

DIFFUSION OF ELECTRONS BACK TO AN EMITTING ELECTRODE IN A GAS

BY IRVING LANGMUIR
GENERAL ELECTRIC COMPANY, SCHENECTADY, N. Y.

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ABSTRACT

An expression is derived for the current i between two electrodes, one of which emits (with uniform current density I_0) electrons with an initial velocity, V_0 , when the electrodes are placed in a gas at such a pressure that the electrons suffer only elastic collisions. If V is the voltage between the electrodes, λ the mean free path of an electron, the current is given by

$$i = (16\pi/3)I_0\lambda C\phi$$

where C is the electrostatic capacitance between the electrodes and

$$\phi = (V/V_0)\ln(1 + V/V_0).$$

If the emitted electrons have a Maxwellian velocity distribution, this equation is applicable with slightly modified values of ϕ , V_0 now being replaced by $T/11600$ volts.

IN SOME recent investigations¹ it was observed that the electron current from a negatively charged electron-emitting electrode in neon at 1 mm pressure did not reach a saturation value, but continued to increase with increasing applied potential, although the currents were so small that space charge effects could be neglected. The conditions of the experiment were such that the electrons, which were probably emitted with a uniform initial velocity, suffered elastic collisions with gas atoms, and the mean free path was so short that the energy gained from the field between collisions was small compared to the initial velocity of electrons. It was thought that under these conditions, electrons leaving the electrode would, as a result of the elastic collisions, diffuse back to the cathode. The following paper contains the derivation of an expression for the current from an electron-emitting electrode as a function of voltage under these conditions.

The motion of slow electrons moving in a gas under the influence of an electric field has been considered theoretically by Hertz² for conditions similar to those stated above except that the electrons were assumed to have zero initial velocity. In these investigations he obtained the following expression for the drift velocity of an electron

$$v_x = \frac{1}{3} \frac{\gamma\lambda}{v} - \frac{v\lambda}{3} \frac{1}{n} \frac{dn}{dx} \quad (1)$$

where v is the velocity of an electron along its path; v_x the drift velocity of an electron in direction of the field; $\gamma = (e/m)(dV/dx)$ is the acceleration of an electron in the direction of the electric field; and λ is the electronic mean free

¹ See a recent letter to the editor. I. Langmuir and C. G. Found, Phys. Rev. **36**, 604 (1930).

² Hertz, Zeits. f. Physik **32**, 278 (1925).

path. In this equation the 2nd term in the right member represents the effect of diffusion, the diffusion coefficient D being

$$D = \left(\frac{1}{3}\right)v\lambda.$$

If the electron velocities were everywhere uniform throughout the gas as has often previously been assumed in considering the mobility of electrons, the effect of the field would be to cause a drift with velocity $v_x = \gamma\lambda/v$.

The 1st term in the right member of Eq. (1), however, is only 1/3 as great as this expression for the mobility. This is due to the fact that in Hertz's derivation the electrons were assumed to make only elastic collisions with gas molecules and thus their velocities increase as they move into regions of more positive potential. There is thus a *thermal effusion* effect by which the electrons tend to move back against the field. The result is that the net drift is only 1/3 as great as if no increase of electron velocity occurred.

Assuming that the electrons start from the electrode (0) with a uniform velocity corresponding to V_0 volts, and that they suffer only elastic collisions with gas atoms, their velocity at any point is given by

$$\left(\frac{1}{2}\right)mv^2 = (V + V_0)e \quad (2)$$

where V is the potential at the point relative to that of the emitting electrode.

Differentiating, we obtain

$$v dv = \frac{e}{m} \frac{dV}{dx} dx = \gamma dx$$

and the concentration gradient is thus

$$\frac{dn}{dx} = \frac{\gamma}{v} \frac{dn}{dv}. \quad (3)$$

Combining Eqs. (1) and (3), the drift velocity is

$$v_x = \frac{\gamma\lambda}{3} \left(\frac{1}{v} - \frac{1}{n} \frac{dn}{dv} \right) \quad (4)$$

It is not, in general, necessary that the maximum concentration gradient should lie in a direction coinciding with that of the maximum potential gradient. For the sake of simplicity, however, let us consider those cases in which at each point these two directions do coincide. The equipotential surfaces are then also surfaces of equal concentration n , and n is thus a single-valued function of V . We shall also restrict ourselves to cases in which the currents are so small that space charge effects are negligible, i.e., the potential distribution is given by integration of Laplace's equation.

Consider now a tube of force of small cross-section reaching from the emitting electrode (0) to a second electrode (1) which is at potential V_1 . Let A be the cross-sectional area of the tube at any distance s measured along the tube and let L be the number of lines of force ($\int E_s dA$) contained in the tube.

