RELATIVISTIC THEORY OF THE PHOTOELECTRIC EFFECT
PART I. THEORY OF THE *K*-ABSORPTION OF X-RAYS

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PART II. PHOTOELECTRIC ABSORPTION OF ULTRAGAMMA RADIATION

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Abstract

Part I: A strict theory of the absorption of x-rays is developed on the basis of relativistic quantum electrodynamics. The theory is applied to the absorption of x-rays by a Dirac electron in the field of a nucleus. Corrections to the non-relativistic theory are appreciable only for heavy elements, where the present calculations give a K-discontinuity twenty percent smaller than the earlier ones. The agreement with experiment is not improved.

Part II: The theory is applied to the calculation of the absorption of quanta whose energy is larger than the proper energy mc^2 of the electron. The cross section for absorption is here given approximately by

$$\sigma \backsim 2 \times 10^{-22} Z^5 \lambda.$$

This result is applied to account for the excess absorption over that predicted by the Klein-Nishina formula found experimentally for the gamma-rays of ThC″ by Chao and Tarrant. The theory is in fairly good agreement with experiment for Cu, but disagrees violently with it for Pb. An examination of the approximations made in deriving and applying the theoretical result shows that they cannot have introduced this discrepancy. There is thus a definite conflict between electrodynamical theory and experiment.

Introduction

The photoelectric cross section for absorption, σ, is connected with the mass absorption coefficient μ, by the relation $\mu = \sigma/\rho$. Various quantum theoretic derivations of σ have been given; in addition to these the literature contains interesting and graphic solutions treating such problems as distribution in angle of the photoelectrons, group velocity of the waves representing the photoelectric current, and besides these the theory has been applied to shells higher than the K-shell. We refer to a few of these papers collectively[1]—due to the difference of treatment, or to the difference in the problem treated we shall make few explicit references.

The problem under discussion may be considered that of a hydrogenic

[1] Wentzel, Zeits. f. Physik **40,** 574; **41,** 828 (1926); Oppenheimer, Zeits. f. Physik **41,** 268 (1926); Phys. Rev. **31,** 349 (1928); Beck, Zeits. f. Physik **41,** 443 (1926); Suguira, J. de Physique **8,** 113 (0000); Sommerfeld and Schur, Ann. d. Physik **4,** 413 (1930); Schur, Ann. d. Physik **4,** 433 (0000); Froehlich, Ann. d. Physik **7,** 109 (4930); Stobbe, Ann. d. Physik **7,** 661 (1939); Szczeniowski, Phys. Rev. **35,** 347 (1930); Bethe, Ann. d. Physik **4,** (1930); T. Muto, I.P.C.R., Tokyo (1931).

atom in its normal state, interacting with a light-quantum field which induces transitions to hyperbolic orbits. This problem, strictly speaking, has one answer—the complete relativistic solution; but for convenience the problem is divided into two parts. If the region of interest does not involve photo-electron velocities that are too great, it is justifiable to assume that the non-relativistic model will suffice.

This is the model employed by Nishina and Rabi,[2] and involves simply the solution of the Schroedinger equation. This result could be expected to hold for those elements for which the binding in the K-shell is not great enough to necessitate bombardment by quanta whose wave-length are short compared to the size of the K-shell, but it should be added that neither this nor the relativistic result should be expected to hold for very light elements, since here the assumption that the effects of the more external electrons are negligible is invalid. We mention that except for very light elements the Nishina-Rabi result is in accord with experience.

To obtain a formula theoretically more general than that of Nishina and Rabi necessitates a complete solution of the relativistic problem. This in the present state of the theory means describing the atom with Dirac's linear Hamiltonian, using retarded potentials and the relativistic theory of the light-quantum field.

Attempts in this direction have not been numerous. It is not difficult to include retardation and evaluate the integrals if a Schroedinger wave equation describes the atom, and this has been done by at least two writers. Although it presents an analytic problem of some interest such a discussion is of course incomplete. An additional contribution has recently been made[3] in which the author employs Dirac wave functions but neglects retardation in discussing transitions for wave-lengths between λ_0 for the K-limit, and $\lambda_0/2$.

In the case of obtaining a solution of the more difficult relativistic problem, we should be interested in comparing it with two sets of experimental data. (1) Although agreement between the Nishina-Rabi result and experiment is qualitatively pretty good there are discrepancies which make a better quantitative check desirable. Consequently we should concern ourselves in seeing whether relativitity properly corrects the non-relativistic theory, and to this end we should compare our result with empirical data for the K-limit and the region immediately adjoining it.* (2) Recent data from the region of gamma and cosmic rays have made it appear not impossible that our previous assumption, that the cross section for absorption is too small to be observable for light as hard as gamma rays, may be unjustified, and that in reality the photoelectric absorption is comparable with the scattering in this region.

[2] Nishina and Rabi, Verh. d. Deut. Phys. Ges. **9**, 6 (1928).

[3] Roess, Phys. Rev. **37**, 533 (1931).

* Such a comparison has been made by Roess (Ref. 3). Since, as will appear in this paper, including the retarded potentials does not improve the accuracy over his model at the K-absorption limit we have not considered it valuable to make the further calculations away from the limit.

Work contributing along these lines has been done by Chao,[4] Tarrant,[5] Gray[6] and Meitner and Hupfeld. In particular the experiments of Chao could be accounted for if the ratio of σ for lead, to the Klein Nishina cross section for scattering[7] were about 0.5 for $\lambda = 4.7$X.U. In the course of the discussion considerably more will be said concerning the latter problem. For the present let us develop the general integrals and dispose of the discussion of the K-limit.

Part I. Theory of the K-Absorption of X-rays

The cross section for absorption.

The Hamiltonian for our problem consists of three terms. The term for the atom

$$H_0{}^m = (\mathrm{s}' \cdot p) - e\phi + mc^2\Gamma_0$$

we take from Weyl.[8] s' is a matrix (analogous to Dirac's α matrix) and will be given later. ϕ is the hydrogenic scalar potential Ze/r. Γ_0 is a matrix[9]; it will not be needed explicitly.

Let a wave function for the normal state of the atom given by this Hamiltonian be ψ_0; let the corresponding energy be $h\nu_0$. Further, let a complete set of wave functions for the energy $h\nu = h\nu' + h\nu_0 > 0$, be ψ_m; we need the index m because there are many states with the same energy, to which a photoelectric transition is possible. Let the ψ_m be normalized to $d\nu$. Then the cross section for photoelectric absorption of a quantum of frequency ν, with vector of propagation parallel to z, and with the electric vector plane-polarized in the x-y plane parallel to the unit vector $\mathbf{e}$, is given by[10],*

$$\sigma = \sum_m \frac{4\pi e^2 c}{h\nu} \left| \int \tilde{\psi}_m (\mathbf{e} \cdot \mathrm{s}') \psi_0 \sin (\kappa z + \beta) d\tau \right|^2. \qquad (1)$$

[4] Chao, P.N.A. **16**, (1930).

[5] Tarrant, Proc. Roy. Soc. **A128**, 345 (1930).

[6] Gray, Proc. Roy. Soc. **A130**, 524 (1931).

[7] Klein, Nishina, Zeits. f. Physik **52**, 853 (1928).

[8] Weyl, Gruppentheorie u. Quantenmechanik, p. 175.

[9] Reference 8, p. 172.

[10] See for instance J. R. Oppenheimer, Phys. Rev. **35**, 461, par. 3, where the analogous result is derived for series transitions from the Heisenberg-Pauli theory. The extension to transitions in the continuous spectrum is trivial.

* The quantum which we are here considering is represented by a standing electromagnetic wave. The physics of our problem would make it more desirable to work with a progressive wave in which the momentum of the quantum was directed along the positive z-axis. In particular the use of such a wave would give an asymmetry in the direction of photoelectric emission about the $x-y$ plane; whereas from formula (1) we obtain a symmetrical result (see Eq. (17)). This may be interpreted as the photoelectric emission produced by a half unit intensity of radiation propagated in the positive z-direction, plus that from a half unit intensity of radiation propagated in the negative z-direction. The total value of the cross section integrated over all the angles is of course correctly given by (1). It would be possible to work directly with a relativistic extension of Dirac's light quantum theory (cf. e.g. I. Waller, Zeits. f. Physik **61**, 1930). When we start, however, with Maxwell's theory, and quantize according to the method of Heisenberg and Pauli we are led directly to the use of standing waves. It is not hard to modify the treatment of Heisenberg and Pauli; but it has seemed desirable to us to use the results of the strict theory in their most accessible form.

