THE MOTION OF A DIRAC ELECTRON IN A MAGNETIC FIELD*

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Abstract

Solutions of the Dirac wave equation representing a free electron moving in a uniform magnetic field are obtained. The functions are similar to those obtained by Landau and by Uhlenbeck and Young as solutions of the Schrödinger equation. A wave packet is constructed representing a beam of electrons passing through a slit. The results agree with the classical predictions to terms of the order of the de Broglie wave-length of the electron divided by the radius of curvature of its classical path. For experimental cases this ratio is of the order 10^{-8} to 10^{-10}. Hence it is concluded that the difference between magnetic deflection measurements of e/m and other determinations cannot be explained as a quantum effect.

A NUMBER of papers[1,2,3,4] have recently appeared treating the motion on an electron in a magnetic field on the basis of quantum mechanics. The purpose of these investigations was to see if the difference in the values of the specific charge of the electron[5] obtained by deflection and by spectroscopic experiments could be explained as a quantum effect. Recent experimental work on free electrons by Perry and Chaffee[6] and by Kirchner[7] give values for the ratio e/m very close to the spectroscopic values. However, neither of these experiments involved deflections in a magnetic field so that a quantum mechanical effect might still be present in the magnetic measurements. In fact, Kirchner suggests that Page's investigation might explain the difference between his own results and the older ones of Wolf.[8]

Page[1] obtains solutions of the Schrödinger equation representing a free electron in a magnetic field. He shows that the mean radius of the electron's path for each of these solutions is less than the classical radius given by $r = mvc/eH$, except that for one solution his mean radius is equal to the classical. He concludes that if a beam of electrons passing through a slit is represented by a combination of his solutions the average radius of curvature of the paths of the electrons will be less than that calculated by the classical formula and that the difference is of the right order of magnitude to explain the observed

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[1] L. Page, Phys. Rev. **36**, 444 (1930).
[2] G. E. Uhlenbeck and L. A. Young, Phys. Rev. **36**, 1721 (1930).
[3] M. S. Plesset, Phys. Rev. **36**, 1728 (1930).
[4] E. H. Kennard, Proc. Nat. Acad. **17**, 58 (1931).
[5] R. T. Birge, Phys. Rev. Sup. **1**, 47 (1929).
[6] Charlotte T. Perry and E. L. Chaffee, Phys. Rev. **36**, 904 (1930).
[7] F. Kirchner, Phys. Zeits. **31**, 1073 (1930).
[8] F. Wolf, Ann. d. Physik **83**, 849 (1927).

discrepancy in e/m. However, he does not show that a finite beam of electrons can be represented by such a combination of his solutions and in particular he ignores a whole set of solutions of this equation obtained by letting his quantum number m take on negative values. For these solutions the mean radius is *greater* than the classical radius. Hence, his work cannot be considered conclusive. Plesset[3] used a second order relativistic wave equation and carried out calculations similar to those of Page. The above remarks apply to his work as well.

Uhlenbeck and Young[2] used a different form of solution of the Schrödinger equation, which was first given by Landau.[9] They calculated the distance which a beam of electrons incident normally would penetrate into a magnetic field and obtained the classical result.

Kennard[4] showed that in any electromagnetic field the center of gravity of a wave packet obeying the Schrödinger equation would move according to classical laws. From this he concluded that the classical expression could be used whenever an experiment consisted in measuring a mean position of a large number of electrons. He was not able to extend his results to the Dirac wave equation.

In the present work the Dirac equation for the electron is used. Solutions for a homogeneous magnetic field are obtained which are analagous to the solutions used by Uhlenbeck and Young. From these solutions a wave packet is constructed which represents a beam of electrons passing through a slit into a magnetic field. The motion of this packet is studied.

Solutions in the Magnetic Field

We shall use the linear Hamiltonian for the electron in the form given in Dirac's quantum mechanics.[10]

$$\{W/c + (e/c)A_0 + \rho_1[\boldsymbol{\sigma}\cdot(\boldsymbol{p} + (e/c)\boldsymbol{A})] + \rho_3 mc\}\psi = 0. \tag{1}$$

For a uniform magnetic field H in the z-direction we can write

$$A_0 = 0,\ A_z = 0;\ A_x = -\tfrac{1}{2}Hy;\ A_y = \tfrac{1}{2}H_x.$$

Putting $w = eH/2c$, Eq. (1) becomes

$$\{W/c + \rho_1[\sigma_x(p_x - wy) + \sigma_y(p_y + wx) + \sigma_z p_z] + \rho_3 mc\}\psi = 0.$$

We shall find solutions of this equation which are much like those used by Uhlenbeck and Young.[2,9] To do this put

$$\psi = e^{i(w/k)xy}e^{i(\eta/k)y}\phi(x). \tag{2}$$

k is Planck's constant divided by 2π. Assuming that ψ is independent of z.

