}}\overset{++}{\boldsymbol{p}_{j\sigma\tau}})e^{-i(\mu_\sigma+\mu_\tau)t}\}\alpha(s\boldsymbol{p}), \end{aligned} \tag{27}$$

the notation being as in (17). The evaluation of sums such as occur here is apt to be a tedious process, but the work can often be halved by noting that the contributions of any given pair of field-states to $\Sigma(s)\int dp$ in these formulas are merely complex conjugates of each other.

It is noteworthy that with respect to all of these formulas the quantum-states of the particle-field system, represented by the various pairs of values of s and $\boldsymbol{p}$, fall into an infinite number of non-combining groups, each characterized by a certain value of the total momentum, $\Sigma(j)\boldsymbol{p}_j+\Sigma(\sigma)\gamma h\nu_\sigma\boldsymbol{n}_\sigma$. It is well known that only states belonging to the same one of these "Compton" groups influence each other's amplitudes, as shown by (17); for this reason in problems on energy or momentum only a single typical group has usually been employed, represented by a single initial state for the field. But it appears likewise from (25) to (27) that only members of the same group combine with each other in producing packet motion, so that many features of this motion, too, can be obtained from the study of one group alone. On the other hand, since concentration in space requires that for any given field state the packet must extend over an appreciable range in $\boldsymbol{p}$, there will be other features which depend essentially upon the co-existence of many different Compton groups. As a matter of fact, in radiation problems the use of a single group has always compelled the resort to a special device of some sort in order to make connection with the spatial intensity of the radiation.

If the field is in any single pure quantum state (e.g., $\alpha(s\boldsymbol{p})=0$ except when $s=s_1$), then it is obvious from (25), (26) and (27) that it has no direct effect upon the motion of the centroid. There may, however, be an indirect effect through values of the α's arising from interaction with the test particle; unfortunately this question, involving the field close to the particle that emits it, lies beyond the reach of the present analysis. In the special case of a particle at rest with the field in the normal or lowest-energy state, however, it is clear from symmetry considerations that the acceleration of the packet centroid must really vanish. This is very satisfactory because it means that the "infinite energy" $\Sigma(\sigma)h\nu_\sigma/2$ associated with this state, whatever its true significance, at least implies no observable electromagnetic field. On the other hand, if one wave-train is "excited" (one $N_{\sigma s}$ not zero), there must be a steady forward acceleration of the packet corresponding to the mean Compton recoil and representing an elementary sort of radiation pressure upon the particle; it seems impossible, however, that the acceleration should include any component oscillating with frequency ν_σ, for there is nothing in the situation to fix the phase of such an oscillation. In this case we have an elementary sort of radiation pressure but in the ordinary sense no electric or magnetic intensity.

These pure quantum states of the field represent, of course, ideal conditions that never actually occur. But the same situation should obtain even with the field in a mixed state whenever the initial spread in the momentum $\boldsymbol{p}$ is small compared to the Compton kicks, of magnitude $h\nu_\sigma/c$, associated with all wave-trains that can contribute appreciably to the centroid motion, as is the case in observations on the Compton effect; for then in each product of two α's in the formulas one factor or the other contains a value of $\boldsymbol{p}$ lying outside the initial range and so vanishes approximately.

The ordinary null-field representing the absence of all radiation, which we always take as a starting point in emission problems, cannot, however, be assumed to correspond to the field's being entirely in its quantum state of lowest energy, for from (17) it is obvious that the presence of charged particles, even if unaccelerated, will not leave the field in this state. The null-field appears to be one containing many different quantum states with amplitudes that are completely disorganized. Such a lack of organization, arising eventually from the various time-factors in (17), prevents the existence of any correlation between the α's that come to be multiplied together in (25) and (27), so that positive and negative values cancel each other on the whole, i.e., we have complete destructive interference among the wave-trains, and the net physical effect is nil.

A physical wave-train must likewise be represented by a combination of field states, only in this case there is a certain type of coordination among their coefficients. Of this an example is obtained in the next section.

THE CLASSICAL CASE

We are now ready to attack the principal problem of the present paper, namely, the manner in which the quantum theory of radiation passes over under suitable conditions into classical theory as a first approximation.