Here the coordinate z is to be measured from the center of mass of the atom; the phase constant β is introduced to take account of the fact that this center of mass may be anywhere between the crest and the trough of our electromagnetic wave. In our results we are to average over β. Further $\mathrm{K}=2\pi\nu/c$. We write for convenience

$$\Gamma = \kappa z + \beta .$$

We introduce two components of $\mathbf{s}'$[11]

$$s_x' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad s_y' = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}. \tag{2}$$

Let us define the integral in Eq. (1) to be I^0. And further as there are two normal states we will identify one of them with the index α, and the other with $\beta \cdot I^{0^2} = I_\alpha{}^{0^2} + I_\beta{}^{0^2}$. Then,

$$I_\alpha{}^0 = \frac{e_x}{2}(I_\alpha{}^R + I_\alpha{}^L) - \frac{ie_y}{2}(I_\alpha{}^R - I_\alpha{}^L)$$
$$= \frac{e_x - ie_y}{2} I_\alpha{}^R + \frac{e_x + ie_y}{2} I_\alpha{}^L$$

where

$$I_\alpha{}^R = \int \tilde{\psi}_k (s_x' + is_y')\psi_\alpha{}^0 \sin \Gamma d\tau \tag{3}$$

$$I_\alpha{}^L = \int \tilde{\psi}_k (s_x' - is_y')\psi_\alpha{}^0 \sin \Gamma d\tau .$$

Then multiplying $I_\alpha{}^0$ by its complex conjugate, the cross product term drops out because the selection rules for the two integrals are different. Averaging over all directions of e_s, we obtain

$$I_\alpha{}^{0^2} = \tfrac{1}{4}(I_\alpha{}^{R^2} + I_\alpha{}^{L^2}) .$$

This equation may be interpreted as a verification of the fundamental notion[12] that out of circularly polarized light we may build up a non-polarized beam, that has the same properties as a non-polarized beam built up from plane-polarized light in which the electric vector may have all possible directions. An exactly similar relation holds for $I_\beta{}^0$. In addition it may be shown by using the wave functions associated with s', that $I_\beta{}^L = I_\alpha{}^R$, $I_\alpha{}^L = I_\beta{}^R$. Consequently we may write

$$I^{0^2} = \tfrac{1}{2}(I_\alpha{}^{R^2} + I_\beta{}^{R^2}) .$$

We have then from Eq. (1)

[11] Reference 8, p. 172.

[12] Dirac, The Principles of Quantum Mechanics, Ch. I.

$$\sigma_\alpha + \sigma_\beta = \sum_k \frac{2\pi e^2 c}{h\nu}[I_\alpha{}^{2R}(k) + I_\beta{}^{2R}(k)].$$

The superscript R is to indicate one kind of polarization. Since we shall be interested in only one kind, R will not be written in explicitly from now on.

$$\sigma = \tfrac{1}{2}(\sigma_\alpha + \sigma_\beta) = \frac{\pi e^2 c}{h\nu} \sum_k (I_\alpha{}^2(k) + I_\beta{}^2(k)). \tag{4}$$

I_α is defined by Eq. (3), and I_β by a similar equation.

The wave functions.

In the following Weyl's wave functions and quantum numbers will be used.[13] The normal states α and β are specified by the quantum numbers $k=1$, $m=0$; $k=1$, $m=-1$ respectively. Both I_α and I_β will give rise to transitions into states of positive and negative values of k, and in each case $\Delta m=1$. The meaning of the quantum numbers j and m can briefly be given as follows. The total angular momentum of the electron in units of $h/2\pi$ is given by $M^2=k^2-1/4=j(j+1)$, where, in accordance with the customary meaning, j is the coupled angular momentum of spin and orbit. $k=\pm(j+1/2)\neq 0$. $k<0$ corresponds to states where spin and orbit are anti-parallel; while for $k>0$ spin and orbit are parallel, except for $k=1$ when the orbital angular momentum is zero. m is analogous to the magnetic quantum number, but is integral. The z-component of the total angular momentum is $h(m+1/2)/2\pi$.

The wave functions for the normal states may be obtained directly out of Weyl.[14] Only their components necessary to express I_α and I_β will be given here.

$$\left.\begin{array}{l}\alpha;\ k = 1,\ m = 0: \quad \psi_2{}^0 = -\psi_4{}^0 = i\alpha \sin\theta e^{i\phi} \\ \beta;\ k = 1,\ m = -1: \psi_2{}^0 = \bar{\psi}_4{}^0 = 1 + \rho_1 - i\alpha\cos\theta\end{array}\right\} \cdot N_0 e^{-rb} r^{\rho_1 - 1}$$

α equals the fine structure constant times the atomic number $=2\pi e^2 Z/hc$
b equals the reciprocal of the first Bohr radius $=4\pi^2 me^2 Z/h^2$

$$\rho_1 = (1-\alpha^2)^{1/2}, \text{ [In general, } \rho = (k^2-\alpha^2)^{1/2}]$$
$$1/N_0 = 4[\pi(1+\rho_1)\Gamma(1+2\rho_1)(2b)^{-1-2\rho_1}]^{1/2}.$$

Weyl's solutions[15] may be expressed in terms of better known functions in their dependence on θ.

$$r\psi_1 = e^{im\phi}(vq + iwp)$$
$$r\psi_2 = e^{i(m+1)\phi}[v(p - q\cos\theta) - iw(q - p\cos\theta)]\csc\theta$$

ψ_3 and ψ_4 are respectively ψ_1 and ψ_2 with the sign of iw changed
$q=\sin^{-m}\theta$ times the sum of Weyl's P and Q.
$p=\sin^{-m}\theta$ times the difference of Weyl's P and Q

[13] For a more complete statement of the formal role of the quantum numbers than is given in this paragraph one should consult par. 40 of reference 8.

[14] Reference 8, pp. 177–178.

[15] Reference 8, p. 184.

Then it is not hard to find that p and q are proportional to Ferrer's[16] $P_k{}^m(\cos\theta)$, and $P_{k-1}{}^m(\cos\theta)$ respectively, and then to show that

$$r\psi_1 = e^{im\phi}\left[vP_{k-1}{}^m(\cos\theta) + iw\frac{k-m}{k+m}P_k{}^m(\cos\theta)\right]$$

$$r\psi_2 = -\frac{e^{i(m+1)\phi}}{k+m}\left[vP_{k-1}{}^{m+1}(\cos\theta) + iwP_k{}^{m+1}(\cos\theta)\right].$$

v and w are the radial functions as in Weyl. Making the appropriate changes they are identical with the solutions given by Gordon.[17] We take over Gordon's solutions, and since we are interested in the continuum only, we make the convention $(1-\epsilon^2)^{1/2} = +i(\epsilon^2-1)^{1/2}$. The wave functions then become

$$\begin{aligned} r\psi_1(k) &= cNe^{im\phi}\left[(k+m)(\epsilon+1)^{1/2}\,x\,P_{k-1}{}^m(\cos\theta)\right.\\ &\quad \left.+\, i(k-m)(\epsilon-1)^{1/2}yP_k{}^m(\cos\theta)\right]\\ r\psi_2(k) &= -\,cNe^{i(m+1)\phi}\left[(\epsilon+1)^{1/2}\,x\,P_{k-1}{}^{m+1} + i(\epsilon-1)^{1/2}yP_k{}^{m+1}(\cos\theta)\right] \end{aligned} \qquad 5(\text{a, b})$$

for positive k. The functions for negative k we distinguish by a prime, indicating that the sign of k has been changed in Eqs. (5, a, b) and that the absolute value of k is considered:

$$\begin{aligned} r\psi_1'(-k) &= -\,cN'e^{im\phi}\left[(k-m)(\epsilon+1)^{1/2}\,x'P_k{}^m\right.\\ &\quad \left.+\, i(k+m)(\epsilon-1)^{1/2}y'P_{k-1}{}^m\right]\\ r\psi_2'(-k) &= cN'e^{i(m+1)\phi}\left[(\epsilon+1)^{1/2}\,x'P_k{}^{m+1} + i(\epsilon-1)^{1/2}y'P_{k-1}{}^{m+1}\right]. \end{aligned} \qquad 5(\text{c, d})$$

In each case to obtain ψ_3 change the sign of iy in ψ_1. The same connection holds between ψ_4 and ψ_2.