The equation for ϕ is

$$\left\{\frac{W}{c} + \rho_1[\sigma_x p_x + \sigma_y(2wx + \eta)] + \rho_3 mc\right\}\phi = 0. \tag{3}$$

[9] L. Landau, Zeits. f. Physik **64**, 629 (1930).

[10] P. A. M. Dirac, Quantum Mechanics, Oxford University Press (1930).

We see that the solution for any value of η can be obtained from that for $\eta = 0$ by replacing x by $(x+\eta/2w)$. That is

$$\phi_\eta(x) = \phi_{n=0}\left(x + \frac{\eta}{2w}\right).$$

Writing out the component equations for the case $\eta = 0$

$$\begin{aligned}
\left(\frac{W}{c} + mc\right)\phi_1 + \frac{k}{i}\frac{d\phi_4}{dx} - i2wx\phi_4 &= 0 \\
\left(\frac{W}{c} + mc\right)\phi_2 + \frac{k}{i}\frac{d\phi_3}{dx} + i2wx\phi_3 &= 0 \\
\left(\frac{W}{c} - mc\right)\phi_3 + \frac{k}{i}\frac{\partial\phi_2}{dx} - i2wx\phi_2 &= 0 \\
\left(\frac{W}{c} - mc\right)\phi_4 + \frac{k}{i}\frac{d\phi_1}{dx} + i2wx\phi_1 &= 0.
\end{aligned} \tag{4}$$

Eliminating ϕ_1, between the first and last of the above gives

$$\frac{d^2\phi_4}{dx^2} + \frac{1}{k^2}\left((W^2/c^2) - m^2c^2 + 2wk - 4w^2x^2\right)\phi_4 = 0.$$

Or putting $\xi = 2x(w/k)^{1/2}$ and

$$\nu = \frac{1}{4wk}(W^2/c^2 - m^2c^2) = \frac{1}{4wk}p^2$$

where p is the total momentum, we obtain

$$\begin{aligned}
\frac{d^2\phi_4}{d\xi^2} + (\nu + \tfrac{1}{2} - \tfrac{1}{4}\xi^2)\phi_4 &= 0 \\
\frac{d^2\phi_1}{d\xi^2} + (\nu - \tfrac{1}{2} - \tfrac{1}{4}\xi^2)\phi_1 &= 0.
\end{aligned} \tag{5}$$

We recognize the first of these as the equation for D_ν given in Whittaker and Watson, Modern Analysis.[11] The second is the equation for $D_{\nu-1}$. Comparing the first of Eqs. (4) with the second recurrence formula for the D_ν's given in Whittaker and Watson

$$D_\nu'(\xi) + \tfrac{1}{2}\xi D_\nu(\xi) - \nu D_{\nu-1}(\xi) = 0$$

we find that if

$$\phi_4(x) = aD_\nu(\xi) \tag{6}$$

[11] Whittaker and Watson, Modern Analysis , Cambridge University Press, Fourth Edition (1927).

then

$$\phi_1(x) = i\frac{((W/c) - mc)}{2(kw)^{1/2}} aD_{\nu-1}(\xi).$$

It can be shown in a similar manner that

$$\begin{aligned} \phi_2 &= iC((W/c) - mc)\phi_4 \\ \phi_3 &= iC((W/c) + mc)\phi_1. \end{aligned} \tag{7}$$

The condition that the z- component of the current be zero at $x=0$ makes C real, and if C is real $S_z=0$ for any value of x.

