A THEORETICAL INVESTIGATION OF THE TRANSMISSION OF LIGHT THROUGH FOG

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Abstract

This theoretical investigation was undertaken in an attempt to explain some unexpected experimental results which are described in detail in another paper. The fog droplets are assumed to be dielectric spheres with an index of refraction of 1.33 and the development is based on the work of Mie and Debye on the pressure of light and the colors of colloidal solutions. The results show that the particle size of the fog is a controlling factor in the transmission characteristics of the fog. By assuming the appropriate particle size a theoretical transmission curve is obtained which closely corresponds to the experimental data. In conclusion, it is pointed out that the King formula is not applicable to the scattering of light by particles as large as fog drops.

THIS theoretical investigation of the transmission of light through fog was undertaken as a result of some rather unexpected and interesting experimental results which have been described in another paper. Instead of the gradual increase in transmission from the blue to the red end of the visible spectrum as observed by several experimenters, a definite maximum was found at about 0.490μ. Measurements showed that the particles of the fog used were somewhat smaller than would be expected in natural fog and in the artificial fogs employed by other investigators. It appeared reasonable to suppose that the observed maximum was dependent on the particle size of the fog and that for the larger particle fogs the maxima were beyond the range of the measurements or in regions where other factors are of greater importance. It is the purpose of this paper to obtain a theoretical confirmation of the experimental results with the hope of extending these results to other regions of the spectrum and to fogs having different particle sizes.

Although this particular problem has apparently not been treated heretofore, the general method of attack has been developed by Mie,[1] Debye,[2] Jobst[3] and others in their work on the colors of colloidal solutions and the pressure of light. A general outline of the solution of the problem is given below but for a detailed exposition the reader is referred to the excellent papers of Mie and Debye.

Consider a sphere of a given material suspended in a medium of a second material. The field equations for this case are expressed in spherical coördinates so that the boundary conditions may be more readily introduced. In

[1] G. Mie, Ann. d. Physik **25**, 377 (1908).
[2] P. Debye, Diss. München, 1908, Ann. d. Physik **30, 57** (1909).
[3] G. Jobst, Ann. d. Physik **76**, 863 (1925).

order that the fields may be represented by simple scalar potentials the total field is arbitrarily split into two partial fields which are so defined that the radial magnetic component of the first and the radial electric component of the second are zero. There are then four partial fields, two within the sphere and two outside each of which may be represented by a scalar potential. For each of these potentials a wave equation of the form

$$\Delta^2 p + m^2 p = 0 \tag{1}$$

may be developed where p is one of the partial potentials. This differential equation is solved by replacing p with the product of three functions each dependent on a single variable and thereby obtaining three differential equations. The solutions of these equations are known and are, respectively, a Hankel function, a Legendre polynomial and a circular function and the solution of the original wave equation is then the product of these three functions. The boundary conditions now may be readily introduced by equating the tangential components of the inner and outer fields at the boundary of the two media.

Now consider a plane polarized incident wave travelling in the direction of the negative Z axis with its electric vector along the X axis. To be consistent with the above developments for the secondary fields of the sphere the field of this incident wave is represented by two scalar potentials expressed in spherical coordinates. The total field external to the sphere consists of the field due to the incident wave plus the field of the wave reflected from the sphere.

Up to this point this outline has very closely followed the development as given by Debye[4] and the reader is referred to this paper for details. Debye was interested in the pressure of light and the remainder of his derivation is not directly applicable to our problem. Mie,[5] however, was studying the colors exhibited by colloidal solutions and we may follow him from this point.

The energy loss due to the sphere is given by the surface integral taken over the sphere, of the time average of the Poynting or energy-flow vector of the total field. By collecting the terms involving only the incident wave, those containing only the reflected wave and lastly those terms containing a combination of the two, the integral may be separated into three components. The value of the integral including only the incident wave taken over the surface of the sphere is evidently equal to zero. The second integral represents the portion of the energy which is reradiated by the sphere. The third integral gives the total energy loss including both the absorbed and the reradiated portions. The absolute value of this integral is given by

$$|A| = \frac{\lambda^2}{2\pi N_a^2} Re \sum_{u=1}^{\infty} (-1)^u (2u+1)(C_u{}^1 + C_u{}^2) \tag{2}$$

where

[4] P. Debye, reference 2.