In the above expressions the quantities occurring are defined as follows:

$$\begin{aligned} f &= x + iy = e^{i\gamma}r^{\rho}e^{-ik_0r}F(\rho+1+in,\, 2\rho+1,\, 2ik_0r)\\ e^{2i\gamma} &= (\rho+in)/(k-ib/k_0)\\ f' &= x' + iy' = f(-\,|\,k\,|\,) = ie^{i\gamma'}(e^{-i\gamma}f)\\ e^{2i\gamma'} &= +\,(\rho+in)/(k+ib/k_0)\\ F(\alpha,\beta,x) &= \sum_{\sigma=0}^{\infty}\frac{\Gamma(\alpha+\sigma)\Gamma(\beta)}{\Gamma(\beta+\sigma)\Gamma(\alpha)\sigma!}x^{\sigma}\\ \epsilon &= E/mc^2. \end{aligned}$$

The following conservation law holds between the energy of the incident light, the energy of the electron in the normal state, and the energy of the photoelectron:

$$\begin{aligned} \kappa' + \rho_1 &= \epsilon, \text{ where } \kappa' = h\nu/mc^2\\ k_0 &= 2\pi mc(\epsilon^2-1)^{1/2}/h\\ n &= b\epsilon/k_0. \end{aligned}$$

[16] Whittaker and Watson, Modern Analysis, p. 323.

[17] Gordon, Zeits. f. Physik **48**, 11 (1928).

The normalization is carried out by the Weyl-Hellinger method.

The asymptotic expression of $f(k)$ for large r is

$$f \sim 2h_1 e^{ik_0 r} r^{in}$$

where,

$$h_1 = \frac{e^{i\gamma}\Gamma(2\rho+1)e^{-\pi n/2}e^{-\pi\rho i/2}}{2\Gamma(\rho+in)\,|\,2k_0\,|^{\,\rho-in}}\,.$$

Therefore, using

$$r\psi_1 = A_1 x + iB_1 y$$

$$r\psi_1 \sim G_1 e^{ik_0 r} r^{in} + \text{complex conjugate},$$

$G_1 = h_1(A_1 + iB_1)$, and A_1 is the coefficient of x in the expression for $r\psi_1$ of Eq. (6).

The normalization is now performed as follows. We use the above expression for ψ_1, and find

$$\lim_{\Delta\nu\to 0}\int_0^\infty \bar{\psi}_1\left\{\int_{\Delta\nu}\psi_1 d\nu\right\} r^2 dr.$$

The sum of these expressions for all four components of ψ is then integrated over the angles and set equal to unity.

One takes the asymptotic expression for ψ since the contribution over any finite part of the radial integration vanishes. This is the physical assertion that the relative time an electron (in a hyperbolic orbit) spends near the nucleus is negligible.

The normalization is further simplified by the smoothness of the integrand in the integral over $d\nu$; all factors involving ν except the exponential may be considered constant, since we will go to the limit $\Delta\nu \to 0$. One obtains

$$\int r^2\bar{\psi}_1\left\{\int_{\Delta\nu}\psi_1 d\nu\right\}dr = \frac{d\nu}{dk_0}\,\pi\,|\,G_1\,|^{\,2} + 0(\Delta\nu).$$

On completing the normalization

$$c^2 = \frac{8\pi m\epsilon(2k_0)^{2\rho-1}e^{\pi n}\,|\,\Gamma(\rho+1+in)\,|^{\,2}}{h\Gamma^2(2\rho+1)}$$

$$N^2 = \frac{(k-m-1)!}{4\pi\epsilon(k+m)!}\;\frac{2k+1}{(4k+1)\epsilon+1}$$

$$N'^2 = \frac{(k-m-1)!}{8\pi\epsilon(k+m)!}\,.$$

Let $N_1 = cN$, $N_1' = cN'$

As a guide in discussing the integrals, before actually expressing them, the physical meaning of the transitions that I_α and I_β of Eq. (3), and the analogous equation, correspond to, can be mentioned here. $I_\alpha(\pm 1)$ will not exist as there are no states such that $|k|-m-1<0$, and for I_α m changes from zero to one. This will assert itself when I_α is seen to contain a factor $(k-1)$. $I_\beta(1)$, since it gives the probability of a transition involving a final state of zero orbital angular momentum, one might expect to be small. It must certainly be a term that approaches zero as the frequency of the incident light approaches zero, for the transition could not occur without some correction not considered in deriving the Schroedinger cross section. It turns out that this transition does occur, and is induced by the retardation factor; its probability is small of the order $(\kappa/b)^2$.

Expression of the integrals.

Using Eqs. (2) and (3) we may write

$$I_\alpha(k) = 2\int (\bar{\psi}_1 k\psi_2{}^{0,\alpha} - \bar{\psi}_3{}^k\psi_4{}^{0,\alpha}) \sin \Gamma d\tau \tag{6}$$
$$= 4\pi i\alpha N_0 N_1 (k+m)(\epsilon+1)^{1/2} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{i\phi(1-m)} d\phi r^{\rho_1} e^{-rb} x$$
$$\sin\theta P_{k-1}{}^m(\cos\theta)\sin\Gamma\sin\theta dr d\theta.$$

This gives immediately $m=1$. We employ the following representation of a wave.[18]

$$\sin\Gamma = \left(\frac{2\pi}{(\kappa r)^3}\right)^{1/2} \csc\theta \sum_{\sigma=0}^{\infty}(\sigma+3/2)P_{\sigma+1}{}^1(\cos\theta)\sin\left(\frac{\pi\sigma}{2}+\beta\right) J_{\sigma+3/2}(\kappa r) \tag{7}$$

integrate over θ, and use the following relation between Bessel functions and confluent hypergeometric functions

$$J_{k-1/2}(\kappa r) = \frac{(\kappa r/2)^{k-1/2}e^{-i\kappa r}}{\Gamma(k+\frac{1}{2})} F(k, 2k, 2i\kappa r).$$

The result is

$$I_\alpha(k) = \frac{-4\pi^{3/2}N_0N_1\alpha k(k^2-1)(\epsilon+1)^{1/2}(\kappa/2)^{k-2}\sin\left(\frac{\pi k}{2}+\beta\right)}{\Gamma(k+\frac{1}{2})}$$
$$\int_0^\infty e^{-r(b+i\kappa)} x r^{k+\rho_1-2}\cdot F(k, 2k, 2i\kappa r)dr.$$

It will be convenient in discussing this and the remaining integrals to express x and y in the explicit forms that will now be given. Using the recursion formula

$$\alpha F(\alpha+1, \gamma+1, x) = (\alpha-\gamma)F(\alpha, \gamma+1, x) + \gamma F(\alpha, \gamma, x),$$

[18] Reference 16, p. 383 (the expression in the reference should be multiplied by two).

we may write

$$x = \frac{\rho e^{-i\gamma'} e^{-ik_0 r} r^{\rho}}{(\rho^2 + n^2)^{1/2}} [F(\rho + in, 2\rho, 2ik_0 r) + gF(\rho + in, 2\rho + 1, 2ik_0 r)]$$

where the following definitions hold: $\gamma' = \tan^{-1} n/\rho - \gamma$,

$$g = \frac{1}{2\rho}\left[k - \rho + i\alpha\left(\frac{\epsilon - 1}{\epsilon + 1}\right)^{1/2}\right] \rightarrow 0\left(\frac{\alpha^2}{4k^2}\right), \text{ as } k_0 \rightarrow 0.$$

By finding the real and imaginary parts of $e^{-ik_0 r}F(\rho+1+in,\ 2\rho+1,\ 2ik_0 r)$, y may be found.

$$y = \frac{r^{\rho} e^{-ik_0 r}}{(\rho^2 + n^2)^{1/2}} [pF(\rho + in, 2\rho, 2ik_0 r) + qrF(\rho + 1 + in, 2\rho + 2, 2ik_0 r)]$$

where,

$$p = \left[\frac{1}{2}\left(\rho^2 - \rho k + \frac{\alpha^2 \epsilon}{\epsilon - 1}\right)\right]^{1/2} \rightarrow 0\left[\frac{\alpha^2}{2(\epsilon - 1)}\right]^{1/2}, \text{ as } k_0 \rightarrow 0$$

$$q = \frac{b\epsilon\left[\frac{1}{2}\left(\rho^2 - \rho k + \frac{\alpha^2\epsilon}{\epsilon - 1}\right)\right]^{1/2} + \rho k_0\left[\frac{1}{2}\left(\rho^2 + \rho k + \frac{\alpha^2\epsilon}{\epsilon + 1}\right)\right]^{1/2}}{\rho(2\rho + 1)}$$

$$\rightarrow 0\left[\frac{b^2\alpha^2}{2\rho^2(2\rho + 1)^2(\epsilon - 1)}\right]^{1/2}, \text{ as } k_0 \rightarrow 0$$

$$y' = y(-k).$$

To obtain x' we use the relation $2x = f + \bar{f}$, and the formula

$$F(\alpha + 1, \gamma, x) - F(\alpha, \gamma, x) = \frac{x}{\gamma} F(\alpha + 1, \gamma + 1, x)$$

and thus obtain

$$x' = -e^{i\gamma'} r^{\rho} e^{-ik_0 r}\left[\frac{k_0 r}{2\rho + 1} F(\rho + 1 + in, 2\rho + 2, 2ik_0 r) + jF(\rho + in, 2\rho + 1, 2ik_0 r)\right]$$

where,

$$j = \frac{\rho - k + i\alpha\left(\frac{\epsilon - 1}{\epsilon + 1}\right)^{1/2}}{2i(\rho + in)}.$$

We may now write out the integrals arising from the two normal states α and β. Before doing so, in order to save writing, we call attention to the factor $\sin(\pi k/2 + \beta)$ that will occur in front of each term. Each term will ultimately have its absolute value squared, when we must average over the phases β. In each case the average value of $\sin^2(\pi k/2+\beta)$ will be $1/2$, and to indicate this we now write $1/2$ for each of these factors.