The classical case, as stated above, is one in which the particles can be represented by wave-packets that preserve a certain particle-like character for a considerable length of time. To avoid disrupting them it appears that the radiation field must be sufficiently coarse-grained to be approximately uniform over the wave-packet for each particle; that is, if Δx is the effective diameter of the packet and λ the wave-length of the radiation, we must have $\Delta x/\lambda \ll 1$. At the same time the spread of momentum Δp must of course satisfy the relation of indetermination, $\Delta x \Delta p > h$ (roughly). Now such a spread of momentum tends, as time goes on, to blur the packet; we may say roughly that the diameter will increase at a rate $\Delta v = \Delta p/m$ and so will become equal to λ in a time $\lambda m/\Delta p$. During this time we shall want to follow the particle during at least several oscillations under the influence of the field. Hence we shall want to have $(\lambda m/\Delta p)\nu = mc/\Delta p \gg 1$, a condition that is easy to satisfy at ordinary speeds. By multiplication of these three inequalities we can obtain others; combining two of the latter with the first one above, we can write as an expression for the relation of indetermination and the conditions for the classical case combined:

$$\Delta x/\lambda \ll 1, \; h\nu/(c\Delta p) \ll 1, \; \lambda \gg h/mc. \tag{28}$$

The second of these inequalities can be interpreted as meaning that the maximum Compton kick due to the field, $h\nu/c$, must be much less than the indefiniteness in momentum; thus it is characteristic of the classical case that the Compton effect is reduced to a slight blurring of the packet which is quite negligible as compared with the effect of the natural degree of indefiniteness in the velocity. The third inequality, on the other hand, sets an actual limit to the smallness of the scale upon which anything like a classical relation between radiation and motion can hold at all. Characteristic quantum phenomena are possible under suitable conditions upon any scale; for instance, in principle the Compton effect might be observed with free electrons and Hertzian waves, the electrons being so closely controlled in velocity for this purpose that their position would be indefinite by miles. On such large scales phenomena may be either approximately classical or more or less quantum-mechanical in character, depending upon conditions. But below the limit of size set by the third inequality above radiation phenomena are bound to present more or less of a non-classical aspect.

Let us now fix our attention upon the radiation emitted by a particle of charge e, mass m, and coordinates $\mathbf{x}$ and $\boldsymbol{p}$, as detected by its effect upon a second particle for which corresponding quantities are e', m', $\mathbf{x}'$, $\boldsymbol{p}'$. To make the first particle radiate we must subject it to an accelerating force, and to preserve classical conditions this force must likewise be practically uniform over the packet; we shall represent it by a potential energy term $V = -\boldsymbol{g}(t)\cdot x$. The complete wave-equation will then be

$$-\epsilon\frac{\partial\psi}{\partial t} = \sum(\sigma)\pi\nu_\sigma\left(Q_\sigma{}^2 + \epsilon^2\frac{\partial^2}{\partial O_\sigma{}^2}\right)\psi + (1/2m)(\epsilon\nabla - \gamma e\sum(\sigma)\boldsymbol{b}_\sigma R_\sigma)^2\psi$$
$$+ (1/2m')(\epsilon\nabla' - \gamma e'\sum(\sigma)\boldsymbol{b}_\sigma R_\sigma{}')^2\psi - \boldsymbol{g}\cdot\mathbf{x}\psi, \tag{29}$$

the primes on ∇' and R' indicating that $\mathbf{x}'$ in place of $\mathbf{x}$ occurs in these operators.

To allow, first, for the effect of V, consider the simpler case:

$$-\epsilon\partial\psi/\partial t = (\epsilon^2/2m)\nabla^2\psi + (\epsilon^2/2m')\nabla'^2\psi - \boldsymbol{g}\cdot\mathbf{x}\psi,\ \psi = \int\alpha(\boldsymbol{p}\boldsymbol{p}')e^{\zeta(\boldsymbol{p}\cdot\mathbf{x}+\boldsymbol{p}'\cdot\mathbf{x}')}dp\,dp'.$$

One easily verifies that

$$\alpha(\boldsymbol{p}') = \alpha_0(\boldsymbol{p} - \boldsymbol{G}_t, \boldsymbol{p}')e^{\zeta\eta},\quad \boldsymbol{G}_t = \int_0^t \boldsymbol{g}dt,$$
$$\eta = -p'^2t/2m' + (1/2m)\left[-p^2t + 2\boldsymbol{p}\cdot\left(\boldsymbol{G}t - \int_0^t \boldsymbol{G}_t dt\right)\right. \tag{30}$$
$$\left. -2\int_0^t dt\quad \boldsymbol{g}\cdot\int_0^t \boldsymbol{g}'(t_2)t_2dt_2\right],$$

is a solution reducing to $\alpha = \alpha_0$ at $t = 0$ (the term $\boldsymbol{g}\cdot\mathbf{x}\psi$ arising in the wave-equation from an integration by parts in p); we have thus for the rate of change of α,

$$\dot\alpha = (0p)\alpha = -\zeta(p^2/2m + p'^2/2m')\alpha - \boldsymbol{g}\cdot\nabla_p\alpha, \tag{31}$$

∇_p denoting a vector whose components are $\partial\alpha/\partial p_k$. The quantity $\boldsymbol{G}_t$ represents, of course, the total change in the mean expectation of momentum, $\bar{\boldsymbol{p}} = \int \boldsymbol{p}\alpha^*\alpha\, dp$, during the time t.