[5] G. Mie, reference 1.

$$C_u^1 = (-1)^u \frac{N_a\psi_u(x)\psi_u'(y) - N_i\psi_u'(x)\psi_u(y)}{N_a\phi_u(x)\psi_u'(y) - N_i\phi_u'(x)\psi_u(y)}$$

$$C_u^2 = (-1)^u \frac{N_i\psi_u(x)\psi_u'(y) - N_a\psi_u'(x)\psi_u(y)}{N_i\phi_u(x)\psi_u'(y) - N_a\phi_u'(x)\psi_u(y)}$$

λ is the wave-length of the incident wave; a is the radius of the sphere; $x=2\pi aN_a/\lambda$; $y=2\pi aN_i/\lambda$; N_a=complex index of refraction of medium outside of sphere; N_i=complex index of refraction of the sphere; $\psi_u(x)=(\pi x/2)^{1/2} J_{u+1/2}(x)$; $\phi_u(x)=(\pi x/2)^{1/2}H^2_{u+1/2}(x)$ and Re=real part of. Primes denote first derivatives with respect to x or y.

It is assumed that the spheres are so far apart that the scattering is incoherent so that if the number of spheres per unit of volume is n we may write for k the coefficient of absorption or the energy loss per unit of volume

$$k = n|A|. \tag{3}$$

For values of x which are large compared to unity Jobst[6] has derived certain approximate expressions for $|A|$ with the aid of Debye's semi-convergent developments of Bessel and Hankel functions.[7] These expressions unfortunately do not apply to the region of principal importance in this investigation and it was necessary to obtain $|A|$ by direct summation of Eq. (2).

For the special case of fog it has been assumed that the fog droplets are dielectric spheres having an index of refraction of 1.33. Hence

$$N_a = 1 \quad N_i = 1.33$$

$$y = 1.33x.$$

Since fog droplets are formed around nuclei of hygroscopic salts, the conductivity is not zero but the inclusion of a finite conductivity would increase the complexity of the expression for $|A|$ to such an extent that numerical evaluation would be practically impossible.

For the purposes of the numerical summation of $|A|$ it is convenient to arrange C_u^1 and C_u^2 as follows

$$C_u^1 = (-1)^u \frac{\psi_u(x)\psi_u'(1.33x) - 1.33\psi_u'(x)\psi_u(1.33x)}{\phi_u(x)\psi_u'(1.33x) - 1.33\phi_u'(x)\psi_u(1.33x)}$$

$$\phi_u(x) = \psi_u(x) + i\chi_u(x)$$

$$\phi_u'(x) = \psi_u'(x) + i\chi_u'(x)$$

where

$$\chi_u(x) = (-1)^u\left(\frac{\pi x}{2}\right)^{1/2} J_{-u-1/2}(x$$

[6] H. Jobst, reference 3.

[7] P. Debye, Math. Ann. 67, 535 (1909).

whence

$$C_u^1 = (-1)^u \frac{B}{B + iD}$$

and

$$ReC_u^1 = (-1)^u \frac{B^2}{B^2 + D^2}$$

where

$$B = \psi_u(x)\psi_u'(1.33x) - 1.33\psi_u'(x)\psi_u(1.33x)$$

and

$$D = \psi_u'(1.33x)\chi_u(x) - 1.33\psi_u(1.33x)\chi_u'(x).$$

An entirely similar expression may be obtained for ReC_u^2. In the processes of summation it was found that available tables of half order Bessel functions were rather limited and it was necessary to compute a number of these functions.