We can write the solutions

$$\begin{aligned} \psi_1 &= e^{i(w/k)xy}e^{i(\eta/k)y}\phi_1\left(x+\frac{\eta}{2w}\right) \\ \psi_2 &= e^{i(w/k)xy}e^{i(\eta/k)y}iC\left(\frac{W}{c} - mc\right)\phi_4\left(x+\frac{\eta}{2w}\right) \\ \psi_3 &= e^{i(w/k)xy}e^{i(\eta/k)y}iC\left(\frac{W}{c} + mc\right)\phi_1\left(x+\frac{\eta}{2w}\right) \\ \psi_4 &= e^{i(w/k)xy}e^{i(\eta/k)y}\phi_4\left(x+\frac{\eta}{2w}\right) \end{aligned} \tag{8}$$

where the ϕ's are given by (6). We might point out that if $\lambda = h/p$ is the de Broglie wave-length and $r = cp/eH$ is the radius of the classical circle then

$$\pi r = \nu\lambda.$$

Hence ν is the number of de Broglie wave-lengths in a classical half circle. Uhlenbeck and Young found that $\pi r = (\nu+\frac{1}{2})\lambda$ when Schrödinger's equation is used instead of Dirac's.

The Solutions in Free Space

Put $p^2 = p_1^2 + p_2^2 = W^2/c^2 - m^2c^2$ where p_1 and p_2 are the momenta in the x and y-directions. The monentum in the z-direction is considered zero. Then for field free space we expect solutions of the form $e^{i/k(p_1x+p_2y)}$. The following set of solutions was found:

$$\begin{aligned} \psi_1 &= \frac{p_1 - ip_2}{(W/c) + mc} A_4 e^{i/k(p_1x+p_2y)} \\ \psi_2 &= A_2 e^{i/k(p_1x+p_2y)} \\ \psi_3 &= \frac{p_1 - ip_2}{(W/c) - mc} A_2 e^{i/k(p_1x+p_2y)} \\ \psi_4 &= A_4 e^{i/k(p_1x+p_2y)} \end{aligned} \tag{9}$$

where A_2 and A_4 are arbitrary constants.

If we consider the functions (9) as representing a beam moving in the $+x$ and $+y$-directions, we can find the representation of a similar beam moving in the $-x$ and $+y$-directions by using different constants B_2 and B_4 and by replacing p_1 by $-p_1$.

Matching Solutions

Let us suppose that we have a uniform magnetic field of strength H in the z-direction for all positive values of x, and that for x negative the field is zero. We wish to match solutions of the type (8) with those of type (9) for $x=0$. We see first that $p_2=\eta$ in order that the solutions be equal for all values of y. We suppose that in the free space there is both an incident beam (p_1 positive) and an emergent beam (p_1 negative). Setting the sum of these solutions equal to the functions in the field with $x=0$ gives

$$\begin{aligned}
&\frac{-p_1(A_4-B_4)+ip_2(A_4+B_4)}{(W/c)+mc} = \phi_1(p_2/2w)\\
&A_2+B_2 = iC((W/c)-mc)\phi_4(p_2/2w)\\
&\frac{-p_1(A_2-B_2)+ip_2(A_2+B_2)}{(W/c)-mc} = iC((W/c)+mc)\phi_1(p_2/2w)\\
&A_4+B_4 = \phi_4(p_2/2w).
\end{aligned} \tag{10}$$

Solving for the B's in terms of the A's gives

$$\begin{aligned} B_4 &= \gamma A_4\\ B_2 &= \gamma A_2 \end{aligned} \tag{11}$$

where

$$\gamma = \frac{1-ip_2/p_1+\dfrac{(W/c)+mc}{p_1}\dfrac{\phi_1(p_2/2w)}{\phi_4(p_2/2w)}}{1+ip_2/p_1-\dfrac{(W/c)+mc}{p_1}\dfrac{\phi_1(p_2/2w)}{\phi_4(p_2/2w)}}. \tag{12}$$

Since the ratio ϕ_1/ϕ_4 is pure imaginary from Eqs. (6) we see that γ is a number divided by its complex conjugate and hence $|\gamma|=1$. Also since D is even or odd according as ν is even or odd, one of the functions ϕ_1, ϕ_4 is odd, the other even. Hence, their ratio is odd. Hence, if we change the sign of p_2, γ becomes $1/\gamma$. That is

$$\gamma(p_1,-p_2) = \frac{1}{\gamma(p_1,p_2)} = \gamma^*(p_1,p_2). \tag{13}$$

Having obtained these solutions we will use them in several ways. We will first consider an infinite beam incident normally and later will construct a wave packet.

