NATURAL UNITS FOR ATOMIC PROBLEMS

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Abstract

This paper deals principally with two systems of units, designated A and B. In both, the velocity of light and the rest mass of the electron are taken as units. In A, the length unit is $h^2/4\pi^2 m_0 e^2$, which is the radius of the first Bohr hydrogen orbit, in B, it is $h/2\pi m_0 c$, which is $(2\pi)^{-1}$ times the Compton shift in scattering at 90°. In each system, the choice of the above units determines the unit of charge through the equation for the Coulomb force, $f=\epsilon^2/r^2$. In system A, the electronic charge is numerically equal to the fine structure constant, α; in system B, the square of the electronic charge is numerically equal to α, and h is equal to 2π. It is shown that many formulas of quantum theory are greatly simplified when written in terms of system B; the coefficients of the Schrödinger and Dirac equations contain only small integers and the electronic charge. The use of these units provides a consistent scheme for simplifying the algebra associated with problems of quantum theory.

§1

We are all familiar with numerous cases in which the mathematical treatment of physical problems is simplified by a suitable choice of units. In the computation of atomic energy levels Hylleraas[1] and others have found it very useful to employ a unit of length of the same order of magnitude as the radius of the first Bohr orbit. Sometimes also, the energy required to ionize the hydrogen atom from its lowest state is taken as the unit of energy. However, when the relativistic wave equations are used, the self-energy of the electron is a more convenient unit. This circumstance led the writer to consider the whole question of natural units for use in problems of atomic and nuclear physics. It soon appeared that rather striking simplifications of the wave-mechanical equations can be achieved by proper choice of units. The determination of the most convenient unit systems was not straightforward, but required considerable searching. The writer's conclusions are presented in this brief paper, which is devoted to methodology, and contains no new physical result.

One of the best-known attempts to simplify the equations of quantum theory and related subjects is that of Planck,[2] who chose new units of length, time, mass and temperature, of such sizes that the velocity of light, the quantum of action, the universal constant of gravitation, and the gas constant are all numerically equal to one. The rounded values of the units of length, time and mass are as follows:

[1] Hylleraas, Zeits. f. Physik **65**, 209 (1930), and earlier papers.
[2] Planck, Heat Radiation, English translation by Masius, p. 175.

Unit of length: 4×10^{-33} centimeter
Unit of time: 1.3×10^{-43} second
Unit of mass: 5.4×10^{-5} gram.

We leave the gas constant out of consideration, since it may always be made unity by properly choosing the unit of temperature.

Obviously, the difficulty with this system is that the new units are not convenient in size for use in atomic problems. The same criticism may be directed against many similar attempts. The writer tried, therefore, to obtain systems of units which fulfill, as far as possible, the following requirements: The fundamental units of length, time and mass must be of such magnitude that atomic lengths, times, and masses are expressible by fairly small numbers, as long as we deal with small quantum numbers; and the coefficients in the principal equations of atomic physics should be simplified when the new units are used. Two of the systems examined will be discussed in detail.

§2

In setting up a system of natural units almost every step consists in abandoning some artificial relation associated with the c.g.s. system. We try to proceed as a Maxwell demon probably would if he were entrusted with the construction of a natural and universal scheme of units. In this section we concern ourselves with two systems, designated A and B. In both, the velocity of light, c, is taken as the unit of velocity. In extranuclear problems, the units of length adopted are as follows:

A: $h^2/4\pi^2 m_0 e^2$, the radius of the first hydrogen orbit if the motion of the nucleus is neglected.

B: $h/2\pi m_0 c$, the Compton wave-length change in scattering at 90°, divided by 2π.

The corresponding units of time are fixed by the choices already made. They are,

A: $h^2/4\pi^2 m_0 e^2 c$.

B: $h/2\pi m_0 c^2$.

In both systems we take the rest mass of the electron as the mass unit. At first sight it might seem better to assign the value unity to e, the electronic charge, but this does not simplify the more important equations of quantum theory as much as the choice we have made.[3]

It will often be convenient to prime a quantity measured in c.g.s. units, the unprimed symbol being reserved for the same quantity measured in the new units. Thus $h' = 6.547 \times 10^{-27}$ erg sec., the conventional value of Planck's constant, while it happens that in system B the numerical value of h is 2π.