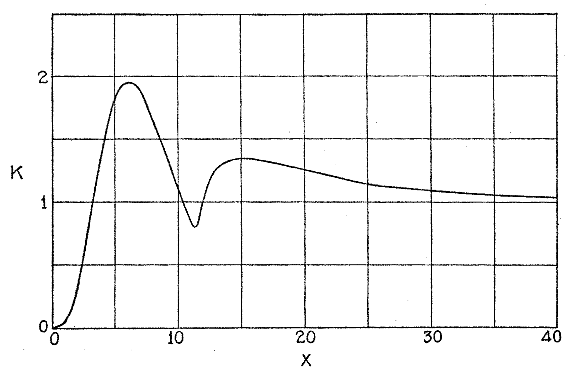

Fig. 1. Curve of K against x, where $x = 2\pi a/\lambda$.

The portion of $|A|$ under the summation sign is a numeric and is a function of x. Instead of multiplying this sum by $\lambda^2/2\pi$ to obtain $|A|$, thereby introducing a new variable, it is more convenient to multiply by $1/x^2$. The resultant expression is a function of x alone and is directly proportional to $|A|$. If this expression is designated by K we have

$$|A| = 2\pi a^2 K. \tag{4}$$

The curve for K as a function of x is given in Fig. 1. For values of x beyond the second maximum K has been computed from Jobst's approximate expressions referred to above. Care must be taken not to join these expressions just beyond the first maximum as this would eliminate the extremely important minimum.

An experimental curve for the transmission of light through an artificial fog is reproduced in Fig. 2. The methods employed in obtaining this curve are covered in detail in another paper.[8] The ordinates of the curve are ratios of the intensity of the light after passing through the fog to the initial inten-

[8] H. G. Houghton, "The Transmission of Visible Light Through Fog."

sity. This ratio will be hereafter called the transmission ratio or simply the transmission. For this case

$$E/E_0 = e^{-kZ} \tag{5}$$

where E is the final intensity, E_0 the initial intensity, Z the length of absorbing medium and k the coefficient of absorption. From Eqs. (3), (4) and (5)

$$E/E_0 = e^{-2\pi n a^2 Z K}. \tag{6}$$

The particles of the fog on which these measurements were made were found to range from about two to three microns in diameter but no values of n were determined. However, it is apparent from an inspection of Figs. 1 and 2, that

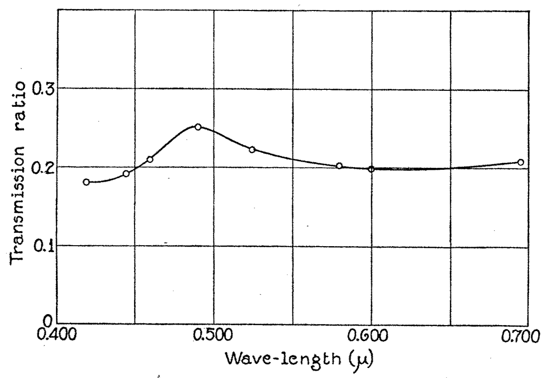

Fig. 2. Experimental curve for the transmission of light through a fog composed of particles, $2.5 \pm 0.5\mu$ in diameter.

the maximum of the experimental curve corresponds to the minimum of the K curve. From the curves

when $\lambda = 0.490\mu$ $E/E_0 = 0.245$

and when $x = 11.2$ $K = 0.825$

also since $x = \frac{2\pi a}{\lambda}$ $2a = \frac{11.2 \times 0.490}{\pi} = 1.75\mu$

This agrees as well as can be expected with the lower end of the observed range of particle diameters.

From Eq. (6)

$$2\pi n a^2 = \frac{\log_e E_0/E}{K}$$

$$= \frac{\log_e (1/.245)}{0.825} = 1.705$$

so that $E_0/E = \log_e^{-1} (1.705K).$

This expression is plotted in Fig. 3 but this curve is not directly comparable with Fig. 2 since the experimental points represent the transmission ratios for finite bands of radiation while the theoretical expression gives a true spectral curve. With the aid of the transmission curves of the filters used a