Then,

$$I_\alpha(k) = \frac{-(2\pi)^{3/2} i\alpha N_1 N_0 k(k^2-1)(\epsilon+1)^{1/2}(\kappa/2)^{k-2} e^{-i\gamma'}\rho}{\Gamma(k+\frac{1}{2})(\rho^2+n^2)^{1/2}} \cdot [S(\mu-2, k, a, 2\rho) + gS(\mu-2, k, a, 2\rho+1)] \quad 8(a)$$

$$I_\alpha(-k) = \frac{-4\pi^{3/2} i\alpha N_1' N_0 k(k^2-1)(\epsilon+1)^{1/2}(\kappa/2)^{k-1}}{\Gamma(k+3/2)} \left[\frac{k_0}{2\rho+1} S(\mu, k+1, a+1, 2\rho+2) + jS(\mu-1, k+1, a, 2\rho+1)\right] \quad 8(b)$$

$$I_\beta(k) = \frac{-(2\pi)^{3/2} i\alpha N_1 N_0 k(\kappa/2)^{k-2}}{\Gamma(k+\frac{1}{2})(\rho^2+n^2)^{1/2}} \left\{ \frac{k\alpha(\epsilon+1)^{1/2}\kappa^2 e^{-i\gamma'}}{4k^2-1} [S(\mu, k+1, a, 2\rho) + gS(\mu, k+1, a, 2\rho+1)] - \alpha e^{-i\gamma'}(\epsilon+1)^{1/2}\rho(k-1) [S(\mu-2, k-1, a, 2\rho) + gS(\mu-2, k-1, a, 2\rho+1)] + \frac{(1+\rho_1)(\epsilon-1)^{1/2}\kappa^2}{2k+1} [pS(\mu, k+1, a, 2\rho) + qS(\mu+1, k+1, a+1, 2\rho+2)] \right\} \quad 8(c)$$

$$I_\beta(-k) = \frac{-2(2\pi)^{3/2} i N_0 N_1' k(\kappa/2)^{k-1}}{\Gamma(k+\frac{1}{2})(\rho^2+n^2)^{1/2}} \left\{ -\frac{(\rho^2+n^2)^{1/2}\kappa^2\alpha(\epsilon+1)^{1/2}(k+1)e^{i\gamma}}{(2k+1)^2(2k+3)} \cdot \left[\frac{k_0}{2\rho+1} S(\mu+2, k+2, a+1, 2\rho+2) + jS(\mu+1, k+2, a, 2\rho+1)\right] + \frac{(\rho^2+n^2)^{1/2} k\alpha(\epsilon+1)^{1/2} e^{i\gamma'}}{2k+1} \left[\frac{k_0}{2\rho+1} S(\mu, k, a+1, 2\rho+2) + jS(\mu-1, k, a, 2\rho+1)\right] - (1+\rho_1)(\epsilon-1)^{1/2} [p'S(\mu-1, k, a, 2\rho) + q'S(\mu, k, a+1, 2\rho+2)] \right\} \quad 8(d)$$

where,

$$S(A, k, a, 2\rho) = \int_0^\infty e^{-r(b+i\kappa+ik_0)} r^A F(k, 2k, 2i\kappa r) F(a, 2\rho, 2ik_0 r) dr \quad (9)$$

and, $\mu = \rho+\rho_1+k$, $a = \rho+in$

The other quantities S are obtained from this by appropriate changes in the parameters.

By expanding both of the confluent functions in this integral in powers of r, integrating over r, and then summing first over powers of $2i\kappa$, one obtains an infinite series, each term of which contains a hypergeometric function. To the latter apply the formula

$$F(\alpha, \beta, \gamma, x) = (1 - x)^{-\beta} F\left(\gamma - \alpha, \beta, \gamma, \frac{-x}{1 - x}\right)$$

and then use the definition

$$F(\alpha, \beta, \gamma, x) = B^{-1}(\beta, \gamma - \beta) \int_0^1 u^{\beta-1}(1 - u)^{\gamma-\beta-1}(1 - xu)^{-\alpha} du.$$

We now define

$$S = \frac{2i\kappa}{b + i\kappa + ik_0}, \quad t = \frac{2ik_0}{b + i\kappa + ik_0} \cdot$$

It is then seen by summing under the sign of integration in the expression obtained as just indicated, that

$$S(\mu - 2, k, a, 2\rho) = \frac{(S/2i\kappa)^{\mu-1}\Gamma(\mu - 1)}{(1 - s)^k B(k, k)}$$
$$\int_0^1 (u - u^2)^{k-1}\left(1 + \frac{su}{1 - s}\right)^{\mu-2k-1} F\left(a, \mu - 1, 2\rho, t + \frac{tsu}{1 - s}\right) du.$$

To the hypergeometric function in the integrand we apply

$$F(\alpha, \beta, \gamma, x) = (1 - x)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta, \gamma, x)$$

so

$$F\left(a, \mu - 1, 2\rho, t + \frac{tsu}{1 - s}\right)$$
$$= \left[1 - t\left(1 + \frac{su}{1 - s}\right)\right]^{2\rho-\mu-a+1} F\left(\bar{a}, 2\rho - \mu + 1, 2\rho, \tau\left(1 + \frac{su}{1 - s}\right)\right).$$

It follows that

$$S(\mu - 2, k, a, 2\rho) = \frac{(S/2i\kappa)^{\mu-1}\Gamma(\mu - 1)(1 - t)^{2\rho-\mu-a+1}}{(1 - s)^k B(k, k)} \cdot$$
$$\int_0^1 (u - u^2)^{k-1}\left(1 + \frac{su}{1 - s}\right)^{\mu-2k-1}\left[1 - \frac{stu}{(1 - s)(1 - t)}\right]^{2\rho-\mu-a+1} \tag{10}$$
$$F\left(\bar{a}, 2\rho - \mu + 1, 2\rho, t\left(1 + \frac{su}{1 - s}\right)\right) du.$$

All of the remaining integrals S may again be obtained from this one by making suitable changes in the parameters.

We shall now treat explicitly the problem at the K-limit.

Let us introduce the following quantities

$$\delta = 2k + 1 - \mu$$
$$\delta' = 2\rho - \mu + 1.$$

Rewriting Eq. (10)

$$S(\mu-2, k, a, 2\rho) = \frac{(s/2i\kappa)^{\mu-1}\Gamma(\mu-1)(1-t)^{\delta'-a}}{(1-s)^k B(k, k)} \cdot \tag{10'}$$

$$\int_0^1 (u-u^2)^{k-1}\left(1+\frac{su}{1-s}\right)^{-\delta}\left(1-\frac{stu}{(1-s)(1-t)}\right)^{\delta'-a}$$

$$F\left(\bar{a}, \delta', 2\rho, t\left(1+\frac{su}{1-s}\right)\right)du, \text{ for } |s| \sim 0(\propto).$$

This integral is now evaluated by taking the factors $(1+su/1-s)^{-\delta}$ and $F(a, \delta', 2\rho, t(1+su/1-s))$ outside of the sign of integration and giving u its average value 1/2.[19] It is then immediate that

$$S(\mu-2, k, a, 2\rho) = \frac{(s/2i\kappa)^{\mu-1}\Gamma(\mu-1)(1-t)^{\delta-a}}{(1-s)^k}$$

$$\left(\frac{1-s}{1-s/2}\right)^{\delta} F\left(a, \delta', 2\rho, t\left(\frac{1-s/2}{1-s}\right)\right)\cdot F\left(a-\delta', k, 2k, \frac{st}{(1-s)(1-t)}\right) \tag{10''}$$

It would be possible now to discuss this S as a function of κ, to determine its contribution to the wave-length law, *et cetera*. However, this paper is concerned with a discussion of σ at the K-absorption limit, and we propose at this point to reduce our expressions so they will apply only for $k_0=0$. (The K-limit).