[3] The essential point is, we cannot make both the elementary charge and the electronic mass *simultaneously* equal to unity, unless we are willing to modify the classical equation for the force between two charges by introducing a constant analogous to the constant of gravitation. The expression $f = \epsilon^2/r^2$, giving the force between two equal charges, ϵ, serves to define the unit of charge. If the numerical values of f and r are unity, then ϵ is the unit of charge. But it is readily verified that the force between two electrons at unit distance is not one in either of the schemes considered here, so the electronic charge cannot be one in these systems.

The unit of action in B is $m_0' \cdot c' \cdot (h'/2\pi m_0'c') = h'/2\pi = (1.042 \pm .0013) \times 10^{-27}$ erg second. This is the quantum of action adopted by Dirac, Weyl and others, in preference to Planck's constant. The use of the symbol h for this quantity has led to so much confusion that a new one seems desirable. Perhaps h_0 would be an acceptable designation. The desire to employ $h'/2\pi$ as the unit of action constituted the writer's reason for choosing $h'/2\pi m_0'c'$ as the new unit of length in system B. Anyone who wishes to employ Planck's constant as the unit of action may do so by taking $h'/m_0'c'$ as the length unit; but then the factor 2π, or some power of it, will appear in many equations.

In Table I we give the c.g.s. numerical values of the new units of certain important physical quantities. These are rounded, since it is not necessary to

TABLE I. *Approximate c.g.s. values of natural units.*

Physical quantity	c.g.s value of unit in system A	c.g.s value of unit in system B
Length	$h'^2/4\pi^2 m_0'e'^2 = 0.5 \times 10^{-8}$	$h'/2\pi m_0'c' = 3.9 \times 10^{-11}$
Time	$h'^2/4\pi^2 m_0'e'^2 c' = 0.17 \times 10^{-18}$	$h'/2\pi m_0'c'^2 = 1.3 \times 10^{-21}$
Velocity	$c' = 3 \times 10^{10}$	$c' = 3 \times 10^{10}$
Mass	$m_0' = 9 \times 10^{-28}$	$m_0' = 9 \times 10^{-28}$
Momentum	$m_0'c' = 2.7 \times 10^{-17}$	$m_0'c' = 2.7 \times 10^{-17}$
Energy	$m_0'c'^2 = 8.1 \times 10^{-7}$	$m_0'c'^2 = 8.1 \times 10^{-7}$
Action	$\alpha h'/2\pi = 1.4 \times 10^{-25}$	$h'/2\pi = 1.0 \times 10^{-27}$
Charge	$e'/\alpha = 6.5 \times 10^{-8}$	$e'/\alpha^{1/2} = 5.6 \times 10^{-9}$

use them in recasting the equations of quantum theory. In the table, α represents the fine structure constant, which is dimensionless. According to Birge,[4] we have

$$\alpha = \frac{2\pi e^2}{hc} = (7.283 \pm 0.006) \times 10^{-3} \tag{1}$$

and

$$\frac{1}{\alpha} = \frac{hc}{2\pi e^2} = 137.29 \pm 0.11. \tag{2}$$

The numerical value of the unit of charge in any system may readily be found with the aid of these relations. In system B, for example, $c=1$, and $2\pi/h=1$. Therefore,

$$e^2 = \alpha. \quad (\textit{system B only}) \tag{3}$$

and the new unit of charge measured in c.g.s. electrostatic units must be $e'/\alpha^{1/2} = 4.770 \times 10^{-10}/8.533 \times 10^{-2} = 5.59 \times 10^{-9}$ e.s.u. Similarly,

$$e = \alpha. \quad (\textit{system A only}). \tag{4}$$

It is reasonable to suppose that in nuclear problems it will prove convenient to use units exactly like those of Table I, except that the mass of the proton, or possibly that of the alpha-particle, will replace the electronic mass in the definitions of the units. Attention may be directed to the quantity e^2/m_0c^2, which has the dimensions of length (it is equal to $\alpha h/2\pi m_0 c$) and is of the same order of magnitude as the classical radius of the Lorentz electron.

[4] Birge, Phys. Rev. Supplement **1**, 1 (1929).

Systems involving the so-called gravitational radius of the electron have also been considered by the writer, but will not be discussed here.