Distribution of Current in the Field for an Infinite Incident Beam

Let us find the current densities inside the field for the case $p_2=0$. From Dirac:

$$-S_x = \psi^*\alpha_x\psi = \psi_1^*\psi_4 + \psi_2^*\psi_3 + \psi_3^*\psi_2 + \psi_4^*\psi_1$$
$$= (1 + C^2p^2)(\phi_1^*\phi_4 + \phi_4^*\phi_1) = 0.$$

Since the product $\phi_4^*\phi_1$ is pure imaginary from (6) and hence $\phi_1^*\phi_4 = -\phi_4^*\phi_1$. Similarly, $-S_y = \psi^*\alpha_y\psi = -i(1+C^2p^2)(\phi_4^*\phi_1 - \phi_1^*\phi_4)$.

If we use our expressions for the &'s in terms of the D's we have

$$S_y = (1 + C^2p^2)\frac{(W/c) - mc}{(kw)^{1/2}}aa^*D_\nu D_{\nu-1}. \tag{14}$$

Since $D_{\nu-1}$ and D_ν are successive solutions of the Weber equation (5), they have a different number of zeros between $x=0$ and $x=0$. (There are no zeros for $x>r$.) Therefore, S_y is negative for some values of x. In fact, the distance between successive regions of negative S_y is of the order of λ, the de Broglie wave-length. This is apparently an effect of spin since Uhlenback and Young found an expression for S_y which is always positive.

Let us find the average x-coordinate of the current, defined by

$$\bar{x} = \frac{\int_0^\infty xS_ydx}{\int_0^\infty S_ydx}.$$

Classically $S_y = 2I/(r^2-x^2)^{1/2}$ and $\bar{x} = \pi v/4$.

In evaluating $\bar{x}$ using the quantum mechanical expressions we shall need the integrals

$$\int_0^\infty D_\nu D_{\nu-1}d\xi \quad \text{and} \quad \int_0^\infty D_\nu D_{\nu-1}\xi d\xi.$$

We shall evaluate these integrals for ν even. The methods are much the same for ν odd and the final value of $\bar{x}$ is exactly the same.

By multiplying the first of Eqs. (5) by $D_{\nu-1}$, the second by D_ν and subtracting and then integrating we find

$$\int_0^\infty D_\nu D_{\nu-1}d\xi = [D_\nu(0)]^2 = \left(\frac{\Gamma(\frac{1}{2})2^{\nu/2}}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu)}\right)^2. \tag{15}$$

Using the recurrence formulae we find

$$\int_0^\infty \xi D_\nu D_{\nu-1}d\xi = \frac{1}{2}\int_0^\infty [D_\nu]^2d\xi = \frac{1}{2}(2\pi)^{1/2}\nu!. \tag{16}$$

Putting into our expression for $\bar{x}$

$$\bar{x} = \tfrac{1}{4}\left(\frac{k}{w}\right)^{1/2} \frac{\left[\left(\frac{\nu}{2}\right)!\right]^2}{\nu!} 2\nu(2\pi)^{1/2}.$$

By using Stirling's formula

$$n! = (2\pi n)^{1/2}\left(\frac{n}{e}\right)^n$$

we obtain

$$\bar{x} = \frac{\pi}{4} r$$

the classical value. Hence the result agrees with the classical value to the extent that Stirling's formula holds. The error in Stirling's formula is of the order $1/\nu$ and since ν is of the order 10^8 to 10^{10} this deviation is entirely negligible. The same results is obtained if we use Uhlenbeck and Young's solutions of the Schrödinger equation.

By using a Wentzel-Brillouin-Kramers[2,12] approximation near $x=r$ (See Uhlenbeck and Young) ϕ_4 and ϕ_1 can be expressed in the form:

$$\omega\left\{2\nu^{-2/3}\left(1 \pm \frac{1}{2\nu}\right)^{2/3}\left(1 - \frac{x}{r(1 \pm \frac{1}{2}\nu)^{1/2}}\right)\right\}. \tag{17}$$

The plus sign being used for ϕ_4 and the minus for ϕ_1 we find that the last maximum of S_y is between

$$x = r(1 + \tfrac{1}{2}\nu^{-2/3} + \tfrac{1}{4}\nu^{-1}) \quad \text{(maximum of } \phi_4)$$

and

$$x = r(1 + \tfrac{1}{2}\nu^{-2/3} - \tfrac{1}{4}\nu^{-1}) \quad \text{(maximum of } \phi_1)$$

and the current will fall to 0.001 times its maximum value in going a distance of the order $3/2r(\nu)^{-2/3}$.

