THERMIONIC EMISSION AND SPACE CHARGE

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Abstract

The theory of thermionic currents for a plane parallel electrode arrangement including space charge effects is developed by using Fermi-Dirac statistics and by taking into account the wave-mechanical nature of the electron. The use of the Wentzel-Kramers-Brillouin approximate solution of the Schrödinger equation leads to results quite analogous to those of the classical calculations. The current-voltage characteristic is but slightly different but the expressions for potential distribution (and hence space charge, field, energy density, etc.) are appreciably modified by wave mechanics.

THE theory of thermionic currents when space charge effects are considered has been handled classically for the case of plane parallel electrodes by Epstein, Fry, Langmuir and Gans.[1] These authors have derived expressions for the voltage-current characteristics and for the potential distribution between electrodes. Applications of these theories can be made to special cases only with the help of numerical tables since the integrals occurring must be evaluated numerically or graphically. It is the purpose of this paper to investigate the modifications in the above theories which result from the use of quantum rather than classical mechanics.

We shall make the assumption that the area of the electrodes is large enough so that we may treat the problem as one-dimensional, and we shall think of the cathode surface as characterized by a sharp jump in potential. This is of course only an approximation to the actual state of affairs as the potential must change continuously from inside the metal to the space outside. Inside the hot cathode the number of electrons per unit volume whose energies lie between E and $E+dE$ ($E=mv_0^2/2$, where v_0 is the conponent of velocity normal to the emitting surface) give rise to an incident current.[2]

$$di = \frac{4\pi mekT}{h^3}\log\left(1 + Ae^{-E/kT}\right)dE \tag{1}$$

where A is determined by

$$2mkT\log A = h^2\left(\frac{3n}{8\pi}\right)^{2/3} = \frac{h^2}{\lambda^2}\cdot \tag{2}$$

Eq. (1) is derived on the assumption of a Fermi distribution of electrons inside the metal. The constants have the usual meaning, n is the number of free electrons per unit volume and λ a de Broglie wave-length.

[1] P. Epstein, Verh. d. Deutschen, Phys. Ges. **21**, 85 (1919); T. C. Fry, Phys. Rev. **17**, 441 (1921); I. Langmuir, Phys. Rev. **21**, 419 (1923); R. Gans, Ann. d. Physik **69**, 385 (1922).

[2] See e.g., A. Sommerfeld, Zeits. f. Physik **47**, 28 (1928).

The electrons escaping from the metal into the region between cathode and anode give rise to a current and space charge which depend on the applied potential difference and on the separation between anode and cathode. If we denote by $D(E)$ the probability that an electron of energy E (in the sense of Eq. (1)) inside the cathode escape from the metal (the so-called transmission coefficient) and by ψ_D the wave function of a transmitted electron from an incident electron beam ψ_i which is normalized to unit current, the total current may be written

$$i = \frac{4\pi mekT}{h^3}\int_0^\infty D(E)\log(1 + Ae^{-E/kT})dE \tag{2}$$

and the potential in the region between cathode and anode $o<x<l$ is determined by Poisson's equation

$$\frac{d^2\phi}{dx^2} = -4\pi\rho = -\frac{16\pi^2 emkT}{h^3}\int_0^\infty \bar{\psi}_D\psi_D \log(1 + Ae^{-E/kT})dE \tag{3}$$

where ψ_D is of course, a function of ϕ. Eqs. (2) and (3) together with the boundary conditions completely determine the current-voltage characteristic and the potential distribution between the electrodes. From the latter one may obtain expressions for the space charge, energy density, and field distributions by differentiation.

We shall employ approximate expressions for ψ_D and $D(E)$ as obtained by the use of the Wentzel-Kramers-Brillouin approximation to the solution of the Schrödinger equation. All the necessary formulae have been given by the author and L. A. Young.[3] Even with the help of these approximations it is not possible to obtain explicit expressions for $D(E)$ and ψ_D which are valid for all values of x and E when $\phi(x)$ is arbitrary, but we must break up the integrations in (2) and (3) into various ranges and use appropriate expressions for $D(E)$ and ψ_D in these ranges.

The problem divides itself into three subdivisions depending on the manner in which the potential varies from cathode to anode. Since according to Eq. (3) $d^2\phi/dx^2$ is always positive, we have the following possibilities:

Case I. ϕ increases monotonically from cathode to anode.

Case II. ϕ decreases monotonically from cathode to anode.

Case III. ϕ possesses one and only one minimum between cathode and anode.