§3

It is now possible to state the relative advantages of the proposed systems. Atomic diameters are of the order unity in system A, and frequencies associated with the outer electron shells are of the same order as the corresponding c.g.s. wave numbers. It appears that the units of system A are well adapted in size to the discussion of the outer electrons and of wave-lengths in the optical region, while system B is more useful in dealing numerically with fast electrons and x-ray phenomena. However, the chief benefits to be derived from the use of these units are associated with the facts that algebra is often simplified, and that many atomic constants have small numerical values. For example, in both systems the numerical measures of a mass and of the corresponding energy are identical, since $c = 1$. The self energy of the electron is unity. Wave number and frequency are numerically identical and further the electrostatic and electromagnetic units of any quantity are the same. The expression for the Lorentz force per unit charge is $E + [vH]$ and many other electrodynamic formulae are less complex than in c.g.s. units.

It is interesting to note the simple forms assumed by certain equations of atomic theory when the system B is employed. All quantities belonging to the electron have values expressible in terms of integers and of the electronic charge, provided the customary expressions for the electronic angular momentum and magnetic moment are rigorously correct. Thus, the angular momentum is numerically $1/2$, and the magnetic moment $e/2$. Of all electronic properties, then, only the charge requires experimental determination, from the standpoint of both our new unit systems.[5] In both of them, its numeric is connected in a simple way with the fine structure constant, as we see from Eqs. (1) to (4).

Further, in system B the energy, mass and momentum of a photon are all given numerically by $2\pi\nu$, and exponential factors in wave functions are simplified. Thus, numerically $e^{-2\pi iEt/h}$ becomes e^{-iEt}. Such examples could be indefinitely multiplied.

We now illustrate the application of system B to several important equations. Heisenberg's commutation relation is

$$pq - qp = \frac{h1}{2\pi i} \cdot \tag{5}$$

But $h = 2\pi$, and (5) may be rewritten as

$$pq - qp = -i1. \quad (\textit{system B only}). \tag{5a}$$

[5] Of course it is not implied that the determination of electronic constants other than the charge can be avoided in connection with *practical* problems. For example, in order to express a length of the order of a few centimeters in terms of the new unit of length, that unit would first have to be found in terms of some familiar unit, such as the centimeter. The new systems are designed to reduce labor for the theoretical and not the experimental physicist.

In this form, the physical units of the right member are submerged, but the algebra of any matrix problem is simplified, and the "lost" constants can easily be reinserted in the final result. It is not necessary to go through laborious transformations in obtaining (5a) or other simplified equations. For example, in the new units as in the old, Schrödinger's equation has the form

$$\Delta\psi + \frac{8\pi^2}{h^2}(E - V)\psi = 0; \tag{6}$$

but in system B, $4\pi^2/h^2 = 1$, numerically, and (6) becomes,

$$\Delta\psi + 2(E - V)\psi = 0. \quad (\textit{system } B \textit{ only}). \tag{6a}$$

The relativistic Schrödinger equation and the Dirac equations are also simplified. To summarize, when we use system B, the general equations of quantum mechanics have rational coefficients, except for the occurrence of the electronic charge, which is numerically equal to $\alpha^{1/2}$.

Eddington[6] has brought forward a theory involving space of sixteen dimensions, which indicates that $1/\alpha$ may be exactly 137, that is, $1+1/2$ (17.16). If this were substantiated, all coefficients in the wave equation would be rational. However, Birge's value of $1/\alpha$, cited in Eq. (2), disagrees with Eddington's 137 by more than twice the probable error. Other interpretations of α have been suggested by Lewis and Adams,[7] by Perles,[8] and by Fürth;[9] but such matters, important as they may be, lie outside our present scope.

In conclusion, it may be worth while to note that the use of the natural units discussed focuses attention on certain unsolved problems of quantum theory. The only empirical constants which occur when we apply the Schrödinger Eq. (6a) to the problem of the hydrogen atom are α and m/M. The former enters through the fact that $V = -Ze^2/r = -Z\alpha/r$, in accordance with Eq. (3). No doubt the value of α is connected with the internal mechanism of the electron, as we can see by examining classical electronic models, while it appears probable that m/M will eventually be interpreted as the ratio of two eigenwerte of a still-to-be-discovered wave equation. Pending the elucidation of these matters, the use of our system B reduces the wave equation to very simple terms.

[6] Eddington, Proc. Roy. Soc. **A126**, 696 (1930) and earlier papers.
[7] Lewis and Adams, Phys. Rev. **3**, 92 (1914).
[8] Perles, Naturwiss. **16**, 1094 (1928).
[9] Fürth, Zeits. f. Physik **57**, 429 (1929).