We have found that if an experiment consists of measuring the *average* x coordinate of the current the difference between classical and quantum mechanical results will be of the order of 1 part in ν while if the *maximum* x coordinate is used the difference will be of the order 1 part in $\nu^{2/3}$. In either case it is too small to observe.

Two Incident Beams

We shall combine two incident beams such as those found above Eqs. (9), with momenta p_1, p_2 and p_1-p_2. The constants A_2 and A_4 will be the same for both beams. This is the first step in the construction of a wave packet. It will be simpler to make a wave packet from these solutions than from the original solutions (9).

We find for the combined beams

$$\psi_{i1} = \psi_1(p_1, p_2) + \psi(p_1, -p_2) = \frac{-2A_4 e^{ip_1x/k}}{\frac{W}{c} + mc}\left\{p_1 \cos\frac{p_2 y}{k} + p_2 \sin\frac{p_2 y}{k}\right\}.$$

Put $\tan\epsilon = p_2/p_1$. Then

$$\begin{aligned}
\psi_{i1} &= \frac{-2A_4 p e^{ip_1x/k}}{\frac{W}{c} + mc}\cos\left(\frac{p_2 y}{k} - \epsilon\right)\\
\psi_{i2} &= 2A_2 e^{ip_1x/k}\cos\frac{p_2 y}{k}\\
\psi_{i3} &= \frac{-2A_2 p e^{ip_1x/k}}{(W/c) - mc}\cos\left(\frac{p_2 y}{k} - \epsilon\right)\\
\psi_{i4} &= 2A_4 e^{ip_1x/k}\cos\frac{p_2 y}{k}\cdot
\end{aligned} \tag{18}$$

Now let us find the functions representing the emergent beams. Put $\gamma = e^{i\delta}$. Then

$$B_4(p_1, p_2) = \gamma A_4 = A_4 e^{i\delta}$$
$$B_4(p_1, -p_2) = \gamma A_4 = Ae^{-i\delta}$$

and similar expressions for B_2. We obtain

$$\begin{aligned}
\psi_{e1} &= \frac{2A_4 p}{(W/c) + mc}\, e^{-ip_1x/k}\cos\left(\frac{p_2 y}{k} + \delta + \epsilon\right)\\
\psi_{e2} &= 2A_2 e^{-ip_1x/k}\cos\left(\frac{p_2 y}{k} + \delta\right)\\
\psi_{e3} &= \frac{2A_2 p}{(W/c) - mc}\, e^{-ip_1x/k}\cos\left(\frac{p_2 y}{k} + \delta + \epsilon\right)\\
\psi_{e4} &= 2A_4 e^{-ip_1x/k}\cos\left(\frac{p_2 y}{k} + \delta\right).
\end{aligned} \tag{19}$$

For p_2 small compared to p_1 the emergent beam is displaced along the y-axis a distance $k\delta/p_2$. Hence, we wish to find δ.

Using Eqs. (6) we find

$$\frac{(W/c) + mc}{p_1}\,\frac{\phi_1(p_2/2w)}{\phi_4(p_2/2w)} = i(\nu)^{1/2}p/p_1\;\frac{D_{\nu-1}[(p_2/p)2\nu^{1/2}]}{D_\nu[(p_2/p)2\nu^{1/2}]} \tag{20}$$

To evaluate δ we must make some approximations. Two independent approximations are involved. We assume that p_2 is small compared to p_1, and the ν is large compared to 1. Let us consider the differential Eq. (5) for the D_ν. If ν is large compared to 1 and ξ is small compared to $2(\nu)^{1/2}$ we see that an approximate solution can be obtained in the form

$$D = a \cos((\nu)^{1/2}\xi + \alpha).$$

Let us again consider the case ν even, then D_ν is an even function and $D_{\nu-1}$ an odd one and we can write

$$\begin{aligned} D_\nu &= a \cos(\nu)^{1/2}\xi \\ D_{-1} &= b \sin(\nu)^{1/2}\xi. \end{aligned} \tag{21}$$