We shall consider the three cases separately.

Case I. Accelerating field; no potential minimum

We shall restrict our discussion to such potential differences which, although large enough to prevent the formation of a potential minimum, are not so large that "field" currents become appreciable. Hence in this case we need consider only those electrons whose energies lie above a certain value W_a, since the transmission coefficient for the others is practically nil and their contribution to the space charge is completely negligible, except possibly for a region of atomic dimensions near the cathode.

In this case we write Eq. (2) as

$$i = \frac{4\pi emkT}{h^3}\int_{W_a}^{\infty} D(E)\log\left(1 + Ae^{-E/kT}\right)dE. \tag{4}$$

Fig. 1 shows a diagram of the barrier at the cathode surface, and the general shape of the potential energy curve. W_i is defined by

$$W_i = kT\log A.$$

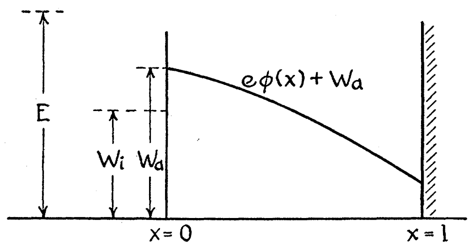

Fig. 1

For potential curves of the type shown in Fig. 1, it has been shown[3] that the transmission coefficient is very nearly

$$D(E) = \frac{4E^{1/2}(E - W_a)^{1/2}}{\left[E^{1/2} + (E - W_a)^{1/2}\right]^2 + \dfrac{h^2e^2(d\phi/dx)^2_{x=0}}{128\pi^2 m(E - W_a)^2}}.$$

Without appreciable error we may neglect the last term in the denominator and write

$$D(E) = \frac{4E^{1/2}(E - W_a)^{1/2}}{\left[E^{1/2} + (E - W_a)^{1/2}\right]^2}. \tag{5}$$

This expression is valid for $E > W_a$ and we shall introduce a slight error by using it for energies down to $E = W_a$. For zero field at the surface (5) is exact for $E \geqq W_a$ and in practice goes very nearly to zero as $E \to W_a$. The error so introduced will tend to make our results somewhat too small.

If now in Eq. (4) we change the variable to

$$y = \frac{E - W_a}{kT}$$

and write

$$\chi = W_a - W_i > 0 \tag{5a}$$

$$\beta = \frac{W_a}{kT}$$

there follows:

$$i = \frac{4\pi me(kT)^2}{h^3}\int_0^{\infty} D(y + \beta)\log\left(1 + e^{-\chi/kT}\cdot e^{-y}\right)dy$$

[3] N. H. Frank and L. A. Young, Phys. Rev. **38**, 80 (1931).

Expanding the logarithm, we obtain

$$i = \frac{4\pi me(kT)^2}{h^3} e^{-\chi/kT} \int_0^\infty D(y+\beta)e^{-y}dy \tag{6}$$

or

$$\left.\begin{aligned} i = i_{\text{sat}} &= \frac{4\pi me(kT)^2}{h^3} e^{-\chi/kT}\bar{D}(\beta) \\ \text{with}\quad \bar{D}(\beta) &= \int_0^\infty D(y+\beta)e^{-y}dy = 4\int_0^\infty \frac{(y+\beta)^{1/2}y^{1/2}e^{-y}dy}{[y^{1/2}+(y+\beta)^{1/2}]^2} \end{aligned}\right\} \tag{7}$$

This is essentially the result obtained by Nordheim[4] and Sommerfeld[2] for saturation current. For all temperatures occurring in practice $\beta = W_a/kT$ is large compared to unity, so that we can evaluate $\bar{D}(\beta)$ by expanding $D(\beta+y)$ in powers of y/β. This expansion is, of course, not valid over the whole range of integration but the presence of the exponential nullifies any errors which might arise from large values of y.

The result of this integration gives:

$$\bar{D}(\beta) = \frac{2\pi^{1/2}}{\beta^{1/2}} - \frac{8}{\beta} + \frac{15\pi^{1/2}}{2\beta^{3/2}} - \cdots$$

or

$$\bar{D}\left(\frac{W_a}{kT}\right) = \frac{2\pi^{1/2}(kT)^{1/2}}{W_a^{1/2}} - \frac{8kT}{W_a} + \frac{15\pi^{1/2}(kT)^{3/2}}{W_a^{3/2}} - \cdots. \tag{8}$$