Returning now to the general case, let us write

$$\psi = h^{-3}\sum(s)\int\int\alpha(s\boldsymbol{p}\boldsymbol{p}')e^{\zeta(\boldsymbol{p}\cdot\mathbf{x}+\boldsymbol{p}'\cdot\mathbf{x}'-W_st)}dp\,dp'. \tag{32}$$

For the present, however, we shall assume that initially only one field state is present, that is, $\alpha(s\boldsymbol{p}\boldsymbol{p}')$ differs from zero only for one chosen value of s and $\int\int\alpha^*(s\boldsymbol{p}\boldsymbol{p}')\alpha(s\boldsymbol{p}\boldsymbol{p}')dp\,dp' = 1$; and in order to restrict our calculation to radiation by the first particle alone we shall consider only the wisps of side-terms that arise out of this initial state through the influence of the first particle as expressed by its term in the Hamiltonian. Adding the g-effect out of (31) in (17), in which we now add $\boldsymbol{p}'$ as a new variable in all α's, we find:

$$\dot{\alpha}(\overset{\pm}{s_\sigma} pp') = \; Op)\alpha(\overset{\pm}{s_\sigma} pp')$$
$$+ (e\zeta/cm)(h/4\pi)^{1/2}(N_{\sigma s} + \tfrac{1}{2} \pm \tfrac{1}{2})^{1/2}\boldsymbol{b}_\sigma \cdot \boldsymbol{p}\alpha(s\overset{\pm}{\boldsymbol{p}_\sigma}\boldsymbol{p}')e^{\pm i\mu_\sigma t}, \tag{33a}$$
$$\dot{\alpha}(\overset{\pm\pm}{s_{\sigma\tau}} pp') = (Op)\alpha(\overset{\pm\pm}{s_{\sigma\tau}} pp')$$
$$- (ie^2/4c^2m)[(N_{\sigma s} + \tfrac{1}{2} \pm \tfrac{1}{2})(N_{\tau s} + \tfrac{1}{2} \pm \tfrac{1}{2})]^{1/2}\boldsymbol{b}_\sigma \cdot \boldsymbol{b}_\tau \alpha(s\overset{\pm\pm}{\boldsymbol{p}_{\sigma\tau}}\boldsymbol{p}')e^{i(\pm\mu_\sigma \pm \mu_\tau)t}, \tag{33b}$$

the latter valid only for $\sigma \neq \tau$. For $\sigma = \tau$ we get such equations as

$$\dot{\alpha}(spp') = (Op)\alpha(spp') - (ie^2/4c^2m) \sum(\sigma)(2N_{\sigma s} + 1)\boldsymbol{b}_\sigma{}^2\alpha(spp'),$$

in which the last term is enormous but after all merely serves to rotate α at high speed in the complex plane; the same sort of rotating term really should appear in all of the $\dot{\alpha}$'s, except for changes in one or two $N_{\sigma s}$'s whose effect vanishes with $\Delta\nu\Delta\omega$, hence this rotation will be without physical effect in the end, and we shall accordingly ignore it from this point on. All other $\dot{\alpha}$'s are zero. Accordingly, by (31) and (30),

$$\alpha(spp') = \alpha_0(s, \boldsymbol{p} - \boldsymbol{G}_t, \boldsymbol{p}')e^{\zeta\eta}. \tag{34}$$

Upon substituting this value of α in the last term in (33a,b) we get, since $\boldsymbol{b}_\sigma \cdot \boldsymbol{p} = \boldsymbol{b}_\sigma \cdot \boldsymbol{p}_\sigma{}^\pm = \boldsymbol{b}_\sigma \cdot \boldsymbol{p}_{\sigma\tau}{}^{\pm\pm}$, equations of the type,

$$\dot{\alpha} = (Op)\alpha + \alpha(spp')F(\boldsymbol{p}, t),$$

whose solution vanishing at $t = 0$ is

$$\alpha = \alpha(spp') \int_0^t F(\boldsymbol{p} - \boldsymbol{G}_t + \boldsymbol{G}_{t_1}, t_1)dt_1.$$