Using Whittaker and Watson's expressions for D_ν we can show that

$$\lim_{\xi\to 0} \frac{D_{\nu-1}(\xi)}{\xi D_\nu(\xi)} = -1.$$

Hence for small ξ

$$\frac{(\nu)^{1/2}D_{\nu-1}(\xi)}{D_\nu(\xi)} = -\tan(\nu)^{1/2}\xi. \tag{22}$$

A better approximation can be obtained by using the Wentzel-Brillouin-Kramer's method[2,12]

$$D_{\nu-1}(\xi) = \frac{-a}{\left[1 - \dfrac{\xi^2}{2(2\nu-1)}\right]^{1/4}} \sin \tfrac{1}{4}\left\{\xi(2(2\nu-1) - \xi^2)^{1/2} + 2(2\nu-1)\sin^{-1}\frac{\xi}{2(\nu-\frac{1}{2}))^{1/2}}\right\}$$

and

$$D_\nu(\xi) = \frac{a}{\left[1 - \dfrac{\xi^2}{2(2\nu+1)}\right]^{1/4}} \cos \tfrac{1}{4}\left\{\xi(2(2\nu+1) - \xi^2)^{1/2} + 2(2\nu+1)\sin^{-1}\frac{\xi}{2(\nu+\frac{1}{2})^{1/2}}\right\}.$$

These hold for considerably larger values of ξ than do Eqs. (21). If we assume ν is large compared to 1, but do not restrict ξ we obtain

$$\frac{(\nu)^{1/2}D_{\nu-1}(\xi)}{D_\nu(\xi)} = -\tan \tfrac{1}{4}\left\{\xi(4\nu - \xi^2)^{1/2} + 4\nu \sin^{-1}\frac{\xi}{2(\nu)^{1/2}}\right\}. \tag{23}$$

Put this expression equal to $-\tan\sigma$ when $\xi = (p_2/p)2(\nu)^{1/2}$. We see that if we expand the argument in terms of ξ and neglect terms of higher order than the 1st, Eq. (23) reduces to (22). This expansion will be considered later.

[12] G. Wentzel, Zeits. f. Physik **38**, 518 (1926); L. Brillouin, C. R. Juli, (1926); H. A. Kramers, Zeits. f. Physik **39**, 828 (1926).

Using Eq. (23) we obtain

$$e^{i\delta} = \frac{1 - i(\tan\epsilon + \sec\epsilon\tan\sigma)}{1 + i(\tan\epsilon + \sec\epsilon\tan\sigma)}\cdot$$

Hence

$$-\tan\frac{\delta}{2} = \tan\epsilon + \sec\epsilon\tan\sigma = \frac{\sin\epsilon\cos\sigma + \sin\sigma}{\cos\epsilon\cos\sigma}\cdot$$

Now $\tan\epsilon = p_2/p_1$ and if p_2 is very small compared to p_1, ϵ is a very small angle. If we neglect ϵ altogether we obtain too small a value for $-\tan\delta/2$ since we decrease the numerator and increase the denominator of the fraction. On the other hand, if we *decrease* the denominator by subtracting $\sin\epsilon\sin\sigma$ and decrease the numerator only by a second order term in by multiplying the last term by $\cos\epsilon$ we obtain

$$-\tan\frac{\delta}{2} = \frac{\sin\epsilon\cos\sigma + \cos\epsilon\sin\sigma}{\cos\epsilon\cos\sigma - \sin\epsilon\sin\sigma} = \tan(\sigma + \epsilon).$$

Neglecting ϵ altogether gives

$$-\tan\frac{\delta}{2} = \tan\sigma.$$

Hence, we can say

$$\sigma < -\frac{\delta}{2} < \sigma + \epsilon. \tag{24}$$

All the above arguments hold for $\cos\sigma$ negative if we interchange the words decrease and increase.