Then

$$dV/ds = L/A. \quad (5)$$

The acceleration is then

$$\gamma = \frac{e}{m} \cdot \frac{L}{A}. \quad (6)$$

The drift current i_A along the tube of force is

$$i_A = Anev_s. \quad (7)$$

Therefore, combining Eqs. (6) and (7)

$$\frac{v_s}{\gamma} = \frac{i_A m}{ne^2 L}. \quad (8)$$

Substituting in Eq. (4) we obtain

$$\frac{3i_A m}{e^2 \lambda L} = \frac{n}{V} - \frac{dn}{dv}. \quad (9)$$

Assuming that the free path λ is the same throughout the length of the tube of force, then since i_A is also constant Eq. (9) may be integrated giving

$$n = \frac{3i_A m v}{e^2 \lambda L} \ln \left(\frac{c}{v} \right) \quad (10)$$

where c is a constant of integration.

To determine the integration constant we can insert the boundary condition that the concentration at the electrode 1 (anode) is zero.

Putting $n=0$ for $v=v_1$ we find $c=v_1$ and Eq. (10) becomes

$$n = \frac{3i_A m v}{e^2 L \lambda} \ln \left(\frac{v_1}{v} \right). \quad (11)$$

At any point in the gas the random current density³ is

$$I_e = \left(\frac{1}{4} \right) nev. \quad (12)$$

Inserting the value of n from Eq. (11) and expressing v in terms of V in accord with Eq. (2) we obtain

$$I_e = \frac{3}{4} \frac{i_A (V + V_0)}{L \lambda} \ln \left(\frac{V_1 + V_0}{V + V_0} \right). \quad (13)$$

In Eq. (9), i_A and L are constant along the tube of force. We have, however, postulated that the equipotential surfaces lying between the two electrodes are identical with the surfaces of equal concentration and therefore n , v and dn/dv are constant over any equipotential surface. We see then by Eq. (9) that the ratio i_A/L , which is constant along any tube, must also have the

³ See I. Langmuir and K. T. Compton, Rev. Mod. Phys. 3, 221 (1931).

same value for all tubes in the field and thus must be equal to i/L_t where i is the total current between the electrodes and L_t is the total number of lines of force passing between the electrodes.

If all the lines of force which emanate from one electrode pass to the other, i.e., if the algebraic sum of the charges of the two electrodes is zero, then

$$L_t = 4\pi V_1 C \tag{14}$$

where C is the electrostatic capacitance between the electrodes in cm.

Replacing i_A and L in Eq. (13) by i and L_t we thus obtain the result

$$I_e = \frac{3}{16\pi} \cdot \frac{i}{\lambda C} \cdot \left(\frac{V + V_0}{V_1} \right) \ln \left(\frac{V_1 + V_0}{V + V_0} \right). \tag{15}$$

Let us denote by I_0 the random current density at the surface of the cathode as calculated from this equation, by placing $V=0$. In order that the presence of the electrode shall be compatible with the conditions that we have assumed in the neighboring gas it is clearly necessary that the electrode should emit electrons (of uniform velocity corresponding to V_0 volts) with a *uniform current density* I_0 which thus corresponds to the saturation current density. The electrons must also be emitted in random directions, i.e., according to Lambert's cosine law.

If we have an emitting electrode which fulfils these conditions, we may thus calculate the current i that passes to the anode:

$$i = \frac{16\pi}{3} I_0 \lambda C \phi \tag{16}$$

where the function ϕ is defined by

$$\phi = \frac{V_1/V_0}{\ln \left(\frac{V_0 + V_1}{V_0} \right)} \tag{17}$$

Values of ϕ for various values of V_1/V_0 are given in Table I.

TABLE I. Currents produced by electrons emitted with uniform velocity. ϕ is defined by Eqs. (16) and (17).

V_1/V_0	ϕ	V_1/V_0	ϕ	V_1/V_0	ϕ
-1.0	0	0.1	1.049	3	2.165
-0.99	0.206	0.2	1.097	4	2.480
-0.9	0.390	0.3	1.140	5	2.79
-0.8	0.496	0.4	1.189	6	3.08
-0.6	0.655	0.6	1.276	8	3.64
-0.4	0.784	0.8	1.360	10	4.16
-0.2	0.895	1.0	1.445	15	5.42
-0.1	0.949	1.5	1.637	20	6.58
0	1.000	2.0	1.820	30	8.75

As the anode voltage is raised the current increases in proportion to ϕ . Since the electrons are emitted with velocities corresponding to V_0 volts, the

anode current falls to zero only when a *retarding* voltage of V_0 volts is applied. With accelerating voltages the current increases slowly but continuously.