Letting k_0 approach zero we see after making a simple confluence in each hypergeometric function of Eq. (10) that

$$S(\mu-2, k, a, 2\rho) = \frac{(s/2i\kappa)^{\mu-1}\Gamma(\mu-1)e^{-\beta}}{(1-s)^k(1-\bar{s}/2)^{\delta}} F(k, 2k, \beta\bar{s})F\left(\delta', 2\rho, \beta\left(1-\frac{\bar{s}}{2}\right)\right) \tag{11}$$

where β is defined as $\beta=2b/(b+i\kappa)$. The remaining integrals are evaluated in exactly the same way (see appendix). In this case, however, where $k_0=0$ the work may be shortened by going to the limit in Eq. (10′). Corresponding to the factors $(1-su)$,$^{-\delta}$ and $F(\delta', 2\rho)$ in the integral just discussed, other factors will occur where δ and δ' will be increased or decreased by one or two. In all cases it will be legitimate for our purposes to proceed in the same way, putting u average equal 1/2.

Thus the equations analogous to Eq. (11) may be written down. These expressions are substituted into Eqs. (8; a, b, c, d) and the square of the ab-

[19] See appendix Part I.

solute value of each $I(\pm k)$ is taken. Then on multiplying by the proper normalizing factor in each case, one obtains the following expressions:

$$I_\alpha^2(k) = H(k)(k^2-1)\,|F(k, 2k, \beta\bar{s})|^2\Big[F(\delta', 2\rho) + \frac{k-\rho}{2\rho}F(1+\delta', 2\rho+1)\Big]^2 \tag{12, a}$$

$$I_\alpha^2(-k) = H(k)\frac{32(\kappa/2)^2b^2\mu^2(\mu-1)^2F^2(k+1)}{(2k+1)^2(2\rho+1)^2(2\rho)^2(b^2+\kappa^2)^2}\Big[F(\delta', 2\rho+2) + \frac{k-\rho}{2}F(\delta', 2\rho+1)\Big]^2 \tag{12, b}$$

$$I_\beta^2(k) = H(k)\frac{b^4}{(b^2+\kappa^2)^2}\Big\{(k-1)\Big[F(\delta', 2\rho) + \frac{k-\rho}{2\rho}F(1+\delta', 2\rho+1)\Big]F(k-1) - \frac{k\mu(\mu-1)(\kappa\,|\,b)^2F(k+1)}{4k^2-1}\Big[F(\delta'-2, 2\rho) + \frac{k-\rho}{2\rho}F(\delta'-1, 2\rho+1)\Big] - \frac{(1+\rho_1)\mu(\mu-1)(\kappa/b)^2F(k+1)}{2\rho(2k+1)}\Big[F(\delta'-2, 2\rho) + \frac{F(\delta'-1, 2\rho+2)}{\rho(2\rho+1)(1+a^2\kappa^2)}\Big]\Big\}^2 \tag{12, c}$$

$$I_\beta^2(-k) = \frac{H(k)(2\kappa)^2\mu^2(\mu-1)^2b^6}{(2\rho+1)^2(2\rho)^2(b^2+\kappa^2)^4}\Big\{\frac{(k+1)(\mu+1)(\mu+2)(\kappa/b)^2}{(2k+1)^2(2k+3)}F(k+2) \cdot\Big[F(\delta', 2\rho+2) + 0\Big(\frac{\alpha^2}{30}\Big)\Big] - \frac{kF(k)}{2k+1}\Big[F(\delta', 2\rho+2) + \frac{(k-\rho)(2\rho+1)(1+a^2\kappa^2)}{2\mu}F(\delta', 2\rho+1)\Big) + \frac{(1+\rho_1)F(k)}{2\rho}\Big[F(\delta', 2\rho+2) + \frac{\rho(2\rho+1)(1+a^2\kappa^2)}{\mu}F(\delta'-1, 2\rho)\Big]\Big\}^2 \tag{12, d}$$

where,

$$H(k) = \frac{8\pi^4N_0^2km(2b)^{2\rho-1}\alpha^2\Gamma^2(\mu-1)(\kappa/2)^{2k-4}}{h\Gamma^2(k+\frac{1}{2})\Gamma^2(2\rho)(b^2+\kappa^2)^{\mu-1-\delta}\,b^{2\delta}\,e_1^{\,4/+a^2\kappa^2}}$$

$a=1/b$, $F(k+m)$ is an abbreviation for $F(k+m, 2k+2m, \beta s)$. Also, the argument of the F's in the brackets has been left out. This argument is $\beta(1-\bar{s}/2)$

in all cases. e_1 is the base of the natural logarithms. We now rewrite Eq. (4), where ν is now the frequency of the K-absorption limit. $\nu = (1-\rho_1)mc^2/h$.

$$\sigma = \frac{\pi e^2}{(1-\rho_1)mc}\left\{ \sum_{k=1}^{\infty}[I_\beta^2(k) + I_\beta^2(-k)] + \sum_{k=2}^{\infty}[I_\alpha^2(k) + I_\alpha^2(-k)]\right\}. \quad (13)$$

By going to the limit $c \rightarrow \infty$ one easily sees that $I_\alpha^2(k) = 3I_\beta^2(k) = 0$ unless $k = 2$; and $I_\beta^2(-k) = (2/3)I_\alpha^2(2)$ if $k = 1$. In this way it becomes obvious that in the absence of spin forces the contributions from the two normal states are equal, and any asymmetry between them vanishes, as it must. Furthermore, for c infinite we have

$$\sigma = \alpha a^2 \frac{2^9\pi^2}{3e_1^4 Z}.$$

As given by Eq. (13) and Eqs. (12), σ, according to our approximations, is correct to within 3 percent for $Z<68$. For $Z>68$ the error may increase, but due to the nature of these approximations we should obtain a fair estimate at least, for all Z.

Results.

Calculations using Eqs. (12) and (13) yield the results of Table I at the K-limit.[20] As indicated, these values of σ are less than those given by the Schroedinger theory, and, with one exception, grow more discrepant with increasing Z. The Schroedinger theory gave fairly good results but was itself slightly low.

TABLE I.

Z	1	11	19	37	55	79
$Z^2\sigma 10^{18}$	6.3	6.3	6.2	5.9	6.0	5.2
$10^{18}Z^2\sigma_{SCH}$	6.3	—	—	—	—	—

From data in Roess' paper his τ_K at the K-limit is from 0.3 percent to 14.6 percent less than the Schroedinger result, for effective Z roughly from 11 to 80. In approximately the same range our result is lower than the Schroedinger result from about 0.3 to 20 percent.

We may say that in this case the more precise model which includes the retarded potentials (Roess does not include the retardation factor in his integrals) does not improve the comparison with experiment, and if anything makes it slightly worse.*† This discrepancy between theory and experiment must be attributed to the inaccuracy of a model which neglects all electrostatic interactions.

[20] It is unfortunate in comparing our results with those of Roess (reference 4) that these calculations are for different values of Z than those he has used. However, due to the obvious inaccuracy of the theory even in this case where retardation is included, and as shown by our results, it does not seem necessary to make a further comparison between our results than can be done qualitatively.

Part II. Photoelectric Absorption of Ultragamma Radiation

Introduction.

In order to discuss the problem mentioned in the introduction to this paper under (2), we should now be obliged to consider the quantities S defined by Eq. (9), for large κ. That such a consideration leads to a grave difficulty it is the purpose of the present section to point out. The nature of this difficulty can be elucidated in a considerably simplified case. Let us in Eq. (9) set $\alpha=0$ and $n=0$. Eq. (9) can then be written, after a simple substitution,

$$S(2k-1, k, k, 2k) = \left(\frac{\kappa k_0}{4}\right)^{-k+1/2} \Gamma^2(k+\tfrac{1}{2}) \int_0^\infty e^{-rb} J_{k-1/2}(\kappa r) J_{k-1/2}(k_0 r) dr.$$

The integral in this equation can now be evaluated[21] with the result

$$\int e^{-rb} J_{k-1/2}(\kappa r) J_{k-1/2}(k_0 r) dr = \frac{(\kappa k_0)^{-1/2}}{\pi} Q_{k-1}\left(\frac{\epsilon}{k_0{}'}\right). \tag{14}$$

In order to evaluate σ it is necessary to find $\Sigma_k I^2$. This sum does not converge uniformly in k; that is the value of k for which the contribution of the terms is a maximum, increases as κ increases. We are thus faced with one of two problems: (a) That of finding the expansion of $Q_k(x)$ about $x=1$, necessarily uniform in k, or, (b) That of finding an asymptotic expansion for $Q_k(x)$, for k large, which is uniform as $x \to 1+0$.