Now let us evaluate σ.

$$\sigma = \tfrac{1}{4}\left(\xi(4\nu - \xi^2)^{1/2} + 4\nu\sin^{-1}\frac{\xi}{2(\nu)^{1/2}}\right).$$

Expanding in powers of $\xi/2(\nu)^{1/2}$ gives

$$\sigma = 2\nu\left\{\frac{\xi}{2(\nu)^{1/2}} - \frac{1}{6}\left(\frac{\xi}{2(\nu)^{1/2}}\right)^3 + \cdots\right\}.$$

Putting $\xi = (p_2/p)2(\nu)^{1/2}$ gives

$$\sigma = 2\nu\frac{p_2}{p}\left\{1 - \frac{1}{6}\left(\frac{p_2}{p}\right)^2 + \cdots\right\}. \tag{25}$$

If p_2 is so small compared to p that the square of p_2/p may be neglected we have

$$\sigma = 2\nu\frac{p_2}{p}\cdot$$

Now $\sin \epsilon = p_2/p$. Hence ϵ is small compared to σ in the same way that 1 is small compared to ν. Since we have already neglected terms of the order $1/\nu$ we shall neglect ϵ compared to σ. This gives

$$\delta = -2\sigma = -4\nu p_2/p. \tag{26}$$

Neglecting the ϵ in expressions (18) and (19) we see that the wave functions are the same for the incident and emergent beams except for (1) the negative signs on ψ_1 and ψ_3 and in the exponent which make the currents be in opposite directions and (2) the phase angle δ in the cosines. Hence, if y_i is the maximum of the incident beam and y_e the maximum of the emergent beam then

$$y_e - y_i = -\frac{k}{p_2}\delta.$$

Putting in the value of from (26) gives

$$y_e - y_i = \frac{4\nu k}{p} = \frac{2\nu\lambda}{\pi} = 2r. \tag{27}$$

We have neglected terms of the order $1/\nu$ hence this result may be in error by a term of order λ. Hence to within distances of the order of a de Broglie wave-length the maximum of the emergent beam will be displaced just twice the radius of the classical circle from the maximum of the incident beam.

Construction of a Wave Packet

The functions we have been considering had a cosine dependence on y and hence extending an infinite distance in both directions. However, by using a Fourier integral over such solutions we can find functions which are negligibly small except in the region: $-\Delta y \leqq y \leqq \Delta y$. These will be of the form:

$$\Psi_{i1} = \int_0^{p_2} \frac{-2A_2(p_2)p}{(W/c)+mc} e^{ipx/k} \cos\frac{p_2 y}{k}\, dp_2 \tag{28}$$

Where P_2 is the maximum value of p_2,

With similar expressions for Ψ_{i2}, Ψ_{i3}, Ψ_{i4}. A_4 is considered a function of p_2. Since each component here will give an emergent beam of the form (17) we can write the functions for the total emergent beam

$$\Psi_{e1} = \int_0^{p_2} \frac{2A_4(p_2)p}{(W/c)+mc} e^{-ip_1x/k} \cos\left(\frac{p_2 y}{k} + \delta\right) dp_2. \tag{29}$$

We have shown that for p_2 sufficiently small δ/p_2 is independent of p_2. Hence the incident beam is reproduced at a distance $y_e - y_i = -k\delta/p_2$ above the point of incidence. This means that if we pass electrons through a slit into a magnetic field they will come out at a distance $2r$ away, the uncertainty being of the order of a de Broglie wave-length.

The functions in the field will be of the form

$$\Psi_1 = e^{iwxy/k} \int_{p2}^{p_2} e^{i\eta y/k} \phi_1\left(x + \frac{\eta}{2w}\right) d\eta \tag{30}$$

and similar expressions for Ψ_2, Ψ_3, Ψ_4, where the A's and ϕ's are connected by the relations (10).

We can now dispense with the device of a field ending abruptly at $x=0$ since the functions (30) will represent a packet of the same form even though the field extends beyond the slits at $x=0$ and hence the functions will have the same form at this point. The use of such a discontinuous field is merely a convenient way of studying the solutions at $x=0$.

We must now examine how small we can make Δy. We want P_2/p to be small compared to unity. At the same time we can conclude either from the theory of Fourier integrals or from Heisenberg's uncertainty relation that:

$$\Delta y \frac{P_2}{k} \approx 1.$$

Hence $\Delta y p/k$ must be large compared to one or Δy is large compared to λ.

$$\Delta y \gg \frac{k}{p} \approx \lambda.$$

This is usually true since a slit 0.1 mm wide would be a very narrow one while λ is of the order of 1 Å. Hence $\Delta y/\lambda \approx 10^6$. This means also that the uncertainties introduced by the approximations used will be small compared to the uncertainties coming from a finite slit width. Hence, when applied to the motion of an electron in a magnetic field quantum mechanics will give the same results as classical for the value of the ratio e/m.

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