Numerically, for tungsten at about 2800°K, $W_a = 10$ volts; $kT \cong 0.2$ volt, so that

$$\beta = \frac{W_a}{kT} \cong 50$$

with this value of β

$$\bar{D}(\beta) = 0.40.$$

This result was found by Nordheim, who later retracted it in the light of calculations made with an image field.[5] It seems to the author that the latter question is open to objection and that the first results, essentially those obtained here, are more nearly correct. It must be further pointed out that the value of $\bar{D}$ here calculated is probably too small as the potential energy probably reaches the value W_a a short distance (order of atomic dimensions) from the cathode surface and then drops as we come into the metal. The effect of this is to reduce the value of β in Eq. (5) so that the value $\bar{D} = 0.50$ which is demanded by experiment seems entirely reasonable. The lowering of the work function due to the applied field (Schottky correction) is neglected in this treatment.

[4] L. Nordheim, Zeits. f. Physik **46,** 833 (1928).

[5] L. Nordheim, Proc. Roy. Soc. **A121,** 626 (1928).

To find the potential distribution for this case we employ Eq. (3) in which we replace the lower limit of the integral by W_a without appreciable error. For ψ_D we use the W–K–B approximation[3] (normalized to unit current),

$$\psi_D = \left(\frac{m}{2}\right)^{1/4}\left(\frac{8\pi^2 m}{h^2}\right)^{1/4} \frac{c \exp\left(i\int_0^x y^{1/2}dx\right)}{y^{1/4}}; \quad y = \frac{8\pi^2 m}{h^2}[E - W_a - e\phi]$$

whence

$$\psi_D\bar{\psi}_D = \left(\frac{m}{2}\right)^{1/2}\frac{c\bar{c}}{(E - W_a - e\phi)^{1/2}} = \left(\frac{m}{2}\right)^{1/2}\frac{D(E)}{(E - W_a - e\phi)^{1/2}} \tag{9}$$

Eq. (3) becomes

$$\frac{d^2\phi}{dx^2} = -\frac{16\pi^2 mekT}{h^3}\left(\frac{m}{2}\right)^{1/2}\int_{W_a}^{\infty}\frac{D(E)\log(1 + Ae^{-E/kT})dE}{(E - W_a - e\phi)^{1/2}} \tag{10}$$

with the notation

$$u = -\frac{e\phi}{kT}, \quad y = \frac{E - W_a}{kT} \quad (e \text{ negative})$$

and

$$g^2 = \frac{32\pi^2 m^2 e^2}{h^3}\left(\frac{2kT}{m}\right)^{1/2} e^{-\chi/kT} \tag{10$'$}$$

Eq. (10) goes over into

$$\frac{d^2u}{dx^2} = \frac{g^2}{4}\int_0^{\infty}\frac{D(y+\beta)}{(y+u)^{1/2}}e^{-y}dy. \tag{11}$$

One integration yields

$$\left(\frac{du}{dx}\right)^2 = g^2[p(u,\beta) - p(0,\beta) + c_1] \tag{12}$$

where we have placed

$$p(u,\beta) = \int_0^{\infty} D(y+\beta)(y+u)^{1/2}e^{-y}dy. \tag{13}$$

The constant c_1 depends on the field at the cathode by virtue of the relation

$$\left(\frac{du}{dx}\right)^2_{x=0} = g^2 c_1$$

and hence is always positive.

Another integration of (12) gives

$$gx = \int_0^u \frac{dz}{(p(z,\beta) - p(0,\beta) + c_1)^{1/2}}. \tag{14}$$

Here we have chosen the second integration constant so that $u=0$ when $x=0$. To determine c_1, we place $x=l$ in Eq. (14) and find ($u=u_1$ at the anode)

$$gl = \int_0^{u_1} \frac{dz}{(p(z, \beta) - p(0, \beta) + c_1)^{1/2}} \tag{15}$$

which is a transcendental equation determining c_1. Inserting this value of c_1 into Eq. (14), we have found the potential u as a function of x. The condition that case I exist, expressed in terms of the potential difference between anode and cathode and of their separation, is obviously from (15)

$$gl \leqq \int_0^{u_1} \frac{dz}{(p(z, \beta) - p(0, \beta))^{1/2}} \tag{15'}$$

since $c_1 \geqq 0$.

The results so far obtained are very similar to those obtained classically. In fact, if we place $D(E)=1$ and introduce i_s instead of g^2 with the help of Eq. (7), the resulting equations are identical with the classical expressions. This is not surprising in view of the fact that for $0<x<l$ the electron density is so low that the Fermi-Dirac and Maxwell-Boltzmann statistics become indistinguishable.