Either of these problems, after considerable study of the contour integral, and of the differential equation, we believe to form a major analytical difficulty; and that the answer to (b) is probably not to be found by the method of steepest descents. It should further be remarked that even in the case problem (a) or (b) is resolved, that, due to the non-uniformity of the function $\Sigma_k I_\alpha{}^2$, it is not unlikely that the function I_α will be such as to make the sum over k difficult, if not impossible.

Just this difficulty persists if n and α are retained in Eq. (9), and, although the analysis is more complicated, the answer would be forthcoming if this simpler case could be treated. In the face of this we have been forced to leave the solution in this form.

This difficulty, and in fact most of the complication encountered in this physically simple problem, can be regarded as having been introduced by the

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* Roess explains that his τ_k, which is assertedly computed for a one-electron atom, does not need to be doubled to give the total absorption of the two K-electrons. A closer examination of Roess' work, shows, however, that his τ_k gives the absorption of an atom in which both normal states are filled, whereas our σ gives the absorption for an atom with one electron, which is with equal probability to be found in either of the two normal states.

† As remarked in the footnote of page 2, no treatment of the wave-length dependence is to be given in this paper. However, it has been verified that retardation does not change the dependence on wave-length in any critical way, and that as nearly as the curve may be described at all by assigning a specific exponent to λ this exponent is never much different from 3. Roess found that the relativistic theory, when retardation was not included, gave something closer to a λ^3 law than was given by the previous less exact theories applied to the K-shell.

[21] G. N. Watson, Theory of Bessel Functions, p. 389.

resolution of the waves representing states in the continuum into spherical harmonics, in order to obtain a solution of Dirac's equation. In the relativistic theory the problem of the hydrogen atom is no longer separable in parabolic coordinates, so that wave functions representing hyperbolic orbits, which take the relativistic change of mass with velocity into account, and which are otherwise analogous to Gordon's wave functions,[22] are not known. We see graphically that the trouble arises from the fact that for very hard radiation there is a strong directional selection for the photoelectrons, while due to the non-separability of the wave equation in any coordinate system that allows this condition to be useful, we are forced to recompose waves resolved into spherical harmonics by solving the above stated problem.

Now for $b/\kappa << 1$ the translational energy of the photoelectrons becomes very large, and we should not expect in the limit of $b/\kappa \sim 0$ that the effect of binding on the photoelectrons would play a very great role. We thus feel tempted to neglect altogether the binding of the photoelectrons, and to apply for their wave functions in the continuum relativistic plane waves.

In neglecting the binding of the photoelectrons we are setting the two quantities n and b/κ equal to zero; the neglect of b/κ will always be justified for sufficiently hard radiation, but for the application to the gamma rays of ThC'' the effect of terms $1/\kappa'$ is appreciable, as we shall see in our formulas (18) and (19). The neglect of n does not become justified as $\kappa' \to \infty$ and we have taken particular care to verify that this effect is inappreciable for light elements, and can introduce even for lead at the very most a factor of three. To do this we have *inter alia* so modified the wave functions that they have the correct asymptotic behavior for an electron moving in the field of the nucleus. This modification introduces a term in the exponent of the wave functions, in addition to the ik_0r term of the plane waves, of the form $in \cdot \log k_0 r$. The result of this investigation is given in the footnote to Eq. (17), and justifies the approximation involved in neglecting the binding entirely.

It would appear impossible to improve substantially this calculation except by a strict evaluation of the cross section by the method outlined in the first paragraph of this section.

Derivation of the results.

We distinguish the two sets of wave functions in the continuum by the labels α' and β'; each set corresponds to a different orientation of spin.

$$\left.\begin{aligned} \psi_1^{\alpha'} &= 1 + \epsilon + k_{0z}' \\ \psi_2^{\alpha'} &= k_{0x}' + ik_{0y}' \\ \psi_3^{\alpha'} &= 1 + \epsilon - k_{0z}' \\ \psi_4^{\alpha'} &= -(k_{0x}' + ik_{0y}') \end{aligned}\right\} \cdot Ne^{i(k_0 \cdot r)}$$

$$\psi_1\beta' = \bar{\psi}_2^{\alpha'},\ \psi_2^{\beta'} = \psi_3^{\alpha'},\ \psi_4^{\beta'} = -\bar{\psi}_1^{\alpha'},\ \psi_4^{\beta'} = \psi_1^{\alpha'}$$

$$(\bar{\psi}^{\alpha'} \cdot \psi^{\beta'}) = 0$$

ϵ as before is the energy in units mc^2. k_0' is the momentum in units mc.

[22] Gordon, Zeits. f. Physik **48**, 180 (1928).

There will be four integrals I corresponding to transitions between the two initial and the two final states. They are $I^{\alpha\alpha'}$, $I^{\alpha\beta'}$, $I^{\beta\alpha'}$, $I^{\beta\beta'}$, and are defined as in Eq. (6) by the equation

$$I^{\alpha\alpha'} = 2\int (\bar{\psi}_1^{\alpha'}\psi_2^{0,\alpha} - \psi_3^{\alpha'}\psi_4^{0,\alpha}) \sin \Gamma d\tau. \qquad (15)$$

By use of the wave equations for the normal states already given, and those adopted for the final states, we have from Eq. (15)

$$I^{\alpha\alpha'} = 4i\alpha N_0 N(1+\epsilon)\int e^{-i(k_0 r)} \sin\theta e^{i\phi} r^{\rho_1 - 1} e^{-rb} \sin\Gamma d\tau$$

$$I^{\alpha\beta'} = 0$$

$$I^{\beta\alpha'} = 4N_0 N\int \left\{k_{0z}'(1+\rho_1) - i\alpha(1+\epsilon)\cos\theta\right\} e^{-rb} r^{\rho_1-1} \sin\Gamma e^{-i(k_0 r)} d\tau$$

$$I^{\beta\beta'} = -4N_0 N(k_{0x}' + ik_{0y}')(1+\rho_1)\int e^{-i(k_0 r)} e^{-rb} r^{\rho_1 - 1} \sin \Gamma d\tau.$$

By the same method of normalization given previously we obtain, normalizing to $dvd(\cos w)d\,\phi$,

$$N^2 = \frac{mk_0}{2h\pi(1+\epsilon)}\,.$$

Consider $I^{\alpha\alpha'}$. Writing $\sin\Gamma$ in the exponential form,

$$I^{\alpha\alpha'} = 2\alpha N_0 N(1+\epsilon)\left[e^{i\beta} I_1 - e^{-i\beta} I_2\right]$$

where,

$$I_1 = \int (x + iy) e^{-rb} r^{\rho_1 - 2} e^{-i\mathbf{r}(k_0 - \kappa)} d\tau$$

$$I_2 = I_1(-\kappa).$$

Next multiplying $I^{\alpha\alpha'}$ up by its complex conjugate and averaging over the phases β, we find that

$$|I^{\alpha\alpha'}|^2 = 4\alpha^2 N_0^2 N^2 (1+\epsilon)^2 [|I_1|^2 + |I_2|^2].$$

Let us now evaluate the expression I_2. To do this it is convenient to transform to coordinates whose z-axis has the direction of the vector $k_0+\kappa$. We define the angle w and the quantity p by the following relations

$$\begin{aligned}(k_0+\kappa)^2 &= k_0^2 + \kappa^2 - 2k_0\kappa\cos\omega \\ &= p^2.\end{aligned} \qquad (16)$$

We transform the integrand according to the scheme

	x'	y'	z'
x	l_1	l_2	l_3
y	m_1	m_2	m_3
z	n_1	n_2	n_3

and obtain

$$I_2 = (l_3 + im_3) \int r^{\rho_1+1} e^{-rb-irp\cos\theta} \cos\theta \sin\theta dr d\theta d\phi .$$