We thus find:

$$\alpha(\overset{\pm}{s_\sigma} pp') = (e\zeta/cm)(h/4\pi)^{1/2}(N_{\sigma s} + \tfrac{1}{2} \pm \tfrac{1}{2})^{1/2}\alpha(s\overset{\pm}{\boldsymbol{p}_\sigma}\boldsymbol{p}') \int_0^t \boldsymbol{b}_\sigma \cdot (\boldsymbol{p} - \boldsymbol{G}_t + \boldsymbol{G}_{t_1})e^{\pm i\mu_\sigma t_1}dt_1, \tag{35}$$

$$\alpha(\overset{\pm\pm}{s_{\sigma\tau}} pp') = - (ie^2/4c^2m) [(N_{\sigma s} + \tfrac{1}{2} \pm \tfrac{1}{2})(N_{\tau s} + \tfrac{1}{2} \pm \tfrac{1}{2})]^{1/2}\boldsymbol{b}_\sigma \cdot \boldsymbol{b}_\tau \alpha(s\overset{\pm\pm}{\boldsymbol{p}_{\sigma\tau}}\boldsymbol{p}') \int_0^t e^{i(\pm\mu_\sigma \pm \mu_\tau)t_1}dt_1, \tag{36}$$

in which all α's denote values at time t and the second equation holds only for $\sigma \neq \tau$.

We now direct our attention to the second particle and find in its space the mean vector potential due to the radiation field emitted by the first particle as represented by (34), (35) and (36). In Eq. (25) we replace the subscript j by a prime throughout and add $\boldsymbol{p}$ as another variable in all α's; the sums $\Sigma(s)\Sigma(\sigma)$ reduce to two sets of terms, one set in which our chosen initial $\alpha(spp')$ functions as $\alpha(spp')$ in (25) and the values of α given by (35) as the α's in the bracket, and a second set, complex conjugates of the first, in which these roles are reversed. The values of α given by (36) yield nothing appreciable. Putting

$$p - G_t + G_{t_1} = p_1,$$

the result can be written:

$$\overline{A'} = -(ie/2cm)\int\int dpdp' \sum(\sigma) b_\sigma e^{i\mu_\sigma t}[(N_{\sigma s}+1)\alpha^*(sp_\sigma^+ p_\sigma'^-)\alpha(spp') - N_{\sigma s}\alpha^*(spp_\sigma'^-)\alpha(sp_\sigma^- p')]\int_0^t b_\sigma'\cdot p_1 e^{-i\mu_\sigma t_1}dt_1 + \text{conjugate}. \quad (37)$$

In p_1 we should have, according to (25), $p_\sigma^\pm$ in place of p, but this is immaterial because $b_\sigma\cdot(p_\sigma^\pm - p) = \pm\gamma h\nu_\sigma b_\sigma\cdot n_\sigma = 0$.

The p and p' integrations may now be carried out first; this requires a bit of packet theory which can be indicated in sufficient detail as follows. Assuming that only the one state s contributes appreciably to the integrals, one easily finds, after a partial integration or two in one set of p's,

$$\bar{x} = \int x\psi^*\psi dq = -\epsilon\int\int \alpha^*\nabla_p\alpha dpdp',$$

$$(av)(x-\bar{x})^2 \equiv \int (x-\bar{x})^2\psi^*\psi dq = \frac{h^2}{4\pi^2}\int\int|\nabla_p\alpha|^2 dpdp' - \bar{x}^2;$$

whereas if we introduce the centroid explicitly by writing $\alpha = \beta e^{-\zeta p\cdot\bar{x}}$,

$$\int\int \beta^*\nabla_p\beta dpdp' = 0, \quad D^2 \equiv (av)(x-\bar{x})^2 = \frac{h^2}{4\pi^2}\int\int|\nabla_p\beta|^2 dpdp'. \quad (38)$$

Similar formulas hold for the second particle as well. The Compton shift of p in the factor $e^{-\zeta p\cdot\bar{x}}$ serves to introduce into the result the location of the particles. The occurrence of the same shift in β, on the other hand, is unimportant under our conditions. For the difference between such an integral as $\int\int b_\sigma\cdot p_1\beta^*(p_\sigma^\pm)\beta(p)dpdp'$ and $(av)b_\sigma\cdot p_1 \equiv \int\int b_\sigma\cdot p_1\beta^*(p)\beta(p)dpdp'$ should be of the order of $|\int\int b_\sigma\cdot p_1\gamma h\nu_\sigma|\nabla_p\beta||\beta|dpdp'|$, which, since the spread of momentum is small in the quasi-classical case and b_σ is a constant, is nearly the same as $\gamma h\nu_\sigma[(av)b_\sigma\cdot p_1]\int\int|\beta\nabla_p\beta|dpdp'$, and by the Schwarz inequality the square of the latter integral cannot exceed $\int\int|\beta|^2 dpdp'\int\int|\nabla_p\beta|^2 dpdp' = \int\int|\nabla_p\beta|^2 dpdp' = 4\pi^2D^2/h^2$ by (38). Thus the original difference is of order $2\pi D[(av)b_\sigma\cdot p_1]/\lambda_\sigma$ and so is small compared with $(av)b_\sigma\cdot p_1$ itself by the first of the inequalities (28). A rigorous treatment of the matter is readily constructed, with further use of the Schwarz inequality, but it is devoid of physical interest and will be omitted.