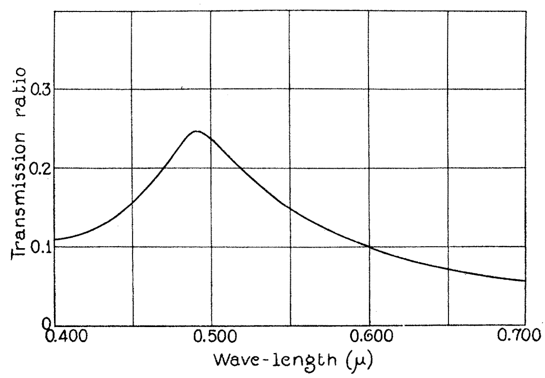

Fig. 3.

theoretical curve may be derived which can be compared directly with the experimental data. This curve is given in Fig. 4, together with the experimental data. The divergence of the two curves is due to the fact that the theoretical curve is for a single particle size while the experimental curve is

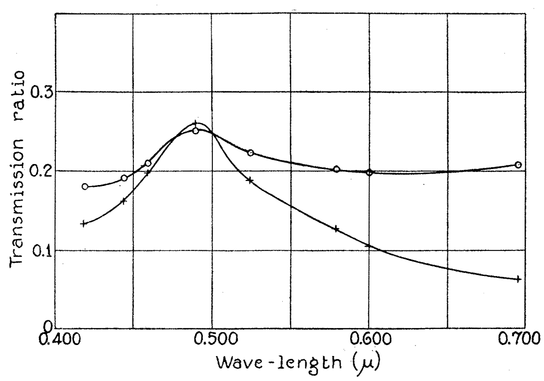

Fig. 4.

for a fog consisting of particles of various sizes within the range of two to three microns. Although a large number of the particles were evidently about 1.75μ in diameter, there were a sufficient number of other sizes present to keep the expermental curve above the theoretical curve. By a proper selection of as few as three particles sizes it was found possible to fit the experimental curve quite closely but in the absence of definite data on the size distribution of the particles such a curve is of doubtful value.

Although no very satisfactory methods of measuring fog particle sizes have been developed, it has generally been assumed that natural fog particles are from five to twenty microns in diameter. For the visible spectrum this corresponds to values of x greater than about twenty. By reference to Fig. 1 it will be seen that K is practically constant in this region and hence little change in transmission would be expected in the case of natural fog. In their measurements on natural fogs, Granath and Hulburt[9] found a very definite increase in transmission for the red end of the spectrum but observed no peaks. Anderson[10] obtained similar results in his work on artificial fogs. If the particle sizes of these fogs were as large as has been assumed, it is hardly possible to explain their results on the basis of this paper.

It has been assumed that the conductivity of the fog particles is negligible but it is possible that in the case of natural fogs the conductivity is sufficient to seriously modify the results. Although it does not appear feasible to obtain a K curve for a finite conductivity by direct summation it is probable that a curve for large values of x can be obtained by some of Jobst's approximate expressions. Work is now being undertaken to determine the probable conductivity of natural fog particles and further theoretical work will await the results of this investigation.

It is also possible that although the larger particles in natural fogs are five to twenty microns in diameter there are a sufficient number of smaller particles present to control the shape of the transmission curve. This can hardly be settled until data are available on the particle size distribution and work is now in progress to develop methods for making such measurements.

Several attempts have recently been made to use a formula given by King[11] as a theoretical expression for the transmission of light through fog. The King formula is essentially Rayleigh's inverse fourth power law for the scattering of light by particles which are small compared to the wave-length of the light. There is no theoretical justification for the application of this formula to the case of fog where the particles are larger than the wave-lengths of light but it may prove useful in some cases as an empirical expression.

The authors desire to take this opportunity to express their thanks to Mr. F. C. Breckenridge of the Bureau of Standards for his many helpful comments and his work in applying the results of this paper to the work of other investigators.

[9] L. P. Granath and E. O. Hulburt, Phys. Rev. **34,** 140 (1929).
[10] S. H. Anderson, Aviation **28,** No. 19, May 10 (1930).
[11] King, Proc. Royal Soc. **A88,** 83 (1913).