Integrating first over the angles we soon find that*

$$I_2 = \frac{\pi^{3/2} i(l_3 + im_3)\Gamma(\rho_1 + 3)}{\Gamma(5/2)} \frac{p}{(p^2 + b^2)^{\rho_1 + 3/2}} F\left(\frac{\rho_1+3}{2}, \frac{1-\rho_1}{2}, \frac{5}{2}, \frac{p^2}{p^2+b^2}\right).$$

The remaining integrals are obtained similarly, and after a few easy calculations, and using the relation

$$\sigma(w) = \frac{\pi e^2 c}{h\nu}\{ | I^{\alpha\alpha'} |^2 + | I^{\alpha\beta'} |^2 + | I^{\beta\alpha'} |^2 + | I^{\beta\beta'} |^2 \}$$

one finds that†

$$\sigma(w) = \frac{4\pi^2 e^4 Z\alpha\Gamma^2(\rho_1 + 2)}{h^3 \nu} \cdot \frac{(2b)^{1+2\rho_1}}{(1 + \rho_1)\Gamma(1 + 2\rho_1)} \frac{mk_0}{1 + \epsilon}$$
$$\cdot \left\{ (1 + \rho_1)^2 k_0^2 (f_{3+}^2 + f_{3-}^2) + \frac{(1 + \epsilon)^2(\rho_1 + 2)^2}{9} (p_+^2 f_{5+}^2 + p_-^2 f_{5+}^2) \right\} \quad (17)$$

$$f_{M\pm} = F\left(\frac{\rho_1 + 3}{2}, \frac{1 - \rho_1}{2}, \frac{M}{2}, \frac{p_\pm^2}{p_\pm^2 + b^2}\right) \cdot (p_\pm^2 + b^2)^{-\frac{\rho_1 + 3}{2}}$$

where p_+ is the p defined by Eq. (16); $p_- = p_+(\pi - w)$.

The above expression for $\sigma(w)$ gives the distribution in angle about the direction of κ. The range of w is $0-\pi$. The hypergeometric function with $M = 5$ varies slowly, as can be seen from a continuation of the function about $1-x$. Similarly in the other case, $M=3$. (See appendix, Part II.)

The angular dependence is then seen to be given, nearly, by the factor

$$\left\{\left(1 + \frac{1}{k_0'^2}\right)^{1/2} - \cos w\right\}^{-4}$$

and the steepness of the curve when $\mathbf{k}_0$ and $\boldsymbol{\kappa}$ are nearly parallel is apparent. For κ infinite it is therefore justified to give both hypergeometric functions their values when k_0 and κ are parallel, in the angular integration. Even for $\kappa' = 5$, as it is for ThC'' gamma rays, it may be shown that the error introduced by this procedure is completely negligible (less than 2 percent). With this understanding we may proceed with the integration over the angles; to do so it is necessary to find

$$\sigma = \int \sigma(w) d \cos w d\phi .$$

* These integrals may be evaluated in terms of elementary functions but we prefer to express them as is here done.

† If the modification due to binding be carried through, its effect can be seen, approximately, by substituting $\rho_1 + in$ for ρ_1 in this formula (excepting of course N_0).

The result of this integration is

$$\sigma = \frac{\pi\alpha_0^2 a_0^2}{4} \frac{\Gamma^2(\rho_1+2)}{(1+\rho_1)\Gamma(1+2\rho_1)} \frac{1}{1+\epsilon}\left[\frac{2\alpha_0^2}{\kappa'(\epsilon-k_0')}\right]^{\rho_1+2}$$
$$\cdot Z^{3+2\rho_1}\left\{(\rho_1+2)\left[\frac{(1+\rho_1)k_0}{(\rho_1+2)\kappa} F\left(\frac{3+\rho_1}{2}, \frac{1-\rho_1}{2}, \frac{3}{2}; x\right)\right]^2\right. \tag{18}$$
$$\left.+(\rho_1+1)\left[\frac{(\rho_1+2)(1+\epsilon)}{3(\rho_1+1)\kappa^1} F\left(\frac{3+\rho_1}{2}, \frac{1-\rho_1}{2}; \frac{5}{2}, x\right]^2\left(1-\frac{2}{3}\alpha^2\right)\right\}.$$

For κ infinite, this result becomes

$$x = (k_0'-\kappa')^2/2\kappa'(\epsilon-k_0')$$

$$\sigma = \frac{\pi a_0^2}{16} \frac{\Gamma^2(\rho_1+2)(2\alpha_0)^{2\rho_1+6}}{(1+\rho_1)\Gamma(1+2\rho_1)} \cdot \frac{Z^{3+2\rho_1}}{\epsilon}\left\{\frac{(1+\rho_1)^2}{2+\rho_1}F^2\left(\frac{1-\rho_1}{2}, \frac{3+\rho_1}{2}, \frac{3}{2}; 1-\alpha^2\right)\right.$$
$$\left.+\frac{(2+\rho_1)^2}{9(1+\rho_1)}F^2\left(\frac{1-\rho_1}{2}, \frac{3+\rho_1}{2}, \frac{5}{2}; 1-\alpha^2\right)\left(1-\frac{2}{3}\alpha^2\right)\right\}. \tag{19}$$

For light elements we may set $\alpha = 0$ and obtain

$$\sigma \sim \frac{88\pi a_0^2\alpha_0^8 Z^5}{3\epsilon}. \tag{20}$$

This formula is radically different from that we should obtain by an extrapolation of the experimental law for softer x-rays, or the earlier non-relativistic calculations. These agree in giving for σ, more or less roughly, it is true,

$$\sigma s = 10^{-2}Z^4\lambda^3 \tag{21}$$

whereas we find

$$\sigma \sim 1.9 \times 10^{-22}Z^5\lambda. \tag{22}$$

For the gamma-rays of ThC″ our result gives a much larger absorption than the extrapolated value (21).[23a] In contrast with the extrapolated formula, (22) gives a photoelectric absorption which falls off almost as slowly with decreasing wave-length as the scattering from a free electron,

$$(\sigma)_{KN} \sim 10^{-15}\lambda \log \epsilon.$$

This result and (18) and (19) are what we must compare with experiment. For sufficiently hard radiation we may use (19) with complete confidence; for the gamma-rays of ThC″, (18) and (19) give appreciably different results, and we shall use the strict result (18).

[23a] One may understand this result physically; the reason for the rapid decrease in σ with $\lambda(\lambda^3)$ in the theory of absorption of ordinary x-rays is that the nucleus has to take up more and more momentum as the frequency of the quantum increases. For very hard light the electrons which are ejected in the direction of the γ-ray beam acquire a momentum which differs from that of the quantum by a constant amount: the momentum which the nucleus has to take up does not increase indefinitely with the energy of the quantum. For this reason we should expect the true value of σ to lie, for hard radiation, higher than the value extrapolated from (21).

Comparison with experiment.

The formulas just given apply only to the K-electrons of a naked nucleus. In using them to obtain the total photoelectric absorption from an atom we shall apply the customary procedure, which may be justified only qualitatively, of supposing each electron in the atom to move in the field of an appropriately screened nucleus. We take the screening constants directly from the spectroscopic values given by Ruark and Urey.[24] There appears to be no more satisfactory treatment of the problem which does not involve a complicated study of the wave functions for all the electrons. It is in this connection of extreme importance to note that the discrepancy between theory and experiment which we find for the absorption of the gamma-rays of ThC'' by Pb, persists even if we neglect entirely the absorption of all but the two K-electrons.

We take therefore the total photoelectric cross section of the atom

$$\sigma_A = \sum_i n_i \sigma(Z - s_i) \tag{23}$$

where $\sigma(Z\text{-}s)$ is the absorption given by (18) for an electron bound to a nucleus of charge $e(Z-s)$, where further n_i is the number of electrons in the atom with screening constant s_i, and where the summation is to be taken over all the electrons of the atom.

The simplest way to compare σ_A with experiment is to find its ratio to the cross section for scattering given by the Klein-Nishina formula. We therefore compute the ratio R *per electron* of the photoelectric cross section to the Klein-Nishina[25] cross section for scattering, (σ) K.N. For ThC'' gamma-rays, this ratio is

$$R = \frac{(\sigma_A/Z)}{(\sigma_E)_{KN}}, \quad \text{for } \lambda = 4.7\ X.U. \tag{24}$$

Chao[26] and Tarrant[27] have observed the total absorption of these gamma-rays of ThC'' in a number of elements, and agree in finding it greater than the value predicted by the Klein-Nishina formula. From their data the quantity R may be directly found. There are of course several possible explanations of the excess absorption which Chao and Tarrant find, an excess absorption which is more marked for the heavier elements. On the other hand the binding of the electrons will surely modify the scattering; it will also give rise to a photoelectric absorption; further, the electrons of the nucleus may absorb or scatter appreciably such hard radiation. (The increase, and according to Chao it is a regular increase, with Z would on this view be hard to understand.) It is here our purpose to discover what part of the effect observed by Chao and Tarrant can be ascribed to photoelectric absorption.