Case II. Retarding field; no potential minimum

For this case the transmission coefficient is practically zero for all electrons in the cathode with energies less than $W_a+e\phi_1$ where ϕ_1 is the negative potential difference between anode and cathode. The expression for the current is according to Eq. (2);

$$i = \frac{4\pi mekT}{h^3} \int_{W_a+e\phi_1}^{\infty} D(E) \log(1 + Ae^{-E/kT}) dE.$$

Changing the integration variable to $y=[(E-W_a-e\phi_1)/kT]$ and expanding the logarithm as before, one obtains

$$i = \frac{4\pi me(kT)^2}{h^3} e^{-\chi/kT} \cdot e^{-e\phi_1/kT} \int_0^{\infty} D\left(y + \beta + \frac{e\phi_1}{kT}\right) e^{-y} dy \tag{16}$$

and placing $u_1 = -e\phi_1/kT$ there follows with the help of Eq. (7)

$$i = i_s e^{u_1} \frac{\overline{D}(\beta - u_1)}{\overline{D}(\beta)} \tag{17}$$

which is the equation of the characteristic for this case. $\overline{D}$ and β are defined as before. To evaluate $\overline{D}(\beta - u_1)$, we can write

$$D(y + \beta - u_1) = 1 - \frac{1}{\beta^2}[(y + \beta - u_1)^{1/2} - (y - u_1)^{1/2}]^4$$

which is equivalent to Eq. (5).

If we now remember that $\beta \gg 1$ and $-u_1 \gg 1$, the latter is necessary if the conditions of case II are fulfilled,* we may write, with

$$\gamma = -\frac{u_1}{\beta}$$

$$D(y+\beta-u_1) \cong 1 - \left[(1+\gamma)^{1/2} - \gamma^{1/2}\right]^4 \left(1 - \frac{y}{2\beta(\gamma(1+\gamma))^{1/2}}\right)^4$$

whence there follows

$$\bar{D}(\beta - u_1) \cong 1 - \left\{(1+\gamma)^{1/2} - \gamma^{1/2}\right\}^4 \left(1 - \frac{2}{\beta(\gamma(\gamma+1))^{1/2}}\right). \tag{18}$$

This expression is very much more nearly equal to unity than that for $\bar{D}(\beta)$ since it represents the effect of the transmission coefficient on the current which is due to electrons whose energies E lie far above W_a. ($E > W_a + e\phi_1$). For these electrons the transmission coefficient differs but little from unity. To obtain a rough idea of the magnitude of $\bar{D}$ $(\beta - u_1)$ let us consider a retarding voltage of about 10 volts. Then

$$-u_1 \cong \beta \cong 50; \gamma = 1$$

so that

$$\bar{D} \cong 1 - 0.03 = 0.97.$$

In finding the potential distribution between the electrodes for this case, we must remember that all electrons whose energies in the cathode lie above W_a contribute to the space charge. We must divide the energy range into two parts:

(a) $$E > W_a + e\phi_1$$

(b) $$W_a + e\phi < E < W_a + e\phi_1.$$

For (a), we have as before (Eq. 9)

$$\psi_D\bar{\psi}_D = \left(\frac{m}{2}\right)^{1/2} \frac{D(E)}{(E - W_a - e\phi)^{1/2}}$$

and for (b) we use[5]

$$\psi_D\bar{\psi}_D = \left(\frac{m}{2}\right)^{1/2} \frac{4E^{1/2}(E - W_a)^{1/2}}{(2E - W_a)(E - W_a - e\phi)^{1/2}} \tag{19}$$

Poisson's equation takes the form

* It is tacitly assumed that the separation of the electrodes is of the order of magnitude of one centimeter.

[5] This expression is obtained from Eq. (30), p. 85, Phys. Rev. **38**, (1931). Eq. (30) should read

$$\psi\bar{\psi} = \frac{4(y_0/y)^{1/2}}{p\left\{1 + \frac{1}{p^2}\left(\frac{\alpha^2}{16y_0^2} + y_0\right)\right\}}$$

and the term in α^2 is neglected compared to y_0

$$\frac{d^2\phi}{dx^2} = -\frac{16\pi^2 emkT}{h^3}\left(\frac{m}{2}\right)^{1/2}\left[4\int_{W_a+e\phi}^{W_a+e\phi_1}\frac{(E-W_a)^{1/2}E^{1/2}\log(1+Ae^{-E/kT})}{(2E-W_a)(E-W_a-e\phi)^{1/2}}dE + \int_{W_a+e\phi_1}^{\infty}\frac{D(E)\log(1+Ae^{-E/kT})}{(E-W_a-e\phi)^{1/2}}dE\right]. \tag{20}$$