If both electrodes emit electrons with velocity V_0 and if the potential of one with respect to the other lies between $-V_0$ and $+V_0$, the current must flow both ways so that the net current is to be obtained by applying Eq. (16) for each electrode in turn and taking the difference of these two currents (or algebraic sum). When the voltage lies outside of the range $-V_0$ to $+V_0$, one or the other of the currents becomes zero so that a single application of Eq. (16) suffices.

It should be kept in mind that Eq. (16) is strictly applicable only at such high pressures that the electron free path is small compared to the distance between the electrodes. Otherwise the boundary conditions which we have used, such as $n=0$ at the anode, are not valid.

We have seen that it was necessary for us to assume that I_0 is uniform over the cathode surface. If this condition is not fulfilled the surfaces of equal concentration no longer coincide with the equipotential surfaces and the problem of the current flow is then far more complicated than for the case we have treated.

It is also clear for the same reason that the foregoing method is not applicable when there are *two anodes* at different potentials. This difficulty is apparent when we consider that according to our procedure n must be placed equal to zero at each of these surfaces, although they are at different potentials. This violates our postulate that n is a single valued function of V . Two anodes at the same potential may, of course, be regarded as a single anode and Eqs. (16) and (17) are then applicable.

Practical applications of Eq. (16) will usually be made with electrodes of simple shapes. Substituting the well-known expressions for the capacitance⁴ in Eq. (16) we obtain:

For *parallel planes*, the net current per unit area flowing between the electrodes is

$$I_x = (4/3)I_0\phi\lambda/a. \quad (18)$$

The current that flows is thus inversely proportional to the pressure and at high pressures may be very small compared to the saturation electron emission I_0 .

For *coaxial cylinders* the net current per unit length is

$$i_L = (8\pi/3)I_0\phi\lambda/\ln(r_1/r_0). \quad (19)$$

Here I_0 is the current density of the electron emission corresponding to saturation from the *emitting electrode* regardless of whether this is the outer or the inner cylinder. The radii of the cylinders are r_1 and r_0 .

For *concentric spheres* the total current is

$$i = (16\pi/3)I_0\phi\lambda r_0 r_1 / (r_1 - r_0). \quad (20)$$

⁴ See p. 198 of reference 3.

It should be noted that if the *saturation current density* I_0 from the inner and the outer cylindrical (or spherical) electrodes are equal, the actual currents remain unchanged if the relative polarity of the electrodes is reversed, although the total saturation currents, if they could be obtained, would be proportional to the areas of the emitting electrodes. This conclusion can readily be tested by experiment.

The source of the electrons from the emitting electrode may be one of many kinds, for example, photoelectric or thermionic. A case of particular interest in connection with a study of the un-ionized (or weakly ionized) regions beyond the end of a neon arc¹ is the liberation of high velocity electrons (*ca.* 10 volts) from the surfaces of electrodes by the action of metastable atoms which diffuse to these electrodes. The metastable atoms are generated throughout the gas by the action of ultraviolet light (broadened resonance lines) from the arc, at a rate proportional to the inverse cube of the distance from the end of the arc. Let us calculate the relative current densities I_0 of emitted electrons from the inner and outer cylinders under these conditions.

If the electrodes are at a considerable distance from the end of the arc and the radiation passes parallel to the axis, then within the space between two cylindrical electrodes of short lengths we may assume a uniform rate of production of metastable atoms. The diffusion of these to the two electrodes is a problem essentially similar to that⁵ of the potential distribution between cylinders if the space contains a uniform space charge ρ .

Imposing the boundary conditions $V=0$ at the surface of both cylinders, we find that the potential gradient at any point of radius r is

$$\frac{dV}{dr} = \pi\rho \left[\frac{(r_2^2 - r_1^2)}{r \ln(r_2/r_1)} - 2r \right] \quad (21)$$

where r_2 and r_1 are the radii of the outer and inner cylinders respectively. By substituting first $r=r_1$ and then $r=r_2$ we find the potential gradients at the surfaces of the two electrodes. The ratio of these two gradients is equal to the ratio of the corresponding rates of arrival of metastable atoms *per unit area* and therefore should be proportional to the ratio I_1/I_2 of the saturation current densities (I_0) of the emitted electrons from the two electrodes. In this way we find

$$\frac{I_1}{I_2} = \frac{r_2[r_2^2 - r_1^2 - 2r_1^2 \ln(r_2/r_1)]}{r_1[r_2^2 - r_1^2 - 2r_2^2 \ln(r_2/r_1)]} \quad (22)$$

For example, if $r_2/r_1=10$ this gives $I_1/I_2 = -2.62$ (the negative sign merely indicates that the currents flow in opposite directions).