Accordingly, writing

$$\alpha(spp') = \beta(spp')e^{-\zeta(p\cdot\bar{x}+p'\cdot\bar{x}')} \quad (39)$$

and $\bar{p}_1 = \int\int p_1\beta^*(spp')\beta(spp')dpdp'$, where $\bar{p}_1 = \bar{p} - G_t + G_{t_1}$ and so represents the value of $\bar{p}$, the average momentum, at time t_1, (37) becomes:

$$\overline{\boldsymbol{A}'} = -\,(ie/2cm)\,\sum(\sigma)\boldsymbol{b}_\sigma e^{i\mu_\sigma(t+\gamma\boldsymbol{n}_\sigma\cdot\bar{\mathrm{x}}-\gamma\boldsymbol{n}_\sigma\cdot\bar{\mathrm{x}}')}\int \boldsymbol{b}_\sigma\cdot\bar{\boldsymbol{p}}_1 e^{-i\mu_\sigma t_1}dt_1 + \text{conjugate}$$

or

$$\overline{\boldsymbol{A}'} = (e/cm)\,\sum(\sigma)\boldsymbol{b}_\sigma \quad \boldsymbol{b}_\sigma\cdot\int_0^t \bar{\boldsymbol{p}}_1 \sin\mu_\sigma(t+\gamma\bar{x}_\sigma-\gamma\bar{x}_\sigma'-t_1)dt_1. \qquad (40)$$

This expression agrees with that in (9) except that $\bar{\boldsymbol{p}}_1/m$ replaces $\boldsymbol{v}$ and the two sets of centroid coordinates replace those of the radiating particle and of the field point respectively. We thus obtain finally the classical expression, (12), for $\boldsymbol{A}$, with $\bar{\boldsymbol{p}}_1/m$ replacing $\boldsymbol{v}$ throughout; the first particle affects the second approximately as if they were classical particles following their respective packet centroids.

The calculation just given is readily generalized to cover the initial presence of more than one field state. The only cases of this sort which are of interest in the present connection are two in which a great many states occur with certain relations as to amplitude and phase. On the one hand, the amplitudes may be entirely disorganized, representing a physical null-field. Then combinations between each of the initial states and the side-terms arising out of the others will destroy each other by interference; each initial state will yield a contribution to $\bar{\boldsymbol{A}}$ like that just obtained but multiplied by the probability of the state, $k_s^2=\iint|\alpha(s\boldsymbol{p}\boldsymbol{p}')|^2dpdp'$, and the total result for emission by a particle will come out as before. On the other hand, the various states may be so correlated in amplitude that they represent in the space of the radiating particle a vector potential $\boldsymbol{A}_0$ coming from other sources. In the latter case intercombinations between each of these initial states and the side-terms arising out of the others serve to introduce into the radiation from our particle the component arising from a term, $-e\boldsymbol{A}_0/cm$, in its velocity; but we shall give no further details.

The results just obtained constitute, unfortunately, only a partial deduction of the classical law, since the law of force, Eq. (24), has not the usual form in terms of derivatives of $\boldsymbol{A}$. Apparently it is necessary to show directly that (24) or (27) leads also, under our conditions, to the familiar formula, $m\ddot{\mathrm{x}}=e\boldsymbol{E}+e\boldsymbol{v}\times\boldsymbol{H}/c$. This is in fact not hard to do, but we shall indicate the details only for the most difficult term, the last one in (24) or the e_j^2 term in (27), representing the combined effect of the velocity due to the vector potential and the magnetic field of the same physical wave-train. This term contains the average of the product $\boldsymbol{A}\times\boldsymbol{H}$, but such an expression cannot in general be separated into the product of the averages, $\bar{\boldsymbol{A}}\times\bar{\boldsymbol{H}}$.

To evaluate this quadratic term it turns out that we have to go to the second approximation for the values of α. We have here a characteristic difference between classical and quantum mechanics, for in classical theory the values of the P's and Q's are obtained once for all in terms of other quantities (Eq. (8a) above). As a matter of fact, contrary to first appearances, a given $\alpha(s)$ does not correspond to an assignment in classical theory of the amplitude of each P and Q; it has reference to a more fundamental wave-oscillation devoid of any classical analogue whatever. A set of amplitudes of the classical P's and Q's corresponds to a wave-packet for the field consisting of many $\alpha(s)$'s in combination. Thus Eqs. (35) and (36) do not really correspond at all to the classical Eqs. (8a).