The use of $W_a+e\phi$ as the lower limit of the first integral is necessary as no electrons whose energy in the cathode is less than $W_a+e\phi$ contribute to the density of charge at the position x where $E=W_a+e\phi(x)$. Changing the integration variable to $y=(E-W_a-e\phi)/kT$ and setting $u=-e\phi/kT$ (20) can be written after the logarithm has been expanded

$$\frac{d^2u}{dx^2} = \frac{g^2e^u}{4}\left[\int_0^{u-u_1}\frac{4(y-u)^{1/2}(y+\beta-u)^{1/2}e^{-y}dy}{y^{1/2}(2y+\beta-2u)} + \int_{u-u_1}^{\infty}\frac{D(y+\beta-u)e^{-y}dy}{y^{1/2}}\right] \tag{21}$$

where g^2 is defined by Eq. (10′). If we define

$$r(u-u_1) = \tfrac{1}{2}\left[\int_0^{u-u_1}\frac{4(y-u)^{1/2}(y+\beta-u)^{1/2}e^{-y}dy}{(2y+\beta-2u)y^{1/2}} + \int_{u-u_1}^{\infty}\frac{D(y+\beta-u)e^{-y}dy}{y^{1/2}}\right] \tag{21′}$$

one integration of Eq. (21) yields

$$\left(\frac{du}{dx}\right)^2 = g^2e^{u_1}\left[\int_0^{u-u_1}e^z r(z)dz + c_2\right]. \tag{22}$$

The constant c_2 is always positive since

$$\left(\frac{du}{dx}\right)^2_{x=l} = g^2e^{u_1}c_2.$$

Another integration leads to the equation for the potential distribution

$$e^{u_1/2}g(l-x) = \int_0^{u-u_1}\frac{ds}{(\rho(s)+c_2)^{1/2}} \tag{23}$$

where we have placed

$$\rho(s) = \int_0^s e^z r(z)dz. \tag{23′}$$

For $x=0$, Eq. (23) gives

$$e^{u_1/2}gl = \int_0^{-u_1}\frac{dz}{(\rho(z)+c_2)} \tag{24}$$

as the transcendental equation determining c_2. Since $c_2 \geqq 0$. Case II occurs when

$$gl \leqq e^{-u_1/2} \int_0^{-u_1} \frac{dz}{(\rho(z))^{1/2}} \cdot$$

Case III. A potential minimum exists

Let us suppose that the minimum occurs at $x = x_m$, so that

$$\frac{du}{dx} < 0 \text{ for } 0 < x < x_m; \quad \frac{du}{dx} > 0 \text{ for } x_m < x < l.$$

(a) For the region $x_m < x < l$, we have the same relations as in case I except that now only those electrons with energies greater than $W_a + e\phi_m$ (rather than W_a) must be considered. Thus we obtain in place of Eq. (11)

$$\frac{d^2u}{dx^2} = \frac{g^2}{4} \int_{-u_m}^{\infty} \frac{D(y+\beta)e^{-y}dy}{(y+u)^{1/2}} \tag{25}$$

where

$$u_m = -\frac{e\phi_m}{kT} \cdot$$

Changing the integration variable to $s = y + u_m$ and integrating (25) once with respect to u, there follows

$$\left(\frac{du}{dx}\right)^2 = g^2 e^{u_m}\left[\int_0^{\infty} D(s+\beta-u_m)(s+u-u_m)^{1/2}e^{-s}ds + c'\right] \tag{26}$$

where c' must be determined so that $du/dx = 0$ when $x = x_m$ (i.e., $u = u_m$). Thus we write

$$\left(\frac{du}{dx}\right)^2 = g^2 e^{u_m}[p(u-u_m, \beta-u_m) - p(0, \beta-u_m)] \tag{26a}$$

where the function $p(s, z)$ is defined by Eq. (13). Another integration yields

$$ge^{u_m/2}(x - x_m) = \int_0^{u-u_m} \frac{dz}{(p(z, \beta-u_m) - p(0, \beta-u_m))^{1/2}} \cdot \tag{27}$$

Here the integration constant has been so determined that $u = u_m$ when $x = l$, we have

$$ge^{u_m/2}(l - x_m) = \int_0^{u-u_m} \frac{dz}{(p(z, \beta-u_m) - p(0, \beta-u_m))^{1/2}} \tag{28}$$

