A NOTE ON LOCAL MOMENTUM IN WAVE MECHANICS

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Abstract

The present paper shows the exact relations between dynamical quantities of classical theory and their wave mechanical analogues. A new quantity *local momentum* is defined and discussed and the Bohr quantum condition is revived in a form compatible with modern theory.

1. THE one-dimensional Schrödinger equation for a particle moving in a force-free field is

$$\psi'' + p^2\psi = 0 \tag{1}$$

where p and p^2 are quantities proportional to the momentum and energy of the particle respectively.[1] This equation possesses the particular solutions

$$\begin{aligned} \psi^+ &= A^+e^{+ipx} \\ \psi^- &= A^-e^{-ipx} \end{aligned} \tag{2}$$

representing the streaming of particles in the positive and negative directions of x. The constants A^+ and A^- are arbitrary and are used to normalize the intensity of the stream of particles to any desired quantity. The corresponding wave equation for a general scalar potential field $V(x)$ is

$$\psi'' + [E - V(x)]\psi = 0. \tag{3}$$

We may assume particular solutions of this equation of the form[2]

$$\psi^\pm = A^\pm(x)e^{\pm i\int P dx} \tag{4}$$

and adopt the terminology *local amplitude* for $A^\pm(x)$ and *local momentum* for $P(x)$. Both $A^\pm(x)$ and $P(x)$ may be taken as real quantities (since any complex number may be written in the form $ae^{i\theta}$ with a and θ real). Substitution of (4) into (3) leads, upon separately equating real and imaginary terms to zero, to the following differential equations for $A^\pm(x)$ and $P(x)$.

$$A^{\pm\prime\prime} - A^\pm P^2 + [E - V(x)]A^\pm = 0 \tag{5a}$$

$$2A^{\pm\prime}P + P'A^\pm = 0. \tag{5b}$$

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[1] Throughout this note units are chosen in such a manner that all equations may be written in their simplest form. It is extremely easy to rewrite all results in the usual system of units.

[2] This substitution is well known in mathematics but its use in wave mechanics has been extremely limited.

The second of these gives an immediate relation between $A^{\pm}$ and P; namely

$$A^{\pm} = P^{-1/2} \tag{6}$$

Eq. (5a) may then be written in terms of $P(x)$ alone

$$P^2 + V - P^{1/2}\frac{d^2}{dx^2}P^{-1/2} = E \tag{7}$$

or if we define the quantity *local kinetic energy* by the relation $T^* = P^2$ we have

$$T^* + V - T^{*1/4}\frac{d^2}{dx^2}T^{*-1/4} = E. \tag{7a}$$

This equation is exact and is the wave mechanical analogue of the classical conservation of energy law.

Now we have as particular solutions of the wave equation corresponding to pure travelling waves in the case of free motion[3]

$$\begin{aligned} \psi^+ &= a^+P^{-1/2}e^{i\int Pdx} \\ \psi^- &= a^-P^{-1/2}e^{-i\int Pdx}. \end{aligned} \tag{8}$$

For the corresponding probability densities we have

$$\begin{aligned} \bar{\psi}^+\psi^+ &= \overline{a^+}a^+/P \\ \bar{\psi}^-\psi^- &= \overline{a^-}a^-/P. \end{aligned} \tag{9}$$

The most general real solution of (3) representing the wave solution for a discrete state of the system may be written

$$\psi = aP^{-1/2}\cos\left[\int_{x0}^{x} Pdx + \text{phase factor}\right]. \tag{10}$$

If the range of x is, for example, from $-\infty$ to ∞ we may show that in order that ψ satisfy the usual boundary conditions we must have

$$\int_{-\infty}^{\infty} Pdx = n\pi \tag{11}$$

where n is a positive integer. Since ψ_n is the n-th characteristic solution of a Sturm-Liouville differential equation it must possess n-1 roots in the finite region of x. This is just the number of roots possessed by $\cos \int Pdx$. It is easy to show that $P(x)$ does not vanish for finite x by a simple study of Eq. (7).

2. The comparison of the preceding equations with classical equations is extremely interesting. Eq. (7a) is to be compared with the classical equation

$$T + V = E$$

in which T is the ordinary kinetic energy. The probability densities (9) become identical with the classical expressions when the *local momentum* is re-

[3] With A and P depending on x either of the Eqs. (8) is capable of describing combinations of progressive and standing waves. As an example write $ae^{ipx}+be^{-ipx}$ in the form $A(x)e^{i\int Pdx}$.

placed by the ordinary momentum. $\int Pdx$ is the wave mechanical action integral and Eq. (11) is the analogue of the most fruitful equation of modern physics—the Bohr-Sommerfeld quantum condition

$$\oint pdx = nh.$$

The local momentum P becomes in the limit (in the sense of the correspondence principle) equal to the classical momentum.

3. As an example the *local momentum* for the two lowest states of the harmonic oscillator is plotted (Fig. 1) as a function of x along with curves of the classical momentum $p(x)$. The scale of x has been so chosen that the $p(x)$ curves which are usually pictured as ellipses are here shown as circles. For all

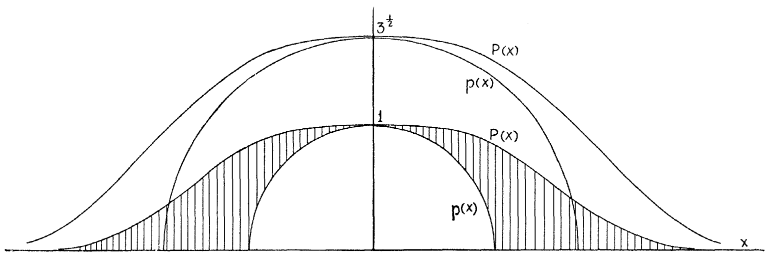

Fig. 1. Comparison of local and classical momentum for the first two states of the harmonic oscillator.

states the area corresponding to the shaded area for the lowest state is $\pi/2$. This alone will show that $P(x)$ and $p(x)$ must approach one another for high quantum states.

4. The present note is not intended to be complete and further investigations are in progress 1*st* on the fundamental nature of P and 2*nd* the application of the results of this paper to various quantum theory problems. In conclusion the writer wishes to express his appreciation to Professor G. E. Uhlenbeck who aroused his interest in the problem of the relation between quantum and classical physics and to Professors J. C. Slater, P. M. Morse and N. H. Frank, of the Massachusetts Institute of Technology, and to M. H. Johnson, Jr., of Harvard University, for their comments and criticism of the manuscript of this note.

Author's Note Added in Proof: Attention should be called to two articles which are related to this note namely H. A. Wilson, Phys. Rev. **35,** p. 948 and W. E. Milne, Phys. Rev. **35,** p. 863. Neither of these writers, it is apparent, realized the rich physical content of their equations.