To obtain the second approximation we insert the values of $\alpha(s_\tau{}^\pm pp')$ as given by (35) for $\alpha(spp')$ on the right of (33a), repeat with σ changed to τ in (33a) instead of in (35), and, adding the results, obtain $\dot{\alpha}(s_{\sigma\tau}{}^{\pm\pm}pp')$; from this $\alpha(s_{\sigma\tau}{}^{\pm\pm}pp')$ can at once be written down just as in obtaining (35) from (33a). The resulting two double integrals with respect to t fit together into one, thus:

$$\int_0^t \boldsymbol{b}_\sigma\cdot\boldsymbol{p}_2 e^{\pm i\mu_\sigma t_2}dt_2\int_0^{t_2}\boldsymbol{b}_\tau\cdot\boldsymbol{p}_1 e^{\pm i\mu_\tau t_1}dt_1+\int_0^t \boldsymbol{b}_\tau\cdot\boldsymbol{p}_2 e^{\pm i\mu_\tau t_2}dt_2\int_0^{t_2}\boldsymbol{b}_\sigma\cdot\boldsymbol{p}_1 e^{\pm i\mu_\sigma t_1}dt_1$$

$$=\int_0^t dt_1\int_0^t \boldsymbol{b}_\sigma\cdot\boldsymbol{p}_1\boldsymbol{b}_\tau\cdot\boldsymbol{p}_2 e^{\pm i\mu_\sigma t_1\pm i\mu_\tau t_2}dt_2.$$

Accordingly

$$\alpha(\overset{\pm\pm}{s_{\sigma\tau}}\boldsymbol{p}\boldsymbol{p}') = -(\pi e^2/c^2m^2h)(N_{\sigma s}+\tfrac{1}{2}\pm\tfrac{1}{2})^{1/2}(N_{\tau s}+\tfrac{1}{2}\pm\tfrac{1}{2})^{1/2}$$

$$\alpha(s\overset{\pm\pm}{\boldsymbol{p}_{\sigma\tau}})\int_0^t \boldsymbol{b}_\sigma\cdot\boldsymbol{p}_1 e^{\pm i\mu_\sigma t_1}dt_1\int_0^t \boldsymbol{b}_\tau\cdot\boldsymbol{p}_1 e^{\pm i\mu_\tau t_1}dt_1. \quad (41)$$

We now go to Eq. (27), replace in it the subscript j by a prime and add p as a variable in α, and then substitute for the α's in the e'^2 term on the right $\alpha(spp')$, $\alpha(s_\sigma{}^\pm pp')$ from (35), and $\alpha(s_{\sigma\tau}{}^{\pm\pm}pp')$ from (41). Most of the terms in $\Sigma(s)\Sigma(\sigma,\tau)$ vanish, but the two series of values $\alpha(s_\sigma+)$ and $\alpha(s_\sigma-)$ combine with each other and with themselves, and $\alpha(s)$ combines with $\alpha(s_{\sigma\tau}{}^{\pm\pm})$. The sixteen expressions thus found we shall not write down; we shall merely state that, after the $\boldsymbol{p}$ shifts have been replaced by centroid factors as was done in calculating $\boldsymbol{A}$, the terms all combine up very nicely and give us for the e'^2 term in (27):

$$(e'/cm')\iint|\beta(s\boldsymbol{p}\boldsymbol{p}')|^2dpdp'(-e'/cm')$$

$$\left[(e/cm)\sum(\sigma)\boldsymbol{b}_\sigma\int_0^t \boldsymbol{b}_\sigma\cdot\boldsymbol{p}_1\sin\mu_\sigma(t+\gamma\bar{x}_\sigma-\gamma\bar{x}_\sigma'-t_1)dt_1\right]$$

$$\times\left[(-2\pi e/c^2m)\sum(\tau)\nu_\tau\boldsymbol{n}_\tau x\boldsymbol{b}_\tau\int_0^t \boldsymbol{b}_\tau\cdot\boldsymbol{p}_1\cos\mu_\tau(t+\gamma\bar{x}_\tau-\gamma\bar{x}_\tau'-t_1)dt_1\right].$$

Comparison with (40) then shows that the first square bracket represents the vector potential emitted by a particle with momentum $\bar{\boldsymbol{p}}_1$, while the second bracket is the curl of this potential taken with respect to the coordinates of the second particle, since curl $[\boldsymbol{b}_\tau f(\bar{x}_\tau')]=\nabla f(\bar{x}_\tau')\times\boldsymbol{b}_\tau=\boldsymbol{n}_\tau\partial f/\partial\bar{x}_\tau'\times\boldsymbol{b}_\tau$. The whole expression thus represents the expected classical component of acceleration, $(e'/cm')(-e'\boldsymbol{A}/cm')\times\boldsymbol{H}=ev_A\times\boldsymbol{H}/cm$, averaged for all momentum constituents of the radiating packet just as if each of these radiated its own complete field. Since this packet was supposed to be very concentrated in momentum we have thus the classical result.