The corresponding equation for concentric spheres is

$$\frac{I_1}{I_2} = -\frac{r_2(r_2 + 2r_1)}{r_1(r_1 + 2r_2)} \quad (23)$$

which gives $I_1/I_2 = -5.72$ when $r_2/r_1=10$.

⁵ See p. 213 of reference 3.

Thus we see that the current density I_0 at the inner electrode is greater than at the outer electrode if the electrons owe their origin to metastable atoms produced uniformly throughout the gas. It therefore follows from Eq. (19) that the net current will be greater (2.62 fold for $r_2/r_1=10$) when the small inner cylindrical electrode is cathode than when the larger electrode is cathode, although if the currents were saturated the current in the former case would be smaller instead of larger (0.26 instead of 2.6 fold).

We have thus far assumed that all the electrons are emitted with the same velocity V_0 volts. Although this may be roughly true for electrons emitted under the influence of metastable atoms, it will certainly not be a satisfactory approximation for thermal emission of electrons. By summation of the currents from each velocity group, we may calculate the net current produced by a Maxwellian velocity distribution.

The total number of electrons per unit volume which have speeds (irrespective of direction) lying between c and $c+dc$ is given by⁶

$$4\pi n(m/2\pi kT)^{3/2} \exp(-mc^2/2kT)c^2dc.$$

Because of the relation expressed by Eq. (12) it follows that the contribution of these electrons to the random current density crossing any given plane is

$$dI = \pi en(m/2\pi kT)^{3/2} \exp(-mc^2/2kT)c^3dc. \quad (24)$$

Applying this to the surface of an electrode which emits electrons with a Maxwellian velocity distribution corresponding to temperature T and current density I_0 , and integrating from $c=0$ to $c=\infty$ we find

$$I_0 = ne(kT/2\pi m)^{1/2} \quad (25)$$

Eliminating n between Eqs. (24) and (25) we obtain

$$dI = I_0 e^{-\eta_0} d\eta_0 \quad (26)$$

where

$$\eta_0 = mc^2/2kT = V_0 e/kT = 11600V_0/T \quad (27)$$

if V_0 is the emission velocity in volts.

To calculate the net current between the two electrodes, we may insert the value of dI in place of I_0 in Eq. (16) and integrate from $\eta_0=0$ to $\eta_0=\infty$.

In this integration the value of ϕ is obtained from Eq. (17) by placing

$$V_1/V_0 = \eta/\eta_0$$

which by Eq. (27) is equivalent to

$$\eta = V_1 e/kT = 11600V_1/T \quad (28)$$

if V_1 is in volts.

⁶ This is obtained from Eq. (71) of Compton and Langmuir's article, *Rev. Mod. Phys.* **2**, 205 (1930), by replacing h by $1/2kT$.

In this way we find

$$i = (16\pi/3)I_0\lambda c\phi_T \tag{29}$$

where

$$\text{or } \left. \begin{aligned} \phi_T &= \eta \int_0^\infty \frac{\epsilon^{-x} dx}{\ln(1 + \eta/x)} \text{ if } \eta > 0 \\ \phi_T &= \eta \int_{-\eta}^\infty \frac{\epsilon^{-x} dx}{\ln(1 + \eta/x)} \text{ if } \eta < 0 \end{aligned} \right\} \tag{30}$$

The reason that the lower limit of integration must be changed when η is negative is that electrons whose initial velocities give $\eta_0 < -\eta$ cannot reach the collector.

TABLE II. Currents produced by electrons emitted with Maxwellian velocities at temperature T ; ϕ_T is defined by Eqs. (29) and (30); η is given by Eq. (28).

η	ϕ_T	η	ϕ_T	η	ϕ_T
-4	0.042	-0.2	0.892	4	2.358
-3	0.100	0	1.000	6	2.908
-2	0.236	+0.1	1.034	8	3.421
-1	0.516	0.5	1.217	10	3.913
-0.8	0.600	1	1.408	20	5.895
-0.6	0.688	2	1.748	30	8.170
-0.4	0.787	3	2.051	50	11.892

The values of η given in Table II have been calculated from Eqs. (30) by Simpson's rule or by series expansion. They are believed to be accurate within one or two units in the last figure. Comparison with Table I shows that for values of η and V_1/V_0 which are equal, the values of ϕ_T and ϕ are very nearly equal when η is greater than about -0.6 . By Eq. (28) we see that this means that, except for large negative voltages, no great error is made if the thermally emitted electrons are considered to have a uniform energy corresponding to $T/11600$ volts.

Experiments which will probably be described in a paper by C. G. Found and the writer have given results which are explicable in terms of the theories here presented. The writer is much indebted to Mr. Found for assistance in the preparation of this paper.