(b) For the region $0 < x < x_m$ we proceed analogously to case II, with the only difference that we replace l by x_m and u_1 by u_m. Hence we write

$$e^{u_m/2}g(x_m - x) = \int_0^{u-u_m} \frac{dz}{(\rho(z))^{1/2}} \tag{29}$$

where $\rho(z)$ is defined by (23′). For $x = 0$, there follows

$$e^{u_m/2}gx_m = \int_0^{-u_m} \frac{dz}{(\rho(z))^{1/2}} \cdot \tag{30}$$

The sum of Eqs. (28) and (30) yields

$$gl = e^{-u_m/2}\left[\int_0^{u_1-u_m} \frac{dz}{[(p(z,\beta - u_m) - p(0,\beta - u_m)]^{1/2}} + \int^{-u_m} \frac{dz}{(\rho(z))^{1/2}}\right]. \tag{31}$$

These last two equations determine the position and magnitude of the potential minimum.

For the current we have in complete analogy to Eq. (4)

$$i = \frac{4\pi mekT}{h^3}\int_{Wa+e\phi_m}^{\infty} D(E)\log(1 + Ae^{-E/kT})dE \tag{32}$$

whence there follows exactly as in case I

$$i = i_s e^{u_m}\frac{\bar{D}(\beta - u_m)}{\bar{D}(\beta)} \tag{33}$$

which is the equation of the characteristic for this case.

Summary and Conclusions

The theory of thermionic currents between plane parallel electrodes has been developed from the Fermi statistics and by taking into account the wave-mechanical nature of the electrons. The results are entirely analogous to those obtained classically. We must distinguish among three cases:

I. Accelerating field; no potential minimum.
II. Retarding field; no potential minimum.
III. The case of the existence of a potential minimum.

The current-voltage characteristics are:

I. $$i = i_s$$

II. $$i = i_s e^{u_1}\frac{\bar{D}(\beta - u_1)}{\bar{D}(\beta)}$$

III. $$i = i_s e^{u_m}\frac{\bar{D}(\beta - u_m)}{\bar{D}(\beta)}\,.$$

If all the $\bar{D}$'s are set equal to unity, we have the classical equations except for the fact that u_m (the minimum value of the potential) is determined by a different equation (Eq. (31)).

Expressions for the potential as function of position between the electrodes have been derived which are quite analogous to the classical functions. For application of these equations it is necessary (just as in the classical case) to perform numerical or graphical integrations. A typical wave-mechanical difference occurs in the case of the expressions for charge density due to those electrons which emerge from the cathode but are turned around between cathode and anode. The density is calculated classically by dividing the current due to these electrons by their velocity and multiplying by 2 (once going out and once returning). Quantum-mechanically, we must not form $\psi\bar{\psi}$ from the expressions for two travelling waves, but must find a ψ-function which represents a standing wave in the region considered and then form $\psi\bar{\psi}$. In so doing, interference effects which are totally foreign to classical calculations appear and modify the expression to be used for this density.

It is furthermore not necessary to carry out the numerical evaluation of the functions which determine the potential distributions in order to see the general difference between our results and the classical results. In every case the density of charge is less in our theory than in the corresponding classical case. Thus the potential distribution curves, although of the same general type as those obtained classically, will have decidedly smaller curvatures. Physically, this is easily understandable since the largest contribution to the classical density came from those electrons which, according to our theory, cannot get out of the metal because of the transmission coefficient $D(E)$.

We have neglected the dependence of the work function on the field and as we have not extended the theory to include strong fields this neglect is not serious. The omission of secondary reflection from the anode is perhaps more serious. Recently, R. S. Bartlett and A. T. Waterman[6] have published a series of papers in which they discuss the equilibrium distribution of electrons in the neighborhood of the surface of a metal and have drawn conclusions therefrom for the case of accelerating and retarding fields. The author does not consider such extension of their results entirely justified. The objections which they raise to the use of Poisson's equation because of the granular nature of the electron gas seem to the author to be largely nullified by the fact that the wave mechanical expression for density is a continuous function of the coordinates. An approximation in the theory here developed is the use of the equilibrium distribution function instead of a modified distribution which exists in the case of steady state flow of current. It can be easily shown that errors so introduced are entirely negligible.

[6] R. S. Bartlett, Phys. Rev. **37,** 959 (1931); A. T. Waterman, ibid. **38,** 1497 (1931).