The waves obtained above are diverging spherical waves, but of course by removing the radiating particle to infinity they can be converted into plane waves. In classical theory such waves can also be obtained more simply merely by assuming just one pair of the Q's and P's to differ from zero by a finite amount. The expression obtained in the latter way is, however, really of the nature of a singular solution; it is not the limiting form of (9) above; and in quantum theory it seems to possess no true analogue.

Apparently the simplest adequate representation of a plane train is the following. Introducing the centroids into (35) and (41) as before by means of (39), let us drop out of the factor, $\exp(-\zeta \boldsymbol{p}\cdot\bar{\mathbf{x}})$, the part that merely cancels in the end and retain only $\exp[-\zeta(\boldsymbol{p}_\sigma{}^{\pm}-\boldsymbol{p})\cdot\bar{\mathbf{x}}]=\exp[\mp i\mu_\sigma\gamma\bar{x}_\sigma]$ and the similar expression in both σ and τ. As the emitting particle recedes to infinity let its charge e increase in proportion to its distance; we then drop it from view as a particle by ignoring $\boldsymbol{p}$ in β, where it merely serves to average over the packet, and writing $-\rho\boldsymbol{n}$ for $\mathbf{x}$, in which we now regard ρ merely as a large number with the dimensions of distance and $\boldsymbol{n}$ as a unit vector. For $\boldsymbol{p}-\boldsymbol{G}_t+\boldsymbol{G}_{t1}=\boldsymbol{p}_1$ let us write $C_1\boldsymbol{l}\cos\mu_0 t_1$; the integrals are then easily evaluated. Omitting, finally, the factors referring explicitly to the second particle, we find in this way from (35) and (41) as the representation of a plane wave-train of frequency $\nu_0=\mu_0/2\pi$, with electric vector in the direction $\boldsymbol{l}$ and ray-direction $\boldsymbol{n}$, the following coordinated set of field states:

$$\alpha(s) = k$$

$$\alpha(s_\sigma^{\pm}) = \frac{\rho C}{2}\left(\frac{\mu_\sigma\Delta\nu\Delta\omega}{ch}\right)^{1/2}\left(N_{\sigma s}+\tfrac{1}{2}\pm\tfrac{1}{2}\right)^{1/2}\boldsymbol{l}\cdot\boldsymbol{l}_\sigma e^{\pm i\gamma\rho\mu_\sigma \boldsymbol{n}\cdot\boldsymbol{n}_\sigma}\left[\frac{e^{i(\mu_0\pm\mu_\sigma)t}-1}{\mu_0\pm\mu_\sigma}+\frac{e^{i(-\mu_0\pm\mu_\sigma)t}-1}{-\mu_0\pm\mu_\sigma}\right] \quad (42)$$

$$\alpha(s_{\sigma\tau}^{\pm\pm}) = \frac{\rho^2C^2}{4}\frac{\mu_\sigma\Delta\nu\Delta\omega}{ch}\left(N_{\sigma s}+\tfrac{1}{2}\pm\tfrac{1}{2}\right)^{1/2}\left(N_{\sigma\tau}+\tfrac{1}{2}\pm\tfrac{1}{2}\right)^{1/2}\boldsymbol{l}\cdot\boldsymbol{l}_\sigma\boldsymbol{l}\cdot\boldsymbol{l}_\tau$$

$$e^{i\gamma\rho\boldsymbol{n}\cdot(\pm\mu_\sigma\boldsymbol{n}_\sigma\pm\mu_\tau\boldsymbol{n}_\tau)}\left[\frac{e^{i(\mu_0\pm\mu_\sigma)t}-1}{\mu_0\pm\mu_0}+\frac{e^{i(-\mu_0\pm\mu_\sigma)t}-1}{-\mu_0\pm\mu_\sigma}\right]\left[\frac{e^{i(\mu_0\pm\mu_\tau)t}-1}{\mu_0\pm\mu_\tau}+\frac{e^{i(-\mu_0\pm\mu_\tau)t}-1}{-\mu_0\pm\mu_\tau}\right]$$

in which ρ is to be made infinite at the end, $C=eC_1/cm\rho$ and is arbitrary, and k is another arbitrary constant that we have introduced for generality; σ and τ are to take on all positive values. The corresponding vector potential for a particle placed in the field with coordinate $\mathbf{x}$ is, from (11), in which we replace $v_\perp(t)$ by $k\boldsymbol{p}_{1\perp}/m$ and then $\boldsymbol{p}_{1\perp}$ by $C_1\boldsymbol{l}\cos\mu_0 t$, assume γv_r negligible, drop the last term as likewise negligible, put $r=\rho+\boldsymbol{n}\cdot\mathbf{x}$, and then let $\rho\to\infty$:

$$\boldsymbol{A} = kC\boldsymbol{l}\cos\mu_0(t-\gamma\boldsymbol{n}\cdot\mathbf{x}-t_0).$$

The intensity of the beam (Poynting's vector) is $\pi\nu_0{}^2k^2C^2/2c$.

If in Eqs. (42) we put $s=0$ and take it to refer to the "normal" state of the field ($N_{\sigma s}=0$), then only states of the type 0, $0_\sigma{}^+$ and $0_{\sigma\tau}{}^{++}$ occur. This simple case is really adequate for the treatment of all practical problems; in the present paper the mere general case of any initial state was preferred merely as a matter of theoretical interest. For problems in energy or momentum one may then, as is usual, simplify the procedure still further by starting with a single state, either 0, or 0_σ^+ for a particular value of σ; but to obtain motional phenomena complete expressions such as those given by (42) must be used.

Radiation and a Free Charged Particle. Photons

The results obtained above suggest the following general picture of the action of monochromatic radiation upon a free electron or other charged particle.

Beginning at one extreme, we may have the particle so closely controlled in momentum that the Compton recoil $h\nu/c$ is much greater than the spread or indefiniteness in momentum of the particle, the position of the latter being then indefinite to the extent of many wave-lengths of the radiation. The principal effect of the radiation in this case is to produce an internal statistical distribution of momentum and a resultant spreading of the packet, which, being proportional to the first power of the time, easily out distances the spreading required by the indetermination principle and so results in the phenomenon of the recoil electrons. This Compton spreading, being confined to the forward hemisphere relative to the incident rays, involves a forward acceleration of the packet centroid; we have thus, in physical terms, radiation pressure but no force corresponding to an electric or magnetic field of the type usually associated with electromagnetic waves. Theoretically such a case can be realized at any frequency whatever.

If we now progressively decrease the indefiniteness of position and at the same time unavoidably increase the indefiniteness of momentum, different quantum states in the packet eventually begin to combine in (25) and (27) and so by "interference" to cause the classical motion of the packet as a whole. At the same time the Compton recoil becomes progressively more and more masked by the initial indefiniteness in momentum of the particle and therefore harder to distinguish from the latter in observation. This change in the phenomenon reaches an advanced stage when the packet "diameter" becomes equal to the wave-length of the radiation.

Finally, when the whole packet becomes very much smaller than a cubic wave-length and at the same time sufficiently well-defined in momentum so that it will stay small during many field-periods, then we have the fully developed classical case. The Compton shift in momentum is now so small that it is altogether lost in the general initial blur of the momentum; the elementary radiation pressure represented by this shift must at the same time pass somehow into the classical radiation pressure. The smallness of packet requisite for this case can be secured initially for any wave-length of radiation, but this condition can be made to last during many periods only if the wave-length considerably exceeds the "Compton wave-length", $\lambda = h/mc$, or 0.0242A for electrons and less in proportion to the mass for heavier ions.

In the centroid motion, when thus fully developed, there is no trace of the billiard-ball action which is characteristic of the Compton effect and which suggests so strongly the existence of photons. One might explain the absence of such discontinuities by pointing out that under those conditions in which the classical motion is actually observable there would always be very many photons acting, so that any discontinuities in the elementary processes would be smoothed out. In the usual photon formulation of radiation theory, however, the photons are associated with the quantum states themselves, a state s with quantum numbers $N_{\sigma s}$ being regarded as one in which there are $N_{\sigma s}$ photons moving in each of the directions $\boldsymbol{n}_\sigma$, whereas we have seen that the centroid motion arises as a combination effect between these states. It seems unlikely that a satisfactory law can ever be found which would represent the centroid velocity as arising in a simple way from many impulses imparted independently by the separate photons. The preferable view seems to be that the photon is, like a wave on the ocean, not an ultimate building block of the world but merely a special form or appearance that sometimes takes shape.