[24] Ruark and Urey, Atoms, Molecules and Quanta, (McGraw-Hill), p. 259.
[25] Reference 7, or see reference 5.
[26] Reference 5.
[27] Reference 5.

Carrying out this comparison, we find from (18), (23), and (24) for Cu

$$R - 1 = 0.08$$

Chao finds for Cu 0.11, and Tarrant 0.12. A set of typical results is given in Table II.

TABLE II.

Z	Obs. Chao	Obs. Tarrant	$R-1$
13	0.06	0.12	0.002
29	.11	.12	.08
30	.11	.006	.08
50	.21	.21	1.3
82	.42	.34	14.

The agreement of this computed value for the photoelectric absorption in Cu with the excess absorption over that given by the Klein-Nishina formula experimentally found by Chao and Tarrant, is in some ways satisfactory, and suggests that the experimental absorption can, even for this very hard radiation, be explained by the photoelectric absorption and scattering of extranuclear electrons. There are, however, very serious difficulties with this interpretation which arise from the following circumstance. Our computed photoelectric absorption increases very much more rapidly (Z^5) with atomic number than that found by Tarrant and Chao. Chao finds roughly a linear dependence on Z; Tarrant, whose values do not agree at all well with those of Chao, finds no regular dependence on Z at all for the excess absorption. (This is what we should expect if the absorption were primarily nuclear.) Our theoretical values for the photoelectric effect give therefore, a smaller a correction for the light elements (e.g. Al) than is needed to explain the empirical values; whereas for heavier elements, like Pb, the theoretical correction is over 25 times too large. It must be emphasized that the empirical values are by no means certain; only in the case of lead, where both Tarrant and Chao find a value very much smaller than the theoretical, can we be sure of a definite discrepancy between theory and experiment.

An examination of the error introduced into our calculations by a partial neglect of the binding of the photoelectrons has convinced us that correction for this error could hardly bring our results into accord with experiment. Only in the case of lead could this correction be appreciable, and even there it should not change our result by more than a factor of three. We believe that a precise calculation based on the strict formula, Eq. (9), would also lead to results that could not be reconciled with the experiments. They would fail to agree with Chao's measurements because of too fast an increase with atomic number; and they would disagree with Tarrant's not only in this, but in giving a quite regular dependence on Z. For if the dependence on Z is really as irregular as that found by Tarrant, the effects can hardly be of extra-nuclear origin, and the photoelectric effect computed by us should be only a small part of the excess absorption observed; whereas for large Z it is in fact greater. We believe, therefore, that the application of the present electrodynamical theory to the absorption of very hard light does not give correct results, There are of course grounds for distrusting the theory in this application, because the energy of the light involved is many times the proper energy of the electron. It is because we seem to have here another breakdown of present electrodynamical theory that we have insisted so much on this comparison with experimental results by no means unambiguous. It is to be hoped that a more complete and rigorous solution of the theoretical problem will be found in the future, and it is very much to be desired that the experiments be repeated and improved.

Appendix

Part I.

To the first order in α^2

$$\delta = \frac{\alpha^2}{2}\left(1 + \frac{1}{k}\right)$$

$$\delta' = \frac{\alpha^2}{2}\left(1 - \frac{1}{k}\right).$$

Since the error involved in this procedure will be of interest to us only for the case $k_0 = 0$, we shall justify the approximation only for this case. It is clear, however, that a similar process will be possible in other restricted parts of the range of k_0.

From Eqs. (12) one sees that the largest contribution to σ comes from the first term of I_α for $k = 2$. For $k = 2$, $F(\delta', 2\rho, \beta(1 - \bar{s}u))$ of Eq. (10′) converges to within 0.007 of its value in two terms ($\alpha < 1/2$); this value is practically independent of $s \sim i\alpha$. Therefore, in a heuristic way we take this factor outside the integral. Now expanding the other factor of interest,

$$(1 - \bar{s}u)^{-\delta} = 1 + \delta\bar{s}u + \frac{\delta(1+\delta)}{2}(\bar{s}u)^2 + \cdots$$

we integrate term by term. In the resulting series

$$B(k_1k)F(k_12k, \beta\bar{s}) + \delta B(k+1, k)F(k+1, 2k+1, \beta\bar{s}) + \cdots$$

the ratio of the first two terms gives u average. We find $|\bar{u}|$ slightly greater than 1/2 (the imaginary part of u average can be neglected), but for our purposes putting average $u = 1/2$ in the factors that have been removed is convenient, and should entail an error of less than 1/2 percent, since the series for the second factor converges to within 0.01 of its value in three terms for $|u| = 3/4$, and the same upper limit on α.

In the remaining integrals the approximation is not always such a good one in the integral itself. Fortunately, however, when in place of the factors above we have the factors $F(\delta' \pm 1)$, or $(1 - \bar{s}u)^{\pm 1 - \delta}$, or similar ones with 2 in place of 1, then, either the approximation just happens to be right to a fair accuracy, or else the integral being considered belongs to a term of σ which is already small because of a factor $(\kappa/b)^2$, or $(\alpha/2)^2$. For example: the other large terms in σ, by Eqs. (12), come from $I_\beta(+2)$, $I_\beta(-1)$. The largest term in $I_\beta(+2)$ comes from $S(\mu-2, k-1, a, 2\rho)$. The $F(\delta')$ factor in this integral is again the same as the one we have just discussed, but the other factor is $(1-\bar{s}u)^{2-\delta}$. The series obtained by expanding this expression converges in four terms to 0.15 of its value, if $|u|$ average is near 1/2. Again integrating term by term, and finding the necessary ratios of the confluent functions in the succeeding terms, we find that putting $|\bar{u}|=\bar{u}^2=\cdots=1/2$ introduces an error <0.02 in this integral.

The remaining large term arises from $I_\beta(-1)$. $S(\mu, k, a+1, 2\rho+2)$ and $S(\mu-1, k, a, 2\rho)$ contribute to this term. The question involved in the first of these is the same as in the term just discussed. The latter involves $(1-\bar{s}u)^{1-\delta}$, and $F(-1, 2\rho, \beta(1-\bar{s}u))$, since $\delta'=0$ for $k=1$. In this case it is better to expand both of these factors under the sign of integration, and obtain

$$c_0+c_1u+c_2u^2+\cdots.$$

Then in a similar way one finds $\bar{u}$, $\bar{u}^2$, $\cdots$. The F in this case vanishes as $\alpha\to 0$.

Now in the contribution to σ, $I_\beta^2(+2)$ has roughly the weight 1/6, and $I_\beta^2(-1)$ the weight 1/3. On squaring we must double the estimated errors. We find that for $k_0=0$, $\alpha<1/2$ the error in σ should be less than 3 percent. The error for small Z should be almost entirely negligible—in general all of our errors vanish with α.

Part II.

The formula for continuation gives in this case,

$$F\left(\frac{\rho_1+3}{2},\ \frac{1-\rho_1}{2},\ \frac{5}{2},\ \frac{p^2}{p^2+b^2}\right)$$
$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{2-\rho_1}{2}\right)\Gamma\left(\frac{4+\rho_1}{2}\right)}F\left(\frac{\rho_1+3}{2},\ \frac{1-\rho_1}{2},\ \frac{1}{2},\ \frac{b^2}{p^2+b^2}\right)$$
$$-2\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{\rho_1+3}{2}\right)\Gamma\left(\frac{1-\rho_1}{2}\right)}\left(\frac{b^2}{p^2+b^2}\right)^{1/2}F\left(\frac{2-\rho_1}{2},\ \frac{4+\rho_1}{2},\ \frac{3}{2},\ \frac{b^2}{p^2+b^2}\right).$$

The difference of this F from unity is seen to be negligible except for the heaviest elements (e.g. Pb), even when $\boldsymbol{k}_0$ and $\boldsymbol{\kappa}$ are parallel. It is to be noted, however, that the correction is in the direction to decrease F, but the decrease in F^2 is not more than 8 percent for lead.

The continuation of $F(\rho_1+3/2, 1-\rho_1/2, 3/2; p^2/p^2+b^2)$ shows that again in this case F changes slowly, and *exceeds* the value unity about 10 percent for Pb when $\boldsymbol{k}_0$ and $\boldsymbol{\kappa}